INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS, SS18

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Abstract

These notes are based on the course *Introduction to Partial Differential Equations* that the author held during the Spring Semester 2017 for bachelor and master students in mathematics and physics at ETH. They are not supposed to replace the several books in literature on Partial Differential Equations. They simply aim at being a guide for what has been done during the course and a help for the preparation to the exam.
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Chapter 1

Generalities on PDEs

This Chapter surveys the principal theoretical issues concerning the solving of partial differential equations. In Section 1.1 we first list several notations we will use throughout these notes. Then we introduce the concept of partial differential equation. In Section 1.2 we discuss briefly well-posed problems for partial differential equations.

1.1 Notation

IR denotes the real numbers, C the complex numbers. We will work in IR^n and n will always denote the dimension. Points in IR^n will generally be denoted by x, y, ξ, η, the coordinates of x are (x_1, ..., x_n). Occasionally x_1, x_2, ... will denote a sequence of points in IR^n rather than coordinates, but this will always be clear from the context.

If Ω is a subset of IR^n, ∂Ω denotes its boundary, and Ω = Ω ∪ ∂Ω its closure.

If x, y ∈ IR^n we set

\[ x \cdot y = \sum_{i=1}^{n} x_i y_i, \]

so the Euclidean norm of x is given by

\[ |x| = (x \cdot x)^{1/2}. \]

We use the following notation for sphere and (open) balls in IR^n with radius
$r > 0$ and center at $x$:

\begin{align*}
S(x, r) &= \partial B(x, r) = \{ y : |y - x| = r \}, \\
B(x, r) &= \{ y : |y - x| < r \}.
\end{align*}

The volume of $B(x, r)$ and the area of $S(x, r)$ are given by

\begin{align*}
|B(x, r)| &= \frac{\omega_n r^n}{n} \quad \text{and} \quad |S(x, r)| = \omega_n r^{n-1},
\end{align*}

where

\begin{align*}
\omega_n &= |S(x, 1)| = \frac{n \pi^{n/2}}{\Gamma(\frac{1}{2} n + 1)}.
\end{align*}

$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ is the Euler gamma function. We also denote

\begin{align*}
\int_{B(x, r)} fdy &= \text{average of } f \text{ over the ball } B(x, r) \\
\int_{\partial B(x, r)} f d\sigma(y) &= \text{average of } f \text{ over the sphere } \partial B(x, r).
\end{align*}

### 1.1.1 Multi-indices and derivatives

i) A multiindex is a vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each component $\alpha_i$ is a nonnegative integer. The order of $\alpha$ is the number $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

For $x \in \mathbb{R}^n$ we set

\begin{align*}
x^{\alpha} &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.
\end{align*}

ii) Given a multiindex $\alpha$ and given $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ we define

\begin{align*}
D^\alpha u(x) &= \frac{\partial^{|\alpha|} u(x)}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}.
\end{align*}

iii) If $k$ is nonnegative integer,

\begin{align*}
D^k u(x) &= \{ D^\alpha u(x) : |\alpha| = k \}.
\end{align*}
Special cases

If $k = 0$, $D^0 u = u$.

If $k = 1$, we regard the elements of $Du$ as being arranged in a vector:

$$Du = \nabla u = (u_{x_1}, \ldots, u_{x_n}) = \text{gradient vector}.$$ 

If $k = 2$ we regard the elements of $D^2 u$ as being arranged in a matrix:

$$D^2 u = \begin{pmatrix}
    u_{x_1 x_1} & \cdots & u_{x_1 x_n} \\
    \vdots & \ddots & \vdots \\
    u_{x_n x_1} & \cdots & u_{x_n x_n}
\end{pmatrix} = \text{Hessian matrix}.$$

v)

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \text{tr}(D^2 u) = \text{Laplacian of } u.$$

vi) We write the divergence of a vector field $F = (F^1, \ldots, F^n)$ as $\text{div}(F) = \sum_{i=1}^{n} F^i_{x_i}$. We observe that $\Delta u = \text{div} (Du)$.

vii) We sometimes employ a subscript attached to the symbols $D, D^2$ etc. to denote the variables being differentiated. For example if $u = u(x,y)$, $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ then $D_x u = (u_{x_1}, \ldots, u_{x_n})$ and $D_y u = (u_{y_1}, \ldots, u_{y_m})$.

viii) Some function spaces. Let $\Omega$ be a domain of $\mathbb{R}^n$, that is an open and connected subset in $\mathbb{R}^n$.

$C(\Omega) := \{ u: \Omega \to \mathbb{R}, \ u \text{ is continuous} \}$

$C^k(\Omega) := \{ u: \Omega \to \mathbb{R}, \ u \text{ is } k\text{-times continuous differentiable} \}$

$C^\infty(\Omega) := \{ u: \Omega \to \mathbb{R}, \ u \text{ infinitely continuous differentiable} \}$

$C_c(\Omega), C^k_c(\Omega), C^\infty_c(\Omega)$ denote functions in $C(\Omega), C^k(\Omega), C^\infty(\Omega)$ with compact support (the support of a function $f$ will be denoted by $\text{spt}(f)$).
1.2 What is a PDE?

Partial differential equations are central objects in the mathematical modeling of natural and social sciences (sound propagation, heat diffusion, thermodynamics, electromagnetism, elasticity, fluid dynamics, quantum mechanics, population growth, epidemiology finance,...etc). They were for a long time restricted only to the study of natural phenomena or questions pertaining to geometry, before becoming over the course of time, and especially in the last century, a field in itself.

A partial differential equation (in short PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

We fix an integer $k \geq 1$ and let $\Omega \subseteq \mathbb{R}^n$ denote an open set.

**Definition 1.2.1** An expression of the form

$$F(x, u(x), Du(x), \ldots, D^k u(x)) = 0, \text{ in } \Omega \quad (1.1)$$

is called a $k^{th}$-order partial differential equation, where

$$F: \Omega \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{m_k} \to \mathbb{R}$$

is given and $u: \Omega \to \mathbb{R}$ is the unknown, ($m_k$ is the number of multi-indices $\alpha$ of order $k$).

We solve the PDE if we find all $u$ verifying (1.1) possibly only among those functions satisfying certain auxiliary boundary conditions on some part $\Gamma$ of $\partial \Omega$. By finding the solutions we mean, ideally, obtaining simple, explicit solutions, or, failing that, deducing the existence and other properties of solutions.
Definition 1.2.2 (Classification of PDE) (i) The PDE (1.1) is called **linear** if it has the form
\[ \sum_{|\alpha| \leq k} a_\alpha(x)D^\alpha u(x) = f(x) \]
for given functions \( a_\alpha \). This linear PDE is **homogeneous** if \( f \equiv 0 \).

(ii) The PDE (1.1) is called **semilinear** if it has the form
\[ \sum_{|\alpha| = k} a_\alpha(x)D^\alpha u(x) + a_0(x, u(x), \ldots, D^{k-1}u(x)) = 0 \]

(iii) The PDE (1.1) is called **quasilinear** if it has the form
\[ \sum_{|\alpha| = k} a_\alpha(x, u(x), \ldots, D^{k-1}u(x))D^\alpha u(x) + a_0(x, u(x), \ldots, D^{k-1}u(x)) = 0 \]

(iv) The PDE (1.1) is called **fully nonlinear** if it depends nonlinearly upon the highest order derivatives.

1.2.1 Examples of PDEs

There is no general theory known concerning the solvability of all partial differential equations. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDEs.

Following is a list of some specific partial differential equations of interest in current research. We will later discuss the origin and interpretation of these PDE.

Throughout \( x \in \Omega \) where \( \Omega \subseteq \mathbb{R}^n \) is an open set, and \( t \geq 0 \). Also \( Du = D_x u \) denotes the gradient of \( u \) with respect to the spatial variable \( x \) and \( u_t \) the derivative with respect to the time variable \( t \).
1. Linear Transport Equation:

\[ u_t + b(x,t) \cdot Du(x,t) = 0, \quad (x,t) \in D \subseteq \mathbb{R}^n \times (0, +\infty), \]

it is a linear first order equation.

2. Laplace equation:

\[ \Delta u(x) = \sum_{i=1}^{n} u_{x_i x_i} = 0, \quad x \in \Omega \subseteq \mathbb{R}^n \]

it is a linear second order equation.

3. Heat or diffusion equation:

\[ u_t(x,t) - k\Delta u(x,t) = 0 \quad (x,t) \in D \subseteq \mathbb{R}^n \times (0, +\infty). \]

it is a linear second order equation.

4. Wave equation:

\[ u_{tt}(x,t) - c^2 \Delta u(x,t) = 0 \quad (x,t) \in D \subseteq \mathbb{R}^n \times (0, +\infty). \]

it is a linear second order equation.

5. Poisson equation:

\[ \Delta u(x) = f(x), \quad x \in \Omega \subseteq \mathbb{R}^n \]

it is a linear second order equation.

6. Nonlinear Poisson equation:

\[ \Delta u(x) = f(u), \quad x \in \Omega \subseteq \mathbb{R}^n \]

it is a semilinear second order equation.

7. Minimal surface equation

\[ \text{div} \left( \frac{Du}{(1 + |Du|^2)^{1/2}} \right) = 0, \quad x \in \Omega \subseteq \mathbb{R}^n \]

it is a quasilinear second order equation.
8. Schrödinger’s equation
\[ iu_t + \Delta u = 0. \]
it is a linear second order equation.

9. Monge-Ampère equation
\[ \text{Det}(D^2u) = f(x) \quad x \in \Omega \subseteq \mathbb{R}^n. \]
it is a fully nonlinear second order equation.

10. Hamilton-Jacobi equation
\[ H(x, u, Du) = 0 \quad x \in \Omega \subseteq \mathbb{R}^n \]
or
\[ u_t + H(x, u, Du) = 0 \quad (x, t) \in D \subseteq \mathbb{R}^n \times (0, +\infty), \]
they are fully nonlinear first order equations.

11. Scalar conservation law
\[ u_t + \text{div}(F(u)) = 0 \quad (x, t) \in D \subseteq \mathbb{R}^n \times (0, +\infty), \]
the vector field \( F(u) \) depends nonlinearly upon \( u \).

1.2.2 Classification of second order semilinear PDEs

The second order semilinear PDEs can be classified in elliptic, parabolic and hyperbolic. To describe such a classification it is convenient to consider the time as a space coordinate, by setting \( x_{n+1} = t \). Set \( N = n + 1 \), the unknown function is written as \( u = u(x_1, \ldots, x_N) \) and the equation (1.1) can be written in the form
\[ \sum_{i,j=1}^{N} A_{ij}(x)u_{x_i x_j} + A_0 = 0 \quad (1.2) \]
where \( A_{ij} \) are the coefficients of a \( N \times N \) symmetric matrix and \( A_0 \) is a function that may depend on \( u \) and its first derivatives. The quadratic term \( \sum_{i,j=1}^{n} A_{ij}(x)u_{x_i x_j} \) is called principal part of the differential equation.
The equation (1.2) is called **elliptic** if the signature\(^{(1)}\) of the matrix \(A_{ij}\) is \((N, 0)\) or \((0, N)\) (namely the matrix is positive definite or negative definite). The (1.2) is called **parabolic** if the signature of the matrix \(A_{ij}\) is \((N - 1, 0)\) or \((0, N - 1)\) and it is called **hyperbolic** if the signature of the matrix \(A_{ij}\) is \((N - 1, 1)\) or \((1, N - 1)\). We notice that the Laplace equation is elliptic, the diffusion equation is parabolic and the wave equation is hyperbolic.

### 1.2.3 Well-posed problems

What is the meaning of solving partial differential equations? Ideally, we obtain explicit solutions in terms of elementary functions. In practice this is only possible for very simple PDEs, and in general it is impossible to find explicit expressions of all solutions of all PDEs. In the absence of explicit solutions, we need to seek methods to prove existence of solutions of PDEs and discuss properties of these solutions.

Hadamard introduced in his book \(^{(2)}\) the notion of *well-posed problems*. A given problem for a partial differential equation is *well-posed* if

(i) there is a solution;

(ii) this solution is unique;

(iii) the solution depends continuously in some suitable sense on the data given in the problem, i.e., the solution changes by a small amount if the data change by a small amount.

Clearly it would be desirable to “solve” a PDE in such a way that (i)-(iii) hold. Notice that we have not specified what we mean by a “solution”. Should we ask, for example that a “solution” \(u\) must be real analytic or at least infinitely differentiable? This might be desirable, but perhaps we are asking too much. It would be wiser to require a solution of a PDE of

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\(^{(1)}\) Recall: the signature of \(A_{ij}\) is the pair \((p, q)\) where \(p\) is the number of positive eigenvalues and \(q\) is the number of negative eigenvalues

order \( k \) to be at least \( k \) times continuously differentiable. Let us informally call a solution with such smoothness a \textit{classical} solution. So by solving a partial differential equation in the classical sense we mean if possible to write down a formula for a classical solution satisfying \((i)\) – \((iii)\) above.

Unfortunately most PDEs cannot be solved in a classical sense and we are forced to abandon the search for smooth classical solutions. We must instead investigate a wider class of candidates for solutions. If from the beginning we demand that our solutions be very regular, then we are usually going to have a really hard time finding them, as our proofs must then necessarily include possibly intricate demonstrations that the functions we are building are in fact smooth enough. A more reasonable strategy is to consider as separate the existence and the regularity problems. The idea is to define for a given PDE a reasonable notion of a weak solution, with the expectation it may be easier to establish its existence, uniqueness and continuous dependence on the data. For various partial differential equations this is the best that can be done. For other equations we can hope that our weak solution may turn out after all to be smooth to qualify as a classical solution. This leads to the question of \textit{regularity} of weak solutions which usually rests upon many intricate calculus estimates.

**Example 1.2.1 (Counterexample of Hadamard)** Consider the problem of finding a harmonic function in \( E \) taking both Dirichlet data and Neumann data on a portion \( \Sigma_1 \) of \( \partial E \) and Dirichlet data on the remaining part \( \Sigma_2 = \partial E \setminus \Sigma_1 \). Such a problem is ill-posed. Even if a solution exists, in general it is not stable in any reasonable topology, as shown by the following example due to Hadamard. The boundary value problem

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0 & \text{in } (-\pi/2, \pi/2) \times (0, +\infty) \\
    u(\pm \pi/2, y) &= 0 & y > 0 \\
    u(x, 0) &= 0 & x \in (-\pi/2, \pi/2) \\
    u_y(x, 0) &= e^{-\sqrt{n}} \cos(nx) & x \in (-\pi/2, \pi/2)
\end{align*}
\]

admits the family of solutions

\[
u_n(x, y) = e^{-\sqrt{n}} \frac{1}{n} \cos(nx) \sinh(ny), \quad n \text{ odd integer.}\]
One verifies that

\[ \| u_n(\cdot, 0) \|_{L^\infty(-\pi/2, \pi/2)} \to 0 \quad \text{as} \quad n \to +\infty \]

and for \( y > 0 \)

\[ \| u_n(\cdot, y) \|_{L^\infty(-\pi/2, \pi/2)} \to +\infty \quad \text{as} \quad n \to +\infty. \]

**Example 1.2.2 (A further example of the well-posedness issue)**

We consider

\[ u_x = c_0 u + c_1(x, y). \]  \hspace{1cm} (1.3)

Since the equation (1.3) contains no derivative with respect to \( y \) we can regard this variable as a parameter. We can write the general solution of (1.3) in the form

\[ u(x, y) = e^{c_0 x} \left[ \int_0^x e^{-c_0 \xi} c_1(\xi, y) d\xi + T(y) \right] \]  \hspace{1cm} (1.4)

where the function \( T(y) \) is determined by the initial condition \( u(x, 0) \).

Suppose \( c_1 \equiv 0 \). Then the solution (1.3) now becomes \( u(x, y) = e^{c_0 x} T(y) \).

i) Suppose that \( u(x, 0) = 2x \), then \( T(y) \) must satisfy \( T(0) = 2xe^{-c_0 x} \), which is of course impossible. Therefore in this case there is no solution.

ii) Suppose that \( u(x, 0) = 2e^{c_0 x} \), then \( T(y) \) should satisfy \( T(0) = 2 \). Thus every function satisfying \( T(0) = 2 \) will provide a solution for the equation with this initial condition. Therefore (1.3) with \( c_1 = 0 \) has infinitely many solutions with the initial condition \( u(x, 0) = 2e^{c_0 x} \).
Chapter 2

First-order PDEs

In this chapter we search for explicit formulas for general first-order nonlinear partial differential equations of the form

\[ F(x, u, Du) = 0 \text{ in } \Omega \quad (2.1) \]

subject to the boundary condition

\[ u = g \text{ on } \Gamma \subset \partial \Omega \quad (2.2) \]

where \( \Omega \subset \mathbb{R}^n \) is open. Here \( F: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \ g: \Gamma \to \mathbb{R} \) are given and they are supposed smooth functions, \( u: \overline{\Omega} \to \mathbb{R} \) is the unknown.

We introduce the basic notion of noncharacteristic hypersurfaces for initial-value problems. We solve initial-value problems by the method of characteristics if the initial values are prescribed on noncharacteristic hypersurfaces.

We also introduce the Hopf-Lax formula for Hamilton-Jacobi equations and study its regularity properties.
Notation. Let us write $F = F(x, z, p)$, for $p \in \mathbb{R}^n$, $z \in \mathbb{R}$, $x \in \Omega$. Thus $p$ is the name of the variable for which we substitute the gradient $Du(x)$ and $z$ is the variable for which we substitute $u(x)$. We set
\[
\begin{cases}
D_p F = (F_{p_1}, \ldots, F_{p_n}) \\
D_z F = F_z \\
D_x F = (F_{x_1}, \ldots, F_{x_n})
\end{cases}
\quad (2.3)
\]

2.1 The Cauchy Problem

The problem (2.1)-(2.2) is called the Cauchy Problem associated to the equation (2.1). A particular case is a problem of the form
\[
\begin{cases}
u_t + G(x, t, u, D_x u) = 0 & \text{in } \Omega = U \times (0, +\infty) \\
u(x, 0) = g(x) & \text{for } x \in U
\end{cases}
\quad (2.4)
\]
where $U \subseteq \mathbb{R}^n$ is an open set and $G: U \times (0, +\infty) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $g: U \to \mathbb{R}$ are given functions. In this case the independent variables are $(x, t)$, where $x = (x_1, \ldots, x_n)$ is the spatial variable, and $t$ is the time variable, $\Gamma = U \times \{0\}$. The equation in (2.4) is associated to an operator $F = \partial_t + G$ where $G$ does not depend on $\partial_t$. In this context the boundary condition (2.2) is called initial condition.

2.1.1 Linear equations with constant coefficients

Homogeneous case

We consider the so-called transport equation with constant coefficients:
\[
u_t + b \cdot Du = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty),
\quad (2.5)
\]
where $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ is a fixed vector and $u: \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}$ is the unknown $u = u(x, t)$. Here $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ denotes a typical point in space and $t \geq 0$ denotes the time.
The equation (2.5) describes for instance the transport of a solid polluting substance along a channel; here $u$ is the concentration of the substance and $b$ is the stream speed.

Which functions $u$ solve (2.5)?

Let us suppose for the moment we are given some smooth solution $u$ and try to compute it. We observe that given $(x,t)$ for every $s \in \mathbb{R}$ we have

$$0 = u_t(x + sb, t + s) + b \cdot Du(x + sb, t + s) = \frac{d}{ds}u(x + sb, t + s) \quad (2.6)$$

Therefore the function $z(s) = u(x + sb, t + s)$ is a constant function of $s$ and consequently for each point $(x,t)$, $u$ is constant on the line through $(x,t)$ with the direction $(b,1) \in \mathbb{R}^{n+1}$. Hence if we know the value of $u$ at any point on each such line, we know its value everywhere.

Let us consider the initial-value problem

$$\begin{cases}
  u_t + b \cdot Du(x,t) = 0 \quad \text{in } \mathbb{R}^n \times [0, +\infty) \\
  u(x,0) = g(x) \quad \text{for } x \in \mathbb{R}^n.
\end{cases} \quad (2.7)$$

Here $b \in \mathbb{R}^n$ and $g: \mathbb{R}^n \to \mathbb{R}$ are known and the problem is to compute $u$. Given $(x,t)$, the line through $(x,t)$ with direction $(b,1)$ is represented parametrically by $(x + sb, t + s)$ $(s \in \mathbb{R})$. This line hits the hypersurface $\Gamma := \mathbb{R}^n \times \{0\}$ when $s = -t$ at the point $(x - tb, 0)$. Since $u$ is constant on the line and $u(x - tb, 0) = g(x - tb)$, we deduce

$$u(x,t) = g(x - tb) \quad (x \in \mathbb{R}^n, \ t \geq 0) \quad (2.8)$$

So if (2.7) has a sufficiently regular solution $u$, it must be given by (2.8). And conversely it is easy to check directly that if $g \in C^1$ then $u$ defined by (2.8) is indeed a solution of (2.7).

The formula (2.8) says that the initial data is propagated along the curves $x = x_0 + bt$ and for this reason the equation (2.5) is known as the transport equation.
Nonhomogeneous problem

Next let us look at the associated nonhomogeneous problem

\[
\begin{aligned}
&\begin{cases}
  u_t + b \cdot Du(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times [0, +\infty) \\
  u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n.
\end{cases}
\end{aligned}
\]

(2.9)

As before fix \((x,t) \in \mathbb{R}^{n+1}\) and set \(z(s) = u(x + sb, t + s)\) for \(s \in \mathbb{R}\). Then

\[
\dot{z}(s) = b \cdot Du(x + sb, t + s) + u_t(x + sb, t + s) = f(x + sb, t + s).
\]

Consequently

\[
\begin{aligned}
  u(x,t) - g(x - bt) &= z(0) - z(-t) = \int_{-t}^{0} \dot{z}(s) \, ds \\
  &= \int_{-t}^{0} f(x + sb, t + s) \, ds \\
  &= \int_{0}^{t} f(x + (s - t)b, s) \, ds;
\end{aligned}
\]

and so

\[
\boxed{
  u(x,t) = g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s) \, ds
}\]

(2.10)
solves the initial problem (2.9).

Remark 2.1.1 Observe that we have derived our solutions (2.8) and (2.10) by in effect converting the partial differential equations into ordinary differential equations. This is a special case of the method of characteristics that we develop later.

2.2 The Method of the Characteristics

2.2.1 A brief description

The method of characteristics was developed in the middle of the nineteenth century by Hamilton. It consists in converting the Cauchy problem
(2.1)-(2.2) into a Cauchy problem for an appropriate system of ODEs (the so-called characteristic equations). The solutions of the characteristic equations parametrize the so-called characteristic curves, which pass through the following \((n - 1)\)-dimensional manifold

\[
\mathcal{S}_0 = \{(x, g(x)), \ x \in \Gamma\} \tag{2.11}
\]

which is called initial manifold.

Let us denote by \(C_p\) the characteristic curve which passes through \(p \in \mathcal{S}_0\). We suppose that for some \(\bar{p} \in \mathcal{S}_0\) the curve \(C_{\bar{p}}\) is not tangent to \(\mathcal{S}_0\). Then by continuity the same holds for all the curves \(C_p\) for \(p\) in a neighbourhood \(\mathcal{N}\) of \(\bar{p}\). The construction of the characteristic equations is made in such a way that the set of the characteristics curves \(C_p\) with \(p\) in a neighbourhood \(\mathcal{N}\) of \(\bar{p}\) constitute a \(n - 1\) dimensional surface corresponding to the graph of a solution of the PDE in consideration. Such a solution is constructed in a neighborhood \(\mathcal{N}\) of \(\bar{p}\).

The existence of the characteristic curves and the fact that \(\mathcal{S}_0\) is a manifold require some regularity assumptions on \(F\) and \(g\). The construction of the characteristic equation depend on the structure of the partial differential equation in consideration. We start by examining this method in the case of quasilinear first order PDEs and we study later the general case of fully nonlinear equations.

### 2.2.2 Applications to quasilinear first order PDEs

For simplicity we start with the case of quasilinear PDEs of the form

\[
c(x, u) \cdot Du + b(x, u) = 0 \tag{2.12}
\]

where \(x = (x_1, x_2) \in \Omega \subseteq \mathbb{R}^2\), \(b \in C^1(\overline{\Omega} \times \mathbb{R})\), \(c \in C^1(\overline{\Omega} \times \mathbb{R})\).

We are interested in studying classical solutions of (2.12), namely functions \(u : \mathcal{D} \to \mathbb{R}\) of class \(C^1\) that verify \(c(x, u(x)) \cdot Du(x) + b(x, u(x)) = 0\) for every \(x \in \mathcal{D} \subseteq \Omega\). The graph of the solution \(u\) is given by

\[
\mathcal{S} = \{(x, u(x)) : x \in \mathcal{D}\}
\]
with $\mathcal{D} = \text{dom} u$. $\mathcal{S}$ is called \textbf{integral surface} for the equation (2.12). Solving the equation (2.12) means to find all the integral surfaces. We recall that if $\mathcal{S}$ is a 2-dimensional surface of $\mathbb{R}^3$ then the normal unit vector to $\mathcal{S}$ at the generic point $q = (x, u(x))$ is given by

$$N_\mathcal{S}(q) = \frac{1}{\sqrt{1 + |Du|^2}}(-Du(x), 1).$$

If we define the vector field $\mathbf{F}: \Omega \times \mathbb{R} \to \mathbb{R}^3$ by setting

$$\mathbf{F}(x, z) = (c(x, z), -b(x, z)) \ \forall x \in \Omega, \forall z \in \mathbb{R}$$

the equation (2.12) can be written in the form

$$N_\mathcal{S}(x, u(x)) \cdot \mathbf{F}(x, u(x)) = 0, \ \forall x \in \mathcal{D}. \tag{2.13}$$

Thus it holds:

**Proposition 2.2.1** Let $\mathcal{S}$ be a graph of a certain function. Then

$\mathcal{S}$ is an integral surface $\iff N_\mathcal{S}(q) \perp \mathbf{F}(q) \iff \mathbf{F}(q) \in T_q \mathcal{S}$

for all $q \in \mathcal{S}$, where $T_q \mathcal{S}$ is the tangent space to $\mathcal{S}$ at $q$.

We call \textbf{characteristic direction} at a point $q \in \Omega \times \mathbb{R}$ the direction identified by $\mathbf{F}$. The relation (2.13) says that the integral surface is tangent at every point to the characteristic direction in that point. This remark suggests to build an integral surface by following the curves that solve the system

$$\dot{\Phi} = \mathbf{F}(\Phi). \tag{2.14}$$

This system of three ODEs is the so-called system of \textbf{characteristic equations}. By setting $\Phi = (x^1, x^2, z)$ we can write the system in the following form

$$\begin{cases}
\dot{x}^1(s) = c_1(x(s), z(s)) \\
\dot{x}^2(s) = c_2(x(s), z(s)) \\
\dot{z}(s) = -b(x(s), z(s))
\end{cases} \tag{2.15}$$
We call characteristic curves the solutions to (2.15). Moreover we call projected characteristic or simply characteristics the projection of the characteristic curves on the domain Ω. By the hypotheses on c and b the system (2.15) has locally a unique solution. Thus for any point \( q \in \Omega \times \mathbb{R} \) there exists only one characteristic passing through it. It holds the following result.

**Theorem 2.2.1**  
An integral surface is the union of characteristic curves, namely \( \mathcal{S} = \bigcup \{ \Phi^*: \Phi^* \text{ characteristic curve for } p \in \mathcal{S} \} \).

**Proof of Theorem 2.2.1.** Let \( \mathcal{S} \) be an integral surface namely \( \mathcal{S} \) is the graph of \( u: \mathcal{D} \subseteq \Omega \rightarrow \mathbb{R} \) solution of (2.12). Let \( q = (x, u(x)) \in \mathcal{S} \) and let us consider the characteristics curve passing through \( q \) namely the the solution \((x(\cdot), z(\cdot))\) of (2.15) such that \( x(0) = x \) and \( z(0) = u(x) \). This solution will be defined in a certain open interval \( I \) containing the origin and such that \( x(s) \in \mathcal{D} \) for all \( s \in I \). We need to show that such a curve is completely contained in \( \mathcal{S} \).

To this purpose we consider the function 
\[
\varphi(s) = z(s) - u(x(s)) \quad \forall s \in I.
\]

We have
\[
\dot{\varphi}(s) = \dot{z}(s) - Du(x(s)) \cdot \dot{x}(s) = -b(x(s), z(s)) - Du(x(s)) \cdot c(x(s), z(s)) = -b(x(s), \varphi(s) + u(x(s))) - Du(x(s)) \cdot c(x(s), \varphi(s) + u(x(s))).
\]

Thus \( \varphi \) is a solution of the Cauchy problem
\[
\begin{cases}
\dot{\varphi} = f(s, \varphi) \\
\varphi(0) = 0,
\end{cases} \tag{2.16}
\]
where \( f \) is defined in a neighborhood of \((0, 0)\) in \( \mathbb{R}^2 \) by:
\[
f(s, y) = -b(x(s), y + u(x(s))) - Du(x(s)) \cdot c(x(s), y + u(x(s))).
\]
The function $f$ is continuous in $I \times \mathbb{R}$ and it is locally Lipschitz continuous with respect to $y$. This guarantees the uniqueness of the solution. It follows that $\varphi(s) \equiv 0$ for every $s \in I$ (the function identically zero is a solution of the Cauchy problem (2.16)). Thus $(x(s), z(s)) \in S$ for every $s \in I$ and this concludes the proof.

Philosophically speaking, one might say that the characteristic curves take with them an initial piece of information of the initial integral surface, and propagate it with them.

**Corollary 2.2.1** *If two integral surfaces intersect at a point, then they intersect along the characteristic curve that passes through that point.*

### 2.2.3 Existence and Uniqueness for the Cauchy Problem

We consider now the Cauchy problem

\[
\begin{cases}
  c(x, u) \cdot Du + b(x, u) = 0 & \text{in } \Omega \\
  u = g & \text{on } \Gamma
\end{cases}
\]  

(2.17)

where $\Gamma \subseteq \partial \Omega$ is a regular curve contained in $\partial \Omega$ and $g: \Gamma \to \mathbb{R}$ is a given function, which is of class $C^1$ is a neighborhood of $\Gamma$.

We are interested in finding local solutions to (2.17) in a neighborhood of $\bar{x} \in \Gamma$, namely classical solutions of the equation (2.12) defined in $D \cap \Omega$ where $D$ is an open neighborhood of $\bar{x}$ and such that $u(x) = g(x)$, for $x \in D \cap \Gamma$. We show the following result:

**Theorem 2.2.2** Let $\bar{x} \in \Gamma$ be such that

\[
c(\bar{x}, g(\bar{x})) \notin T_{\bar{x}} \Gamma,
\]

(2.18)

then the problem (2.17) has a unique local solution in a neighborhood of $\bar{x}$. 

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The hypothesis (2.18) is called transversality condition and it says that the projected characteristic through $\bar{x}$ is not tangent to $\Gamma$. In this case we also say that the curve $\Gamma$ is noncharacteristic for the Cauchy Problem (2.17) at the point $\bar{x}$.

**Proof of Theorem 2.2.2.** Let $\varphi: \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}^2$, $r \mapsto (\varphi^1(r), \varphi^2(r))$ be a regular ($\varphi'(r) \neq 0$) $C^1$ parametrization of $\Gamma$, such that $\varphi(r) = \bar{x}$. We define $h(r) = g(\varphi(r))$ for all $r \in \mathcal{I}$. The boundary condition can be written as $u(\varphi(r)) = h(r)$ for all $r \in \mathcal{I}$. For every $r \in \mathcal{I}$ the Cauchy problem

$$
\begin{align*}
\dot{x}^1(s) &= c_1(x(s), z(s)) \\
\dot{x}^2(s) &= c_2(x(s), z(s)) \\
\dot{z}(s) &= -b(x(s), z(s)) \\
x^1(0) &= \varphi^1(r) \\
x^2(0) &= \varphi^2(r) \\
Z(0) &= h(r)
\end{align*}
$$

(2.19)

admits a unique solution defined in a neighborhood of 0 and it may depend on $r$. If $r$ belongs to an open set $W \subseteq \mathcal{I}$ containing $\bar{r}$ and such that $\overline{W} \subseteq \mathcal{I}$ is compact, then the domain of the solution of (2.19) can be taken independent of $r$.\(^{(1)}\) Let us denote by $(X(r, \cdot), Z(r, \cdot))$ the solution of (2.19) which is defined in an interval $J$ containing 0. From the theory of ODEs it follows that $X: \mathcal{W} \times J \to \mathbb{R}^2$, and $Z: \mathcal{W} \times J \to \mathbb{R}$ are $C^1(\mathcal{W} \times J)\(^{(2)}\)$ and satisfy

$$
\begin{align*}
X_s(r, s) &= c(X(r, s), Z(r, s)) \\
Z_s(r, s) &= -b(X(r, s), Z(r, s)) \\
X(r, 0) &= \varphi(r) \\
Z(r, 0) &= h(r).
\end{align*}
$$

(2.20)

\(^{(1)}\)In the qualitative theory of ODEs the following result holds: Given an open interval $\mathcal{I} \subset \mathbb{R}$ and an open set $D \subset \mathbb{R}^\nu$ and $f: \mathcal{I} \times D \to \mathbb{R}^\nu$ continuous and locally Lipschitz continuous with respect to the second variable, then for every $K \subset \mathcal{I} \times D$ compact, there is $\delta > 0$ such that for every $(x_0, t_0) \in K$ the Cauchy problem $\dot{u}(t) = f(t, u)$, $u(t_0) = x_0$ admits a unique solution in $(t_0 - \delta, t_0 + \delta)$.

\(^{(2)}\)**Differentiable dependence on the data of the solution of the Cauchy Problem:** Given an open interval $\mathcal{I} \subset \mathbb{R}$ and an open set $D \subset \mathbb{R}^\nu$ and $f: \mathcal{I} \times D \to \mathbb{R}^\nu$ continuous. Suppose that for every $(x_0, t_0) \in \mathcal{I} \times D$ there exists a unique solution $\phi(\cdot, t_0, x_0)$ of the Cauchy Problem $\dot{u}(t) = f(t, u)$, $u(t_0) = x_0$ which is defined in $(t_0 - \delta, t_0 + \delta)$ with $\delta > 0$ independent of $(t_0, x_0)$. If $f$ is of class $C^1$ in $\mathcal{I} \times D$ then the map $(t, t_0, x_0) \mapsto \phi(t, t_0, x_0)$ is of class $C^1$ in $(t_0 - \delta, t_0 + \delta) \times \mathcal{I} \times D$.  

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Claim 1: There is an open neighborhood $\tilde{W} \subseteq W$ of $\bar{r}$ and an open interval $\tilde{J} \subseteq J$ containing 0, such that the function $(r, s) \mapsto X(r, s)$ is invertible from $\tilde{W} \times \tilde{J}$ into $X(\tilde{W} \times \tilde{J}) =: D$. Its inverse function is $C^1$.

Proof of the Claim 1. We verify that $X$ satisfies the hypotheses of the Inverse Function Theorem\(^{(3)} \) in $(\bar{r}, 0) \in W \times J$. We know that $X \in C^1$ in an open neighborhood of $(\bar{r}, 0)$. Moreover

$$DX(\bar{r}, 0) = (D\varphi(\bar{r}), c(X(\bar{r}, 0), Z(\bar{r}, 0))) = (D\varphi(\bar{r}), c(\bar{x}, g(\bar{x}))).$$

We observe that the matrix $DX(\bar{r}, 0)$ is invertible by the transversality condition. The claim is thus proved.

Hence there are functions $S, R: D \to \mathbb{R}$ of class $C^1(D)$ such that

$$x = X(R(x), S(x)), \text{ for all } x \in D.$$

We define $u: D \to \mathbb{R}$ by setting

$$u(x) = Z(R(x), S(x)), \text{ for all } x \in D.$$

Claim 2: $u$ is a local solution of (2.12).

Proof of the Claim 2.

We observe that $u \in C^1(D)$, since it is the composition of functions of class $C^1$. If $x \in D \cap \Gamma$ then $S(x) = 0$ and if we set $r = R(x)$ we have $\varphi(r) = X(r, 0) = x$ and $u(x) = g(\varphi(r)) = g(x)$. We can represent the graph of $u$ through the maps $R$ and $S$:

$$S = \{(x, u(x) : x \in D \} = \{(X(r, s), Z(r, s)) : (r, s) \in \tilde{W} \times \tilde{J}\}.$$

We observe that the tangent space $T_pS$ at the point $p = (X(r, s), Z(r, s))$ is generated by the vectors $((X_r(r, s), Z_r(r, s)))$ and $((X_s(r, s), Z_s(r, s)))$. Hence

$$F(p) = (c(X(r, s), Z(r, s)), -b(X(r, s), Z(r, s)))$$

\(^{(3)}\text{Inverse Function Theorem: Let } A \subset \mathbb{R}^n \text{ be an open set and } x_0 \in A \text{ and } f: A \to \mathbb{R}^n \text{ be of class } C^1 \text{ such that } Df(x_0) \text{ is invertible } (Jac_f(x_0) \neq 0). \text{ Then there exits an open neighborhood } \hat{A} \text{ of } x_0 \text{ such that } f: \hat{A} \to f(\hat{A}) \text{ is invertible and the inverse is of class } C^1.\)
we get that $F(p) \in T_pS$. Therefore $u$ is a solution of the equation (2.12).

**Claim 3: Uniqueness of the maximal solution.**

**Proof of the Claim 3.** We suppose that the solution we have built has a maximal domain $D \cap \overline{\Omega}$ (namely it cannot be extended in a larger domain). We take another solution $\tilde{u}$ with $\text{dom}(\tilde{u})=\tilde{D} \cap \overline{\Omega}$. We set

$$S_0 = \{(x, u(x)) : x \in \Gamma \cap D\}, \quad \tilde{S}_0 = \{(x, \tilde{u}(x)) : x \in \Gamma \cap \tilde{D}\}.$$  

By the maximality of $u$ we have $\tilde{S}_0 \subseteq S_0$. Then by the Corollary 2.2.1 we have $\tilde{S} \subseteq S$. Thus the maximal solution is unique.

We can conclude the proof of Theorem 2.2.2.

**Remark 2.2.1** We observe that Theorem 2.2.2 does not provide an explicit formula for the solution of the Cauchy Problem but a constructive way to get the solution that can be adopted in several specific case.

### 2.3 Derivation of characteristic ODEs in the general case

We return to the general equation

$$F(x, u, Du) = 0 \quad \text{in } \Omega$$  \hspace{1cm} (2.21)

subject to the boundary condition

$$u = g \quad \text{on } \Gamma \subset \partial \Omega$$  \hspace{1cm} (2.22)

where $\Omega \subset IR^n$ is open.

Also in this case we are going to convert the PDE into an appropriate system of ODEs. Suppose $u$ solves (2.21)-(2.22) and fix any point $x \in \Omega$. We would like to calculate $u(x)$ by finding some curves lying within $\Omega$, connecting $x$ with a point $x_0 \in \Gamma$ and along which we can compute $u$ at
the one end \( x_0 \). We hope then to be able to calculate \( u \) all along the curve, and so particularly in \( x \).

**Finding the characteristic ODEs.** Let us suppose the characteristic curve is described parametrically the function \( x(s) = (x^1(s), \ldots, x^n(s)) \), the parameter \( s \) lying in some subinterval of \( \mathbb{IR} \). Assuming \( u \) is a \( C^2 \) solution of (2.21), we define also \( z(s) := u(x(s)) \) and in addition we set \( p(s) := Du(x(s)) \). So \( z(s) \) gives the values of \( u \) along the curve and \( p(s) \) records the values of the gradient \( Du \). We must choose the function \( x(\cdot) \) in such a way that we can compute \( z(\cdot) \) and \( p(\cdot) \).

For this we differentiate \( p(s) \) and we get

\[
\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \dot{x}^j(s). \tag{2.23}
\]

Now we differentiate the PDE (2.21) with respect of \( x_i \):

\[
\sum_{j=1}^{n} F_{p_j}(x, u, Du) u_{x_i x_j} + F_z(x, u, Du) u_{x_i} + F_{x_i}(x, u, Du) = 0. \tag{2.24}
\]

We are able to employ this identity to get rid of the second derivatives terms in (2.23) if we set for \( j = 1, \ldots, n \)

\[
\dot{x}^j(s) = F_{p_j}(x(s), z(s), p(s)). \tag{2.25}
\]

We evaluate (2.24) at \( x = x(s) \) and we obtain the identity

\[
\sum_{j=1}^{n} F_{p_j}(x(s), z(s), p(s)) u_{x_i x_j} + F_z(x(s), z(s), p(s)) \dot{p}^i(s) \tag{2.26}
\]

\[
+ F_{x_i}(x(s), z(s), p(s)) = 0.
\]

By combining (2.25), (2.26) and (2.23) we get

\[
\dot{p}^i(s) = -F_z(x(s), z(s), p(s)) \dot{p}^i(s) - F_{x_i}(x(s), z(s), p(s)). \tag{2.27}
\]

Finally

\[
\dot{z}(s) = \sum_{j=1}^{n} p^j(s) F_{p_j}(x(s), z(s), p(s)). \tag{2.28}
\]
By summarizing we get the following system of $2n+1$ ODEs:

\[
\begin{align*}
(a) \dot{p}(s) &= -D_z F(x(s), z(s), p(s)) - D_x F(x(s), z(s), p(s)) \\
(b) \dot{z}(s) &= D_p F(x(s), z(s), p(s)) \cdot p(s) \\
(c) \dot{x}(s) &= D_p F(x(s), z(s), p(s)).
\end{align*}
\]  

This system of $2n+1$ first-order ODE comprises the characteristic equations of the nonlinear PDE (2.21). The functions $p(\cdot), z(\cdot), x(\cdot)$ are called the characteristics. We refer to $x(\cdot)$ as the projected characteristic: it is the projection of the full characteristics $(p(\cdot), z(\cdot), x(\cdot))$ onto the region $\Omega \subseteq \mathbb{R}^n$.

We have shown:

**Theorem 2.3.1 (Structure of characteristic ODEs)** Let $u \in C^2(\Omega)$ solve the nonlinear, first-order partial differential equation (2.1). Assume $x(\cdot)$ solve the ODE (2.29)(c), where $p(\cdot) = Du(x(\cdot)), z(\cdot) = u(x(\cdot))$. Then $p(\cdot)$ solves the ODE (2.29)(a) and $z(\cdot)$ solves the ODE (2.29)(b), for those $s$ for which $x(\cdot) \in \Omega$.

**Remark 2.3.1** The key step in the derivation is the setting $\dot{x} = D_p F$ so that the terms involving second derivatives drop out. We obtain closure of the system, and in particular we are not forced to introduce ODEs for higher derivatives of $u$.

### 2.3.1 Examples

We consider some special cases for which the structure of the characteristic equations is simple. We show how we can sometimes actually solve the characteristic ODEs and thereby explicitly compute solutions of certain first-order PDEs, subject to appropriate boundary conditions.
a. F linear

\[ F(x, u, Du) = b(x)u(x) + c(x) \cdot Du = 0. \]

Then \( F(x, z, p) = b(x)z + c(x) \cdot p \) and so

\[ D_p F = c(x). \]

The equation (2.29)(c) becomes

\[ \dot{x}(s) = c(x(s)), \]

an ODE involving only the function \( x(\cdot) \). Furthermore equation (2.29)(b) becomes

\[ \dot{z}(s) = c(x(s)) \cdot p(s) = -b(x(s))z(s). \]

In summary

\[ \begin{cases} 
  \dot{x}(s) = c(x(s)) \\
  \dot{z}(s) = -b(x(s))z(s)
\end{cases} \]

(2.30)

comprise equations for the linear, first-order PDE (the equation for \( p(\cdot) \) is not needed).

a. F quasilinear

This case has been already considered in the section 2.2.2:

\[ F(x, u, Du) = b(x, u(x)) + c(x, u(x)) \cdot Du = 0. \]

Then \( F(x, z, p) = b(x, z) + c(x, z) \cdot p \) and so

\[ D_p F = c(x, z). \]

The equation (2.29)(c) becomes

\[ \dot{x}(s) = c(x(s), z(s)). \]

Furthermore equation (2.29)(b) becomes

\[ \dot{z}(s) = -b(x(s), z(s)). \]
Consequently
\[
\begin{align*}
\dot{x}(s) &= c(x(s), z(s)) \\
\dot{z}(s) &= -b(x(s), z(s))
\end{align*}
\tag{2.31}
\]
are the characteristic equations for the quasilinear first-order PDE (once again the equation for \( p() \) is not needed).

**Example 2.3.1** We consider the simpler case of a boundary-value problem for a semilinear PDE:
\[
\begin{align*}
\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} &= u^2 \quad \text{in} \{x_2 > 0\} \\
u &= g \quad \text{in} \{x_2 = 0\}
\end{align*}
\tag{2.32}
\]
In this case \( c = (1, 1) \) and \( b = -z^2 \). Then (2.31) becomes
\[
\begin{align*}
\dot{x_1}(s) &= 1, \quad \dot{x_2}(s) = 1 \\
\dot{z}(s) &= z^2
\end{align*}
\tag{2.33}
\]
Consequently
\[
\begin{align*}
x_1(s) &= x_0 + s, \quad x_2(s) = s \\
\dot{z}(s) &= \frac{z_0}{1 - sz_0} = \frac{g(x_0)}{1 - sg(x_0)}
\end{align*}
\tag{2.34}
\]
where \( x_0 \in \mathbb{R}, \ s \geq 0 \), provided the dominator is not zero. Fix a point \((x_1, x_2) \in \{x_2 > 0\}\) so that \((x_1, x_2) = (x^1(s), x^2(s)) = (x_0 + s, s)\), that is \(x_0 = x_1 - x_2\) and \(s = x_2\). Then
\[
u(x_1, x_2) = u(x^1(s), x^2(s)) = z(s) = \frac{g(x_0)}{1 - sg(x_0)}
\]
This solution makes sense only if \(1 - x_2g(x_1 - x_2) \neq 0\). In the case \(g(x_1) = \cos(x_1)\) then the solution to (2.32) is given by \(u(x_1, x_2) = \frac{\cos(x_1-x_2)}{1-x_2\cos(x_1-x_2)}\). In this case the first blow-up occurs at \(x_2 = 1\) at the points \(x_1\) such that \(\cos(x_1 - 1) = 1\), namely \(x_1 = 1 + 2k\pi, \ k \in \mathbb{Z}\), see Fig. 2.1, (in the picture \(x_2 = t\)).
Example 2.3.2 We consider the problem

\[
1) \begin{cases}
    xu_x + yu_y = x + ye^y & \text{in } \Omega \\
    u(x, y) = e^x & \text{on } \Gamma.
\end{cases}
\]

where \( \Omega = \{y \neq x + 1\} \) and \( \Gamma = \{y = x + 1\} \).

Solution.

i) Parametrization of the boundary condition. The graph of the line \( y = x + 1 \) can be parametrized by \( \varphi(r) = (r, r + 1) \). We set \( h(r) = u(r, r + 1) \).

ii) Characteristic equations.

\[
\begin{align*}
\dot{x}(s) &= x(s) \\
\dot{y}(s) &= y(s) \\
\dot{z}(s) &= x(s) + y(s)e^{y(s)}
\end{align*}
\]

The initial conditions for the system (2.35) are

\[
\begin{align*}
x(0) &= r \\
y(0) &= r + 1 \\
z(0) &= e^r
\end{align*}
\]
The general solution is
\[
\begin{align*}
    x(r, s) &= re^s \\
    y(r, s) &= (r + 1)e^s \\
    z(r, s) &= re^s + \exp((r + 1)e^s) + (1 - e)e^r - r.
\end{align*}
\]
(2.37)

Now we want to invert \((r, s) \mapsto (x(r, s), y(r, s))\).

(iii) Invertibility of the function \((r, s) \mapsto (x(r, s), y(r, s))\). We have to solve the system
\[
\begin{align*}
    x &= re^s \\
    y &= (r + 1)e^s
\end{align*}
\]
(2.38)
The solution is
\[
\begin{align*}
    r &= r(x, y) := \frac{x}{y-x} \\
    s &= s(x, y) := \log(y - x)
\end{align*}
\]
(2.39)
in the half-plane \(\{y > x\}\).

(iv) Writing of the solution in function of \(x\) and \(y\). The solution is
\[
    u(x, y) = z(r(x, y), s(x, y)) = x + e^y + (1 - e) \exp\left(\frac{x}{y-x}\right) - \frac{x}{y-x}.
\]

We observe that even if the coefficients of the PDE are \(C^\infty(\mathbb{R}^2)\) as the initial condition, the solution may not be defined in all \(\mathbb{R}^2\).

**Example 2.3.3 (Failure of the transversality condition)** The following example demonstrates the case where the transversality condition fails along some interval.
\[
\begin{align*}
    u_x + u_y &= 1 \quad \text{in } \mathbb{R}^2 \setminus \{y = x\} \\
    u(x, x) &= x.
\end{align*}
\]
(2.40)
This problem has infinitely many solutions.

**Proof.** You first observe that the transversality condition is violated identically. However the characteristic direction is \((1, 1, 1)\), and so it is the direction of the initial curve. Hence, the initial curve is itself a characteristic curve. Set now the problem
\[
\begin{align*}
    u_x + u_y &= 1 \\
    u(x, 0) &= g(x)
\end{align*}
\]
(2.41)
for an arbitrary $g$ satisfying $g(0) = 0$. The solution of (3.64) is easily found to be $u(x, y) = y + g(x - y)$.

Notice that the Cauchy problem
\[
\begin{aligned}
\begin{cases}
  u_x + u_y = 1 \\
  u(x, x) = 1
\end{cases}
\end{aligned}
\]
(2.42)
on the other hand, is not solvable. To see this observe that the transversality condition fails again, but now the initial curve is not a characteristic curve.

c) F fully nonlinear

In the general case, we must integrate the full characteristic equations, if possible.

Example 2.3.4 Consider the fully nonlinear problem
\[
\begin{aligned}
\begin{cases}
  u_{x_1} u_{x_2} = u & \text{in } \{x_1 > 0\} \\
  u = x_2^2 & \text{in } \{x_1 = 0\}.
\end{cases}
\end{aligned}
\]
(2.43)
Here $F(x, z, p) = p_1 p_2 - z$, and hence the characteristic ODEs (2.29) become
\[
\begin{aligned}
\begin{cases}
  \dot{p}_1 = p_1, & \dot{p}_2 = p_2 \\
  \dot{z} = 2p_1 p_2 \\
  \dot{x}_1 = p_2, & \dot{x}_2 = p_1.
\end{cases}
\end{aligned}
\]
(2.44)
We integrate these equations to find
\[
\begin{aligned}
\begin{cases}
  x_1(s) = p_2^0 (e^s - 1), & x_2(s) = x_2^0 + p_1^0 (e^s - 1) \\
  z(s) = z_0 + p_1^0 p_2^0 (e^{2s} - 1) \\
  p_1^0(s) = p_1^0 e^s, & p_2^0(s) = p_2^0 e^s,
\end{cases}
\end{aligned}
\]
(2.45)
where $0 \neq x_2^0 \in \mathbb{R}$, $s \in \mathbb{R}$ and $z_0 = (x_2^0)^2$.

We must determine $p_1^0 = (p_1^0, p_2^0)$. Since $u = x_2^0$ on $\{x_1 = 0\}$, $p_2^0 = u_{x_2}(0, x_2^0) = 2x_2^0$. Furthermore the PDE $u_{x_1} u_{x_2} = u$ itself implies $p_1^0 p_2^0 = z_0 = (x_2^0)^2$ and so $p_1^0 = \frac{z_0}{2}$. Consequently the formulas above become
\[
\begin{align*}
\begin{cases}
x^1(s) = x^0_2(e^s - 1), \quad x^2(s) = \frac{x^0_2}{2}(e^s + 1) \\
z(s) = (x^0_2)^2 e^{2s} \\
p^1(s) = \frac{x^0_2}{2}e^s, \quad p^2(s) = 2x^0_2 e^s.
\end{cases}
\end{align*}
\] (2.46)

Fix a point \((x_1, x_2) \in \Omega\). Select \(s\) and \(x_0\) so that
\[
(x_1, x_2) = (x^1(s), x^2(s)) = (2x^0_2(e^s - 1), \frac{x^0_2}{2}(e^s + 1))
\]
This inequality implies \(x^0_2 = \frac{4x_2 - x_1}{4}\) and \(e^s = \frac{x_1 + 4x_2}{4x_2 - x_1}\) and so
\[
u(x_1, x_2) = u(x^1(s), x^2(s)) = z(s) = (x^2_2)^2 e^{2s} = \frac{(4x_2 + x_1)^2}{16}.
\]

We observe that since \(x^0_2 \neq 0\) then it should be \(x_1 \neq 4x_2\). Therefore \(u\) is a solution of the equation in (2.43) in \(\{x_1 > 1\} \setminus \{x_1 \neq 4x\}\). The triple \((x^0, z^0, p^0) = ((0, 0), 0, (0, 0))\) does not verify the transversality condition.

### 2.4 Local existence: the general case

We want to solve the problem (2.21)-(2.22), at least in a small region near an appropriate portion \(\Gamma \subset \Omega\). It is convenient first to change variables, so as to flatten out part of the boundary \(\partial \Omega\): we first fix any point \(x^0 \in \partial \Omega\). We can find \(\Phi, \Psi: \mathbb{R}^n \to \mathbb{R}^n\) such that \(\Psi = \Phi^{-1}\) and \(\Phi\) straightens out \(\partial \Omega\) near \(x^0\). Given \(u: \Omega \to \mathbb{R}\) we set \(\tilde{\Omega} = \Phi(\Omega), \quad v(y) = u(\Psi(y))\). If \(u \in C^1(\Omega)\) is a solution of (2.21)-(2.22) then \(v\) solves
\[
\begin{align*}
\begin{cases}
G(y, v, Dv) = 0 & \text{in } \tilde{\Omega} \\
v(y) = g(\Psi(y)) & \text{on } \Phi(\Gamma).
\end{cases}
\end{align*}
\]
where
\[
G(y, v, Dv) = F(\Psi(y), v(y), Dv(y)D(\Phi(\Psi(y)))).
\]
The point is that if we change variables to straighten out the boundary near \(x^0\) the boundary problem (2.21)-(2.22) converts into a problem having the same form.
Therefore we may assume without loss of generality that $\Gamma$ is flat near $x^0$, lying in the plane $\{x_n = 0\}$. We will denote $x = (\tilde{x}, x')$ where $\tilde{x} = (x_1, \ldots, x_{n-1})$. We are going to construct the solution at least near $x^0$ by using the characteristic ODEs and for this we must discover appropriate initial conditions

$$x(0) = x^0, \quad z(0) = z^0, \quad p(0) = p^0. \quad (2.47)$$

If the curve $x(s)$ passes through $x^0$, we should insist that

$$z^0 = g(\tilde{x}^0). \quad (2.48)$$

Since $u(x_1, \ldots, x_{n-1}, 0) = g(x_1, \ldots, x_{n-1})$ we may differentiating and find

$$u_{x_i}(x^0) = g_{x_i}(\tilde{x}^0), \quad \forall i = 1, \ldots, n - 1.$$ 

As we also want the PDE (2.21) to hold, we should require that $p^0$ satisfies

$$\begin{cases} 
\dot{p}^0_i = g_{x_i}(\tilde{x}^0), & \forall i = 1, \ldots, n - 1 \\
F(x^0, z^0, p^0) = 0 
\end{cases} \quad (2.49)$$

The conditions (2.48),(2.49) are called **compatibility conditions**. A triple $(x^0, z^0, p^0) \in \mathbb{R}^{2n+1}$ satisfying (2.48) and (2.49) is called **admissible**.

Now let $(x^0, z^0, p^0)$ be an admissible triple. Since we need to solve the characteristic ODEs for nearby points as well, we ask if we can somehow perturb $(x^0, z^0, p^0)$ keeping the compatibility conditions. In other words, given a point $y = (y_1, \ldots, y_{n-1}, 0) \in \Gamma$ with $y$ close to $x^0$ we solve the characteristics ODE

$$\begin{cases} 
(a) \dot{p}(s) = -D_z F(x(s), z(s), p(s))p(s) - D_x F(x(s), z(s), p(s)) \\
(b) \dot{z}(s) = D_p F(x(s), z(s), p(s)) \cdot p(s) \\
(c) \dot{x}(s) = D_p F(x(s), z(s), p(s)) 
\end{cases} \quad (2.50)$$

with the initial conditions

$$p(0) = q(y), \quad z(0) = g(\tilde{y}), \quad x(0) = y. \quad (2.51)$$
Our task is to find a function $q(\cdot)$ such that
\[ q(x^0) = p^0 \] (2.52)
and $(y, g(\tilde{y}), q(y))$ is admissible, namely it satisfies
\[ \begin{cases} 
q^i(y) = g_{x_i}(\tilde{y}) & \forall i = 1, \ldots, n - 1 \\
F(y, g(\tilde{y}), q(y)) = 0,
\end{cases} \] (2.53)
for all $y \in \Gamma$ close to $x^0$. With this regard the following result holds.

**Lemma 2.4.1 (Noncharacteristic boundary conditions)**

There exists a unique solution $q(\cdot)$ of (2.52)-(2.53) for all $y \in \Gamma$ close to $x^0$ provided
\[ F_{p_n}(x^0, z^0, p^0) \neq 0. \] (2.54)

We say that the admissible triple $(x^0, z^0, p^0)$ is noncharacteristic if (2.54) holds.

**Proof of Lemma 2.4.1.** We apply the Implicit Function Theorem to the mapping $G: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ with
\[ \begin{cases} 
G^i(y, p) = p_i - g_{x_i}(y), & i = 1, \ldots, n - 1 \\
G^n(y, p) = F(y, g(\tilde{y})), p.
\end{cases} \]

We observe that $G(x^0, p^0) = 0$ and
\[ D_pG(x^0, p^0) = \begin{pmatrix} 1 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & 1 & 0 \\
F_{p_1}(x^0, z^0, p^0) & \cdots & F_{p_n}(x^0, z^0, p^0) \end{pmatrix}. \]

Therefore $\det(D_pG(x^0, p^0) = F_{p_n}(x^0, z^0, p^0) \neq 0$. The Implicit Function Theorem thus ensures we can uniquely solve the identity $G(y, p) = 0$ for $p = q(y)$ provided $y$ is close to $x^0$. \qed
Remark 2.4.1 If $\Gamma$ is not flat near $x^0$, the condition that $\Gamma$ be non characteristic reads

$$D_p F(x^0, z^0, p^0) \cdot \nu(x^0) \neq 0,$$

(2.55)

$\nu(x^0)$ denoting the outward unit normal to $\partial \Omega$ at $x^0$.

The condition (2.55) is the so-called transversality condition that we have already introduced in (2.18).

Let $(x^0, z^0, p^0)$ be an admissible noncharacteristic triple, then the following result holds:

\begin{lemma} [Local Invertibility]
Assume we have the non-characteristic condition $F_{p_n}(x^0, z^0, p^0) \neq 0$. Then there exist an open interval $I \subset \mathbb{R}$ containing 0, a neighborhood $W$ of $x^0$ in $\Gamma \subset \mathbb{R}^{n-1}$ and a neighborhood $V$ of $x^0$ in $\mathbb{R}^n$ such that for each $x \in V \cap \overline{\Omega}$ there exist unique $s \in I$, $\tilde{y} \in W$ such that

$$x = x(s, \tilde{y}).$$

The map $x \mapsto (s(x), \tilde{y}(x))$ is $C^2(V \cap \overline{\Omega})$.\end{lemma}

\begin{proof}
We have $x(0, x^0) = x^0$. Moreover $x(0, \tilde{y}) = (y, 0)$, $y \in \Gamma$, and so

$$\frac{\partial x^j}{\partial y_i} (0, x^0) = \begin{cases} 
\delta_{ij} & (j = 1, \ldots, n-1) \\
0 & (j = n).
\end{cases}$$

From the characteristic ODEs (2.50) it follows that $\frac{\partial x^j}{\partial s} (0, x^0) = F_{p_j}(x^0, z^0, p^0)$. Therefore $\det(Dx(0, x^0)) = F_{p_n}(x^0, z^0, p^0) \neq 0$ and the result follows from the Inverse Function Theorem.

In view of Lemma 2.4.2 for each $x \in V \cap \overline{\Omega}$ we can locally solve the equation.

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\[
\begin{cases}
  x = x(s, \tilde{y}) \\
  \text{for } \tilde{y} = \tilde{y}(x) \text{ and } s = s(x).
\end{cases}
\]

Finally let us define

\[
\begin{cases}
  u(x) := z(s(x), \tilde{y}(x)) \\
  p(x) := p(s(x), \tilde{y}(x))
\end{cases}
\]  \tag{2.56}

for \(x \in V \cap \overline{\Omega}, \tilde{y} = \tilde{y}(x)\) and \(s = s(x)\).

**Theorem 2.4.1 (Local Existence Theorem)** The function \(u\) defined in (2.56) is a \(C^2\) solution of the equation

\[
F(x, u(x), Du(x)) = 0, \quad x \in V \cap \Omega
\]

with the boundary condition

\[
u(x) = g(x), \quad x \in \Gamma \cap V.
\]

For the proof of Theorem 2.4.1 we refer the reader to Evans’s book.

Next we show an example where the projected characteristics emanating from distinct points in \(\Gamma\) may intersect outside the domain of the solution. Such an occurrence usually signals the failure of our local solution to exist within all of \(\Omega\).

### 2.4.1 Applications

We turn now to various special cases, to see how the local existence theory simplifies.

**a) \(F\) linear.**

We consider the case

\[
F(x, u, Du) = c(x) \cdot Du(x) + b(x)u(x) = 0.
\] \tag{2.57}

The noncharacteristic assumption (2.55) at a point \(x^0 \in \Gamma\) becomes

\[
c(x^0) \cdot \nu(x^0) \neq 0.
\] \tag{2.58}
Furthermore if we specify the boundary condition
\[ u = g, \quad \text{on } \Gamma, \] (2.59)
we can solve (2.53) for \( q(y) \) if \( y \in \Gamma \) is near \( x^0 \) and then apply Theorem 2.4.1 to construct a unique solution of (2.57) and (2.59) in some neighborhood \( V \) of \( x^0 \). Observe that the projected characteristics emanating from distinct points on \( \Gamma \) cannot cross, owing to the uniqueness of the solution of the initial value problem for the ODE \( x(s) = c(x(s)) \).

\textbf{b) \( F \) quasilinear.}

We consider the case
\[ F(x, u, Du) = c(x, u) \cdot Du(x) + b(x, u(x)) = 0. \] (2.60)
The noncharacteristic assumption (2.55) at a point \( x^0 \in \Gamma \) becomes
\[ c(x^0, z^0) \cdot \nu(x^0) \neq 0, \] (2.61)
where \( z^0 = g(x^0) \). If we specify as previously the boundary condition
\[ u = g, \quad \text{on } \Gamma, \] (2.62)
we can solve (2.53) for \( q(y) \) if \( y \in \Gamma \) is near \( x^0 \) and then apply Theorem 2.4.1 to construct a unique solution of (2.57) and (2.62) in some neighborhood \( V \) of \( x^0 \).

\textit{In contrast with the linear case it is possible that the projected characteristics emanating from distinct points in \( \Gamma \) may intersect outside \( V \); such an occurrence usually signals the failure of our local solution to exists within all of \( \Omega \).

\textbf{Example 2.4.1 (Characteristics for conservation law)} \ We consider the scalar conservation law:
\[ u_t + \text{div}(\mathbf{F}(u)) = u_t + \mathbf{F}'(u) \cdot Du = 0, \] (2.63)
in \( \Omega = \mathbb{R}^n \times (0, +\infty) \) subject to the initial condition
\[ u = g \quad \text{on } \Gamma = \mathbb{R}^n \times \{ t = 0 \} \] (2.64)
Here $F: \mathbb{R} \rightarrow \mathbb{R}^n$, $F = (F^1, \ldots, F^n)$ and we set $t = x_{n+1}$. Writing $q = (p, p_{n+1})$ and $y = (x, t)$ we have

$$G(y, z, q) = p_{n+1} + F'(z) \cdot p.$$  

The transversality condition is clearly satisfied at each point $y^0 = (x^0, 0) \in \Gamma$. The characteristics equations are:

$$\begin{cases}
(a) \dot{x}^i(s) = F^i(z(s)), & i = 1, \ldots, n \\
(b) \dot{x}^{n+1}(s) = 1 \\
(c) \dot{z}(s) = 0.
\end{cases} \tag{2.65}$$

Hence $x^{n+1}(s) = s$ in agreement with our having written $x_{n+1} = t$ above. In other words, we can identify the parameter $s$ with the time $t$. We have $z(s) = z^0 = g(x^0)$. Thus

$$x(s) = F'(g(x^0))s + x^0.$$  

Thus the projected characteristics $y(s) = (x(s), s) = (F'(g(x^0))s + x^0, s)$ are straight lines, along which $u$ is constant.

Suppose now we apply the reasoning to a different initial point $w^0 \in \Gamma$, where $g(x^0) \neq g(w^0)$. The projected characteristics may possibly intersect at some time $t > 0$. Since $u \equiv g(x^0)$ on the projected characteristics through $x^0$ and $u \equiv g(w^0)$ on the projected characteristics through $w^0$, an apparent contradiction arises. The resolution is that the initial-value problem (2.63)-(2.64) does not have a smooth solution existing all times $t > 0$. There is the need to extend the classical solution to a kind of “weak” or “generalized” solution.

In this case since $t = s$ we have

$$u(t, x(t)) = z(t) = g(x(t) - tF'(z^0)) = g(x(t) - tF'(u(t, x(t))).$$

Hence

$$u = g(x - tF'(u)).$$

This gives a solution provided
1 + t g'(x - t F'(u)) \cdot F''(u) \neq 0.

We consider the particular case of the Burgers’ equation:

\[
\begin{align*}
& \left\{ \begin{array}{l}
 u_t + uu_x = 0 \quad \text{in } \mathbb{R} \times (0, +\infty), \\
 u(x, 0) = g(x) \quad \text{on } \mathbb{R}.
\end{array} \right.
\end{align*}
\]  

(2.66)

The characteristic ODEs and the corresponding initial values are given by

\[
\begin{align*}
& \left\{ \begin{array}{l}
 \dot{x}(s) = u \\
 \dot{t}(s) = 1 \\
 \dot{u}(s) = 0 \\
 x(0) = x_0, \ t(0) = 0, \ u(0) = g(x_0).
\end{array} \right.
\end{align*}
\]  

(2.67)

Here, \(x, t\) and \(u\) are all treated as functions of \(s\). By solving for \(t\) and \(u\) first and then for \(x\), we obtain

\[
\begin{align*}
x(s) &= g(x_0) s + x_0, \ t(s) = s, \ u(s) = g(x_0).
\end{align*}
\]

By eliminating \(s\) from the expression of \(x\) and \(t\) we have

\[
x(t) = g(x_0) t + x_0.
\]  

(2.68)

By the implicit function theorem, we can solve for \(x_0\) in terms of \((x, t)\) in a neighborhood of the origin in \(\mathbb{R}^2\). If we denote such a function by \(x_0 = x_0(x, t)\) we have a solution \(u = g(x_0(x, t))\), for any \((x, t)\) sufficiently small. We may also write the solution implicitly by \(u = g(x - ut)\). Let \(\gamma_{x_0}\) be the characteristic curve given by (2.68). It is a straight line in \(\mathbb{R}^2\) with a slope \(\frac{1}{g(x_0)}\) along which \(u\) is the constant \(g(x_0)\). For \(x_0 < x_1\) with \(g(x_0) > g(x_1)\), two characteristic curves \(\gamma_{x_0}\) and \(\gamma_{x_1}\) intersect at \((X, T)\) with

\[
T = -\frac{x_0 - x_1}{g(x_0) - g(x_1)}.
\]

Hence \(u\) cannot be extended as a smooth solution up to \((X, T)\) even as a continuous function. Such a positive \(T\) always exists unless \(g\) is nondecreasing. In a special case where \(g\) is strictly decreasing, any two characteristic curves intersect.
1. Let \( g(x) = -x \). In this case \( \gamma_{x_0} \) is given by \( x = x_0 - x_0 t \), and the solution on this line is given by \( u = -x_0 \). We note that each \( \gamma_{x_0} \) contains the point \( (x, t) = (0, 1) \) and hence any two characteristics intersect at \( (0, 1) \). Then, \( u \) cannot be extended up to \( (0, 1) \) as a smooth solution. In fact we have \( x_0 = \frac{x}{1-t} \) and therefore \( u \) is given by

\[
u(x, t) = \frac{x}{t-1}, \quad \forall (x, t) \in IR \times (0, 1).
\]

Clearly \( u \) is not defined at \( t = 1 \).

2. Let \( g(x) = \frac{1}{1+x^2} \). In this case the projected characteristics are

\[
x(t) = x + \frac{1}{1+x^2} t.
\]

We have

\[
u(x(s), t(s)) = u(x + \frac{1}{1+x^2} s, s) = \frac{1}{1+x^2}.
\]

(2.69)

We observe that \( x \mapsto x + \frac{t}{1+x^2} \) is injective if and only if \( t < T = \frac{8}{\sqrt{27}} \).

Actually

\[
\frac{d}{dx} \left( x + \frac{t}{1+x^2} \right) > 0 \iff t < \min_{x>0} \frac{(1+x^2)^2}{2x} = \frac{8}{\sqrt{27}}.
\]

For \( t > \frac{8}{\sqrt{27}} \) the derivative changes sign. This means that the projected characteristics intersect in some point and therefore the problem (2.66) has not anymore \( C^1 \) solutions.

We can see that \( \lim \inf_{t \to T^-} u_x(x, t) = -\infty \). Indeed from (2.69) we get

\[
\frac{d}{dx} u(x + \frac{1}{1+x^2} t, t) = u_x(x + \frac{1}{1+x^2} t, t) \partial_x \left( x + \frac{1}{1+x^2} t \right)
\]

\[
= -\frac{2x}{(1+x^2)^2}.
\]

Moreover

\[
\partial_x \left( x + \frac{1}{1+x^2} t \right) \bigg|_{(x=\frac{x_0}{2}, t=\frac{8}{27})} = \left( 1 - \frac{2xt}{(1+x^2)^2} \right) \bigg|_{(x=\frac{x_0}{2}, t=\frac{8}{27})} = 0.
\]
Then if we set $\bar{x} = \frac{\sqrt{3}}{3}$ we obtain
\[
\lim_{t \to T^-} u_x(\bar{x} + \frac{1}{1 + \bar{x}^2} t, t) = \lim_{t \to T^-} -\frac{2\bar{x}}{(1 + \bar{x}^2)^2}(1 - \frac{2\bar{x}t}{(1 + \bar{x}^2)^2})^{-1} = -\infty.
\]

\textbf{c) F fully nonlinear.} We consider the case of Hamilton Jacobi Equations of the form
\[
u_t + H(x, Du) = 0.
\] (2.70)

The characteristic equations for the Hamilton-Jacobi equation are
\[
\begin{align*}
(a) \dot{p}(s) &= -D_x H(x(s), p(s)) \\
(b) \dot{z}(s) &= D_p H(x(s), p(s)) \cdot p(s) - H(x(s), p(s)) \\
(c) \dot{x}(s) &= D_p H(x(s), p(s)),
\end{align*}
\] (2.71)

The first and the third of these equalities
\[
\begin{align*}
\dot{p}(s) &= -D_x H(x(s), p(s)) \\
\dot{x}(s) &= D_p H(x(s), p(s)),
\end{align*}
\] (2.72)

are called Hamiltonian’s equations. Observe that the equation for $z(\cdot)$ is trivial once $x(\cdot)$ and $p(\cdot)$ have been found by solving Hamilton’s equations. The initial-value problem for the Hamilton-Jacobi equation does not in general have a smooth solution $u$ lasting for all times $t > 0$.  

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Exercise 2.4.1 Find the solutions to the following problems.

1) \[
\begin{array}{l}
u_t + 2tu_x = 0 \\
u(x,0) = e^{-x^2}.
\end{array}
\]
Sol. \(u(x,t) = e^{-(x-t)^2}\).

2) \[
\begin{array}{l}
u_t + 3u_x = 0 \\
u(x,0) = \sin(x)
\end{array}
\]
Sol. \(u(x,t) = \sin(x-3t)\).

3) \[
\begin{array}{l}
u_t - x^2tu_x = 0 \\
u(x,0) = x + 1.
\end{array}
\]
Sol. \(u(x,t) = \frac{(2-t^2)x+2}{2-xt^2}\).

4) \[
\begin{array}{l}
x_2u_{x_1} - x_1u_{x_2} = u \quad \text{in } \Omega \\
u(x_1,0) = g(x_1) \quad \text{on } \Gamma.
\end{array}
\]
where \(\Omega = \{x_1 > 0, x_2 > 0\}\) and \(\Gamma = \{x_1 > 0, x_2 = 0\}\).

5) \[
\begin{array}{l}
u_{x_1} + u_{x_2} = u^2 \quad \text{in } \Omega \\
u(x_1,0) = g(x_1) \quad \text{on } \Gamma.
\end{array}
\]
where \(\Omega = \{x_2 > 0\}\) and \(\Gamma = \{x_2 = 0\}\).

6) \[
\begin{array}{l}
x_1u_{x_1} + u_{x_2} = 2x_1u \quad \text{in } \Omega \\
u(x_1,0) = g(x_1) \quad \text{on } \Gamma.
\end{array}
\]
where \(\Omega = \{x_1 \in \mathbb{R}, x_2 > 0\}\) and \(\Gamma = \{x_2 = 0\}\).

7) Let \(\alpha \in \mathbb{R}\) and \(h = h(x)\) be a continuous function in \(\mathbb{R}\). Consider
\[
\begin{array}{l}
x_2u_{x_1} + x_1u_{x_2} = \alpha u \quad \text{in } x_2 > 0 \\
u(x_1,0) = h(x_1) \quad \text{on } x_2 = 0.
\end{array}
\]

a) Find all points on \(\{x_2 = 0\}\) where \(\{x_2 = 0\}\) is characteristic. What is the compatibility condition on \(h\) at these points?
b) Away from the points in a), find the solution of the initial-value problem. What is the domain of this solution in general?

c) For the case $h(x) = x$ and $\alpha = 1$, check whether this solution can be extended over the points in a).

### 2.5 Introduction to Hamilton Jacobi Equation

In this section we study the Hamilton-Jacobi equation:

$$
\begin{cases}
  u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
  u = g & \text{in } \mathbb{R}^n \times \{t = 0\}
\end{cases}
$$

(2.73)

Here $u: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is the unknown, $u = u(x,t)$, and $Du = D_x u = (u_{x_1}, \ldots, u_{x_n})$.

The Hamiltonian $H: \mathbb{R}^n \to \mathbb{R}$ and the initial function $g: \mathbb{R}^n \to R$ are given.

Our goal is to find an appropriate weak or generalized solution, existing for all times $t > 0$, even after the method of characteristic failed. Observe that the characteristic equations associated to (2.73) can be reduced to the system

$$
\begin{cases}
  \dot{x}(s) = D_p H(p(s)) \\
  \dot{z}(s) = D_p H(p(s)) - H(p(s)) \\
  \dot{p}(s) = -D_x H(p(s)) = 0
\end{cases}
$$

(2.74)

As for conservation laws, the problem (2.73) has not a smooth solution $u$ lasting for all times $t > 0$.

**Example 2.5.1**

$$
\begin{cases}
  |u'(t)|^2 = 1 & \text{in } (-1, 1) \\
  u(-1) = u(1) = 0
\end{cases}
$$

(2.75)

*It is clear that by a simple application of Rolle Theorem this problem has not $C^1$ solution.*

Hereafter we make the following assumptions on $H$. 

---

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(A1) $H$ smooth

(A2) $H$ convex

(A3) $H$ is coercive, namely it satisfies

$$\lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty.$$ 

2.5.1 Legendre transform, Hopf-Lax formula

a. Legendre transform. Let $L: \mathbb{R}^n \to \mathbb{R}$ satisfy (A2) and (A3).

**Definition 2.5.1** The Legendre transform of $L$ is

$$L^*(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(q) \}.$$ (2.76)

We observe that the “sup” in (2.76) is really a “max”, that is there exists some $q^*$ for which

$$L^*(p) = p \cdot q^* - L(q^*)$$

and the mapping $q \mapsto p \cdot q - L(q)$ has a maximum at $q = q^*$. But then $p = DL(q^*)$, provided $L$ is differentiable at $q^*$. Hence the equation $p = DL(q)$ is solvable for $q$ in terms of $p$, $q^* = q(p)$. Therefore

$$L^*(p) = p \cdot q(p) - L(q(p)).$$

**Theorem 2.5.1** Assume $L$ satisfies (A2) and (A3). Then

(i) the mapping $p \mapsto L^*(p)$ is convex and

$$\lim_{|p| \to +\infty} \frac{L^*(p)}{|p|} = +\infty.$$

(ii) We have $L = L^{**}$
b. Hopf-Lax formula

We now consider the functional

$$J[w(\cdot)] := \int_0^t L(\dot{w}(s))ds$$

(2.77)

defined for functions $w \in \mathcal{A} := \{w(\cdot) \in C^1([0,t]; \mathbb{R}^n) | w(0) = y, w(t) = x\}$.

A basic problem in the calculus of variation is to find a curve $x(\cdot) \in \mathcal{A}$ which minimizes the functional $J$.

One can show that if $x(\cdot)$ minimizes (2.77) and if $p(s) = D_qL(\dot{x}(s))$ (the so-called generalized momentum corresponding to the position $x(\cdot)$) then $x(\cdot)$ and $p(\cdot)$ satisfies the ODEs

$$\begin{cases}
\dot{x}(s) = D_pH(p(s)) \\
\dot{p}(s) = -D_xH(p(s)
\end{cases}$$

(2.78)

where $H = L^*$. Moreover the mapping $s \mapsto H(p(s))$ is constant.

In view of the link between a calculus variation problem and the characteristics equations of the associated Hamilton-Jacobi equation we conjecture that there is also a connection between the Hamilton-Jacobi PDE and a calculus variation problem. To this purpose we consider the following functional (which takes into account the initial condition for our PDE):

$$\int_0^t L(\dot{w}(s))ds + g(w(0)).$$

(2.79)

where $L = H^*$. We set

$$u(x,t) = \inf \{ \int_0^t L(\dot{w}(s))ds + g(y) : w(0) = y, w(t) = x \},$$

(2.80)

the infimum taken over all $C^1$ functions $w(\cdot)$ with $w(t) = x$.

The function $u$ is called the value function associated to the minimization problem. We propose to investigate the sense in which $u$ defined in (2.80) solves (2.73). We henceforth suppose also $g: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous.

We note that formula (2.80) can be simplified:
**Theorem 2.5.2 (Hopf-Lax formula)** If $x \in \mathbb{R}^n$ and $t > 0$, then the solution $u = u(x,t)$ of the minimization problem (2.80) is

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \{ tL \left( \frac{x-y}{t} \right) + g(y) \}.$$ (2.81)

**Definition 2.5.2** We call the expression on the right end side of (2.81) the Hopf-Lax formula.

**Proof of Theorem 2.5.2.**

1. Fix any $y \in \mathbb{R}^n$ and define $w(s) := y + \frac{s}{t}(x-y)$ ($0 \leq s \leq t$). The definition of $u$ implies

$$u(x,t) \leq \int_0^t L(w(s))ds + g(y) = tL \left( \frac{x-y}{t} \right) + g(y),$$

and so

$$u(x,t) \leq \inf_{y \in \mathbb{R}^n} \{ tL \left( \frac{x-y}{t} \right) + g(y) \}.$$

2. On the other hand, if $w(\cdot)$ is any $C^1$ function satisfying $w(t) = x$, we have

$$L \left( \frac{1}{t} \int_0^t \dot{w}(s)ds \right) \leq \frac{1}{t} \int_0^t L(\dot{w}(s))ds,$$

by Jensen Inequality. Thus if we write $y = w(0)$ we find

$$tL \left( \frac{x-y}{t} \right) + g(y) \leq \int_0^t L(\dot{w}(s))ds + g(y);$$

and consequently

$$\inf_{y \in \mathbb{R}^n} \{ tL \left( \frac{x-y}{t} \right) + g(y) \} \leq u(x,t).$$

3. We have so far shown

$$u(x,t) = \inf_{y \in \mathbb{R}^n} \{ tL \left( \frac{x-y}{t} \right) + g(y) \};$$
and we leave it as an exercise to prove the infimum above is really a minimum. □

We are going to show that the formula provides a reasonable weak solution of the initial-value problem (2.73).

First we record some preliminary observations.

**Lemma 2.5.1** For each \( x \in \mathbb{R}^n \) and \( 0 \leq s \leq t \) we have

\[
u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ (t - s) (x - y) + u(y, s) \right\}.
\]

In other words, to compute \( u(\cdot, t) \), we can calculate \( u \) at time \( s \) and then use \( u(\cdot, s) \) as the initial condition on the remaining time interval \([s, t] \). It is nothing that the so-called Dynamic Programming Principle.

**Lemma 2.5.2** The function \( u \) is Lipschitz continuous in \( \mathbb{R}^n \times [0, +\infty) \).

Now Rademacher’s Theorem asserts that a Lipschitz function is differentiable almost everywhere. Consequently in view of Lemma 2.5.2 the function \( u \) defined by the Hopf-Lax formula is differentiable for a.e. \((x, t) \in \mathbb{R}^n \times (0, +\infty) \).

It is easy to see that, in general, \( u \) fails to be everywhere differentiable, even if \( L \) and \( g \) are differentiable.

**Example 2.5.2** Let us consider the problem (2.73) with \( n = 1 \), \( L(q) = q^2/2 \) and

\[
g(x) = \begin{cases} 
-x^2 & \text{if } |x| < 1 \\
1 - 2|x| & \text{if } |x| \geq 1.
\end{cases}
\]

In this case

\[
u(x, t) = \begin{cases} 
-x^2 & \text{if } t < 1/2 \text{ and } |x| < 1 - 2t \\
1 - 2(|x| + t) & \text{if } |x| \geq 1 - 2t \geq 0 \text{ or if } t \geq 1/2
\end{cases}
\]

Therefore \( u(x, t) \) is not differentiable in \((0, t) \) with \( t \geq 1/2 \).
The function $u$ defined by the Hopf-Lax formula (2.81) is differentiable a.e. in $\mathbb{R}^n \times (0, +\infty)$ and solves the initial-value problem

$$
\begin{aligned}
&\left\{
\begin{array}{ll}
  u_t(x, t) + H(Du(x, t)) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\
  u(x, 0) = g(x) & \text{on } \mathbb{R}^n \times \{t = 0\}
\end{array}
\right.
\end{aligned}
$$

**Proof of Theorem 2.5.3.**

1. We show that $u = g$ on $\mathbb{R}^n \times \{t = 0\}$.

By choosing $y = x$ in the representation formula (2.81) we discover

$$
u(x, t) \leq tL(0) + g(x).$$

Furthermore

$$
u(x, t) = \min_{y \in \mathbb{R}^n}\{tL\left(\frac{x - y}{t}\right) + g(y)\}$$

$$\geq g(x) + \min_{y \in \mathbb{R}^n}\{tL\left(\frac{x - y}{t}\right) - Lip(g)|x - y|\}$$

$$= g(x) + \min_{z \in \mathbb{R}^n}\{tL(z) - tLip(g)|z|\}$$

$$= g(x) - \max_{z \in \mathbb{R}^n}\{-tL(z) + tLip(g)|z|\}$$

$$= g(x) - t \max_{|w| \leq Lip(g)} \max_{z \in \mathbb{R}^n}\{w \cdot z - L(z)\}$$

$$= g(x) - t \max_{|w| \leq Lip(g)} H(w).$$

Thus

$$|u(x, t) - g(x)| \leq Ct,$$

where $C := \max(|L(0)|, \max_{|w| \leq Lip(g)} |H|)$.

2. Let $(x, t) \in \mathbb{R}^n \times (0, +\infty)$ be a differentiability point of $u$. We show that

$$u_t(x, t) + H(Du(x, t)) = 0.$$

2i) Fix $q \in \mathbb{R}^n$, $h > 0$. Owing to Lemma 2.5.1

$$
u(x + hq, t + h) = \min_{y \in \mathbb{R}^n}\{hL\left(\frac{x + hq - y}{h}\right) + u(y, t)\}$$

$$\leq hL(q) + u(x, t).$$
Hence
\[ \frac{u(x + hq, t + h) - u(x, t)}{h} \leq L(q). \]

Let \( h \to 0^+ \), to compute
\[ q \cdot Du(x, t) + u_t(x, t) \leq L(q). \]

This inequality is valid for all \( q \in \mathbb{R}^n \), and so
\[ u_t(x, t) + H(Du(x, t)) = u_t(x, t) + \max_{q \in \mathbb{R}^n} \{ q \cdot Du(x, t) - L(q) \} \leq 0. \quad (2.82) \]

The first equality holds since \( H = L^* \).

2ii) Now we choose \( z \) such that \( u(x, t) = tL(\frac{x - z}{t}) + g(z) \). Fix \( h > 0 \) and set \( s = t - h, y = \frac{s}{t} x + (1 - \frac{s}{t}) z \). Then \( \frac{x - z}{t} = \frac{y - z}{s} \) and thus
\[
\begin{align*}
    u(x, t) - u(y, s) & \geq tL(\frac{x - z}{t}) + g(z) - [sL(\frac{y - z}{s}) + g(z)] \\
    & = (t - s)L(\frac{x - z}{t}).
\end{align*}
\]

Let \( h \to 0^+ \) to compute
\[
\frac{x - z}{t} \cdot Du(x, t) + u_t(x, t) \geq L(\frac{x - z}{t}).
\]

Consequently
\[
\begin{align*}
    u_t(x, t) + H(Du(x, t)) & = u_t(x, t) + \max_{q \in \mathbb{R}^n} \{ q \cdot Du(x, t) - L(q) \} \\
    & \geq u_t(x, t) + \frac{x - z}{t} \cdot Du(x, t) - L(\frac{x - z}{t}) \\
    & \geq 0
\end{align*}
\]

This complete the proof. \( \square \)

**Remark 2.5.1** The problem of solving (2.73) almost everywhere is not enough to characterize the value function. Indeed such a problem can have more than one solution in the class of Lipschitz continuous functions as the next example shows.
Example 2.5.3 The problem
\[
\begin{cases}
  u_t + \frac{1}{2}u_x^2 = 0 & \text{in } \mathbb{R} \times (0, \infty) \\
  u = 0 & \text{in } \mathbb{R} \times \{t = 0\}
\end{cases}
\] (2.83)

admits the solution \( u \equiv 0 \). However for any \( a > 0 \), the function \( u_a \) defined as

\[
  u_a(x,t) = \begin{cases}
    0 & \text{if } |x| \geq at \\
    a|x| - a^2 t & \text{if } |x| < at
  \end{cases}
\]

is a Lipschitz function satisfying the equation a.e. and \( u_a(x,0) = 0 \).

Exercise 2.5.1 By using the Hopf-Lax formula solve the following initial value problems

1. \[
\begin{cases}
  u_t + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
  u = |x| & \text{in } \mathbb{R}^n \times \{t = 0\}
\end{cases}
\] (2.84)

Solution. Set \( H(p) = \frac{1}{2}|p|^2 \). One can easily show that \( H^*(p) = \frac{1}{2}|p|^2 \).

The Hopf-Lax formula gives
\[
  u(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ \frac{|x-y|^2}{2t} + |y| \right\}
\] (2.85)

Observe that the function \( f_x(y) := \frac{|x-y|^2}{2t} + |y| \) satisfies \( \lim_{|y| \to +\infty} f_x(y) = +\infty \). Therefore \( f_x \) admits an absolute minimum.

We observe that \( y = 0 \) is the only point where \( f \) is not differentiable and \( f_x(0) = \frac{|x|^2}{2t} \).

If \( y \neq 0 \) then \( Df_x(y) = -2 \frac{(x-y)}{2t} + \frac{y}{|y|} = 0 \) if and only if \( |y| = |x| - t \), with \( |x| - t > 0 \). Moreover \( \frac{x-y}{t} = \frac{y}{|y|} \) and \( f_x(y) = |x| - \frac{t}{2} \).

Therefore if \( |x| - t > 0 \) \( u(x,t) = \inf_{y \in \mathbb{R}^n} \{f_x(y)\} = |x| - \frac{t}{2} \) otherwise \( u(x,t) = \inf_{y \in \mathbb{R}^n} \{f_x(y)\} = f_x(0) = \frac{|x|^2}{2t} \).

2. \[
\begin{cases}
  u_t + \frac{1}{2}|Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
  u = -|x| & \text{in } \mathbb{R}^n \times \{t = 0\}
\end{cases}
\] (2.86)

Sol. \( u(x,t) = -|x| - t/2 \).
Chapter 3

Laplace Equation

The Laplace operator is probably the most important differential operator and has a wide range of applications. We study the mean-value property of harmonic functions. We demonstrate that the mean-value property presents an equivalent description of harmonic functions. Due to this equivalence, the mean-value property provided another tool to study harmonic functions. We discuss the fundamental solution of the Laplace equation. We introduce the important notion of Green’s functions which are designed to solve Dirichlet boundary-value problems. We discuss the regularity of harmonic function using fundamental solution. We finally solve the Dirichlet problem for the Laplace equation in a large bounded domains by Perron’s method.

3.1 Basic Properties

Among the most important of all partial differential equation is Laplace’s equation.

\[ \Delta u = 0 \quad \text{in} \ \Omega, \quad (3.1) \]

where \( \Omega \subseteq \mathbb{R}^n \) is an open set, \( u : \Omega \rightarrow \mathbb{R} \) is the unknown. Recall that \( \Delta u = \sum_{i=1}^{n} u_{x_i x_i} \).
Definition 3.1.1 A function \( u \in C^2(\Omega) \) satisfying (3.1) is called a harmonic function.

It is useful to have a physical model in mind when thinking of \( \Delta \) and perhaps the simplest among several comes from the theory of electrostatics. According to Maxwell’s equations, an electrostatic field \( E \) in space (a vector field representing the electrostatic force on a unit positive charge) is related to the charge density in space, \( f \), by the equation \( \text{div} E = f \) and also satisfies \( \text{curl} E = 0 \) (in dimension \( n \), \( \text{curl} E \) is the matrix \( (\partial_i E^j - \partial_j E^i) \)). The second condition says that at least locally \( E \) is the gradient of a function \(-u\), called the electrostatic potential. We therefore have \( \Delta u = -\text{div} E = f \).

1) A function \( u(x) = x \cdot Ax \), where \( A \) is an \( n \times n \) matrix, is harmonic if and only if \( \text{trace}(A) = \sum_{i=1}^n A_{ii} = 0 \).

2) Harmonic functions and holomorphic functions

In 2-D we identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) by associating to every pair \((x, y) \in \mathbb{R}^2\) the complex number \( z = x + iy \). The following result holds

Proposition 3.1.1 i) If \( f: \Omega \to \mathbb{C} \) is holomorphic in \( \Omega \) then \( u = \Re(f) \) and \( v = \Im(f) \) are harmonic in \( \Omega \).

(ii) If \( \Omega \) is a simply connected open subset of \( \mathbb{R}^2 \) and \( u: \Omega \to \mathbb{R} \) is harmonic, then there exists \( v: \Omega \to \mathbb{R} \) such that \( u + iv \) is holomorphic. \( v \) is called the harmonic conjugate of \( u \).

Proof of Proposition 3.1.1. i) Since \( f \) is holomorphic, it is infinitely differentiable

and hence so are \( u \) and \( v \). In particular, \( u \) and \( v \) have continuous second partial derivatives. Moreover \( u \) and \( v \) satisfy the Cauchy-Riemann equations \( u_x = -v_y \) and \( u_y = v_x \) in \( \Omega \). Hence

\[
    u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0
\]

(1) A function \( f: \Omega \subset \mathbb{C} \to \mathbb{C} \) is holomorphic in \( \Omega \) if \( f \) is complex differentiable at every point \( z_0 \in \Omega \).
Note that in the last step we used the fact that \( v \) has continuous second partial derivatives. The proof that \( v \) satisfies the Laplace equation is completely analogous.

ii) Consider the differential form \( \omega(x, y) = -u_y(x, y)\,dx + u_x(x, y)\,dy \).
We observe that \( \omega \in C^1(\Omega) \) and it is closed. Since \( \Omega \) is simply connected, \( \omega \) admits a primitive, namely there exists a function \( v \) such that \( v_x = -u_y \) and \( v_y = -u_x \). Given \( z_0 = (x_0, y_0) \in \Omega \) the function \( v \) is defined by

\[
v(z) = v(x, y) = \int_\gamma \omega \,d\ell,
\]
where \( \gamma \) is any curve connecting \( z_0 \) and \( z \). Since \( \Omega \) is simply connected \( v \) is well-defined, namely it does not depends on \( \gamma \). By construction the function \( f = u + iv \) is holomorphic. We can conclude the proof. \( \Box \)

Since for all \( k \in \mathbb{N} \) the function \( f(z) = z^k \) is holomorphic in \( \mathbb{C} \), there exist two homogeneous polynomials of degree \( k \) which are harmonic functions in \( \mathbb{C} \).

**Invariance of the Laplace Equation**

We say that the Laplace equation is invariant with respect a group \( G \) of \( C^2 \) diffeomorphisms in \( \mathbb{R}^n \) if for every \( g \in G \) and for every harmonic function \( u \) in a open set \( \Omega \) \( u \circ g \) is harmonic in \( g^{-1}\Omega \).

1. **Invariance by translations.** If \( u \) satisfies \( \Delta u = 0 \) in \( \Omega \), then \( v(x) = u(\tau_a(x)) = u(x + a) \) satisfies \( \Delta v = 0 \) in \( \Omega - a \).

2. **Invariance by dilations.** If \( u \) satisfies \( \Delta u = 0 \) in \( \Omega \), then \( v(x) = u(\delta_\alpha(x)) = u(\alpha x) \) satisfies \( \Delta v = 0 \) in \( \alpha^{-1}\Omega \).

3. **Invariance by orthogonal transformations.** Let \( A \) be an orthogonal matrix \( (AA^T = A^TA = Id, A^T \) is the transpose of \( A ) \). If \( u \) satisfies \( \Delta u = 0 \) in \( \Omega \), then \( v(x) = u(Ax) \) solves \( \Delta v = 0 \) in \( A^T\Omega \). In particular the Laplace equation is invariant by rotations.
4. **Invariance by inversions.** In dimension $n \neq 2$ the Laplace equation is not invariant with respect to the inversions. If $u$ satisfies $\Delta u = 0$ in $\Omega$ (where $\Omega$ does not contain the origin), then $v(x) = u(\frac{x}{|x|^2})$ is harmonic only in dimension 2. In dimension $n > 2$ we have to multiply $v$ by a suitable factor. Precisely if $u$ satisfies $\Delta u = 0$ in $\Omega$ then $v(x) = |x|^{2-n}u(\frac{x}{|x|^2})$, $x \in \Omega^- := \{\frac{x}{|x|^2}, x \in \Omega, x \neq 0\}$ satisfies

$$\Delta v(x) = |x|^{-(n+2)}\Delta u(\frac{x}{|x|^2}).$$

In particular we deduce that since the constant $u(x) = 1$ is harmonic then the function $v(x) = |x|^{2-n}$ is harmonic in $\mathbb{R}^n \setminus \{0\}$.

5. **Conformal invariance in dimension 2.** Two domains $\Omega_1$ and $\Omega_2$ are called conformally equivalent if there exists a diffeomorphism $f: \Omega_1 \to \Omega_2$ such that

$$|f_x| = |f_y| \quad \text{and} \quad f_x \cdot f_y = 0 \quad \text{in } \Omega_1. \quad (3.3)$$

A function $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ of class $C^1$ satisfying the conditions (3.3) is called conformal. A function $f \in C^1(\Omega, \mathbb{R}^2)$ is conformal if and only if either $f$ or its conjugate $\bar{f}$ is holomorphic.

**Proposition 3.1.2** Let $\Omega_1$ and $\Omega_2$ be conformally equivalent and let $f: \Omega_1 \to \Omega_2$ be a conformal diffeomorphism. Let $u: \Omega_1 \to \mathbb{R}$ and $v: \Omega_2 \to \mathbb{R}$ such that $u = v \circ f$. Then $u$ is harmonic in $\Omega_1$ if and only if $v$ is harmonic in $\Omega_2$.

**Proof.** We set $f(x_1, x_2) = (\varphi(x_1, x_2), \psi(x_1, x_2))$, $y_1 = \varphi(x_1, x_2)$, $y_2 = \psi(x_1, x_2)$ and $z = x_1 + ix_2$. We assume that $f$ is holomorphic (the proof in the case $f$ is anti-holomorphic is analogous). The following estimate holds:

$$\Delta u(x_1, x_2) = \partial_{y_1,y_1}v(y_1, y_2)[\varphi_x^2 + \varphi_y^2] + \partial_{y_1,y_2}v(y_1, y_2)[\psi_x^2 + \psi_y^2]$$

$$+ \partial_{y_1,y_1}v(y_1, y_2)[\varphi_x \psi_y + \varphi_y \psi_x]$$

$$+ \partial_{y_1}v(y_1, y_2)(\Delta \varphi) + \partial_{y_2}v(y_1, y_2)(\Delta \psi). \quad (3.4)$$

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From the fact that \( f \) is holomorphic it follows that \( \varphi_{x_1} = \psi_{x_2} \) and \( \varphi_{x_2} = -\psi_{x_1} \). Therefore

\[
\Delta \varphi = 0 \quad \text{and} \quad \Delta \psi = 0
\]

\[
[\varphi_{x_1} \psi_{x_1} + \varphi_{x_2} \psi_{x_2}] = 0
\]

\[
[\varphi_{x_1}^2 + \varphi_{x_2}^2] = [\psi_{x_1}^2 + \psi_{x_2}^2] = |f'(z)|^2 = \frac{1}{2}|Df|^2
\]

where \( f'(z) \) is the complex derivative of \( f \) and \( |Df|^2 = |f_x|^2 + |f_y|^2 = 2|f_x|^2 \).

Thanks to the Riemann Mapping Theorem\(^{(2)}\) and the invariance of the Laplace equation with respect to conformal maps it is equivalent to study the Laplace equation in a general simply connected domain of \( \mathbb{R}^2 \) and in a the unit disc.

### 3.2 Fundamental solution

One good strategy for investigating any partial differential equation is first to identify some explicit solutions. Moreover in looking for explicit solutions it is often wise to restrict attention to classes of functions with certain symmetry properties. Since Laplace’s equation is invariant under rotations, it consequently seems advisable to search for radial solutions, that is functions of \( r = |x| \). We attempt to find a solution \( u \) of the Laplace equation having the form \( u(x) = f(r) \).

\[
f''(r) + \frac{n-1}{r} f'(r) = 0 \tag{3.5}
\]

Multiplying this equation (3.5) by \( r^{n-1} \) we get

\[
(r^{n-1} f'(r))' = 0.
\]

Consequently for \( r > 0 \) we have

\(^{(2)}\)In complex analysis, the Riemann mapping theorem states that if \( \Omega \) is a non-empty simply connected open subset of the complex number plane \( \mathbb{C} \) which is not all of \( \mathbb{C} \), then there exists a biholomorphic mapping \( f \) (i.e. a bijective holomorphic mapping whose inverse is also holomorphic) from \( \Omega \) onto the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \),
\[
\begin{align*}
f(r) &= a \log r + b \quad n=2 \\
f(r) &= \frac{a}{r^{n-2}} + b \quad n > 2.
\end{align*}
\]

We note that \( f(r) \) has a singularity at \( r = 0 \) as long as it is not a constant.

### Definition 3.2.1

The function

\[
\Phi(x) = \begin{cases} 
- \frac{1}{2\pi} \log(|x|) & \text{if } n = 2 \\
\frac{1}{(n-2)\omega_n} |x|^{2-n} & \text{if } n > 2
\end{cases}
\]

defined for \( 0 \neq x \in \mathbb{R}^n \), is the fundamental solution of Laplace’s equation.

We recall that \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

The reason for the particular choice of the constants above will be clear later. We will sometimes write \( \Phi(x) = \Phi(|x|) \) to emphasize that the fundamental solution is radial.

We first observe that

\[
|\partial_{x_i} \Phi(x)| \leq \frac{C}{|x|^{n-1}} \quad \text{and} \quad |\partial_{x_i x_j}^2 \Phi(x)| \leq \frac{C}{|x|^n},
\]

for some \( C > 0 \).

#### 3.2.1 The Poisson Equation in \( \mathbb{R}^n \)

By construction for any \( y \in \mathbb{R}^n \) \( x \mapsto \Phi(x - y) \) is harmonic as a function of \( x, x \neq y \). Let us take now \( f: \mathbb{R}^n \to \mathbb{R} \) and note that \( x \mapsto \Phi(x - y)f(y), \ x \neq y \) is harmonic for each \( y \in \mathbb{R}^n \), and thus the sum of finitely many such expressions built for different point \( y \).

This reasoning might suggest that the convolution \( u(x) = (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y)f(y)dy \) will solve the Laplace equation.
Nevertheless we cannot just compute
\[
\Delta u = \int_{\mathbb{R}^n} \Delta_x \Phi(x - y) f(y) \, dy
\]  
(3.6)
since \( D^2 \Phi(x - y) \) is not summable near the singularity \( x = y \). We must proceed more carefully in calculating \( \Delta u \).

**Theorem 3.2.1** (Solution of the Poisson’s equation) Suppose \( f \in C^2_c(\mathbb{R}^n) \) (namely \( C^2 \) with compact support) and let \( u = \phi * f \). Then

(i) \( u \in C^2(\mathbb{R}^n) \)

(ii) \( -\Delta u = f \) in \( \mathbb{R}^n \).

**Proof.**

1. We have
\[
u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy = \int_{\mathbb{R}^n} \Phi(y) f(x - y) \, dy,
\]

We observe that \( \Phi \in L^1_{loc}(\mathbb{R}^n) \) and by applying the properties of convolution (see Theorem A.1.1) we get that \( u \in C^2(\mathbb{R}^n) \) and
\[
\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial f}{\partial x_i}(x - y) \, dy,
\]  
(3.7)
\[
\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} \Phi(y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x - y) \, dy.
\]  
(3.8)

2. Since \( \Phi \) blows up at 0 we need to isolate the singularity inside a small ball. So fix \( \varepsilon > 0 \). Then
\[
\Delta u(x) = \int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x - y) \, dy + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_x f(x - y) \, dy.
\]  
(3.9)

Now
\[
|I_\varepsilon| \leq C \|D^2 f\|_{L^\infty} \int_{B(0,\varepsilon)} |\Phi(y)| \, dy \leq \begin{cases} C \varepsilon^2 |\log(\varepsilon)| & (n = 2) \\ C \varepsilon^2 & (n \geq 3) \end{cases}
\]  
(3.10)
An integration by parts\(^{(3)}\) yields
\[
J_{\varepsilon} = \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \Phi(y) \Delta_y f(x - y) dy
\]
\[
= - \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} D\Phi(y) D_y f(x - y) dy
+ \int_{\partial B(0, \varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x - y) d\sigma(y)
=: K_{\varepsilon} + L_{\varepsilon},
\] (3.11)
where \(\nu\) denote the inward unit normal along \(\partial B(0, \varepsilon)\). We can check that
\[
|L_{\varepsilon}| \leq C\|Df\|_{L^\infty} \int_{\partial B(0, \varepsilon)} |\Phi(y)| d\sigma(y) \leq \begin{cases} C\varepsilon|\log(\varepsilon)| & (n = 2) \\ C\varepsilon & (n \geq 3) \end{cases}
\] (3.12)
By integrating once again by part in \(K_{\varepsilon}\) we get
\[
K_{\varepsilon} = \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \Delta \Phi(y) f(x - y) dy - \int_{\partial B(0, \varepsilon)} \frac{\partial \Phi}{\partial \nu} f(x - y) d\sigma(y)
\]
\[
= - \int_{\partial B(0, \varepsilon)} \frac{\partial \Phi}{\partial \nu} f(x - y) d\sigma(y),
\]
\[
= - \frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B(x, \varepsilon)} f(y) d\sigma(y)
\] (3.13)
\[
= \int_{\partial B(x, \varepsilon)} f(y) d\sigma(y) \to - f(x), \quad \text{as } \varepsilon \to 0.
\]
By combining (3.9)-(3.13) and letting \(\varepsilon \to 0\), we find \(-\Delta u(x) = f(x)\) and we can conclude. \(\square\)
Remark 3.2.1 i) We sometimes write

$$-\Delta \Phi = \delta_0, \text{ in } \mathbb{R}^n$$

$\delta_0$ denoting the Dirac measure on $\mathbb{R}^n$ giving the unit mass to the point 0. We recall that a function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies the equation $-\Delta u = \delta_0$ in a distributional sense if for every test function $\phi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} u(y) \Delta \phi(y) dx = -\phi(0).$$

ii) Theorem 3.2.1 is in fact valid under some less smoothness assumptions on $f$, (see e.g. [3]).

3.3 Sub- and super-harmonic functions

Definition 3.3.1 $u \in C^2(\Omega)$ is said sub- (resp. super-) harmonic if $\Delta u(x) \geq 0$ (resp. $\leq 0$) for every $x \in \Omega$.

Example 3.3.1 a) $u(x) = |x|^2$ is sub-harmonic and $\Delta u(x) = 2n$

b) If $u \in C^2(\Omega)$ is sub-harmonic and $f \in C^2(\mathbb{R}, \mathbb{R})$ is such that $f' \geq 0$ and $f'' \geq 0$, then $f \circ u$ is sub-harmonic in $\Omega$.

c) Let $u \in C^2(\Omega)$ be harmonic. Then the function $v(x) = u^2(x)$ is sub-harmonic.

Next we show some properties of sub/super harmonic functions.
Theorem 3.3.1 *(Weak Maximum Principle)* Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set and \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be a sub-harmonic function. Then

\[
\max_{\Omega} u = \max_{\partial \Omega} u.
\]

Proof. Step 1. We first suppose that \( \Delta u(x) > 0 \) in \( \Omega \), and let \( x_0 \in \overline{\Omega} \) be such that \( u(x_0) = \max_{\Omega} u(x) \). We clearly have \( x_0 \in \partial \Omega \), otherwise \( \Delta u(x_0) \leq 0 \) if \( x_0 \in \Omega \). In this case we have \( \max_{\Omega} u = \max_{\partial \Omega} u \).

Step 2. Suppose now that \( \Delta u(x) \geq 0 \) in \( \Omega \). We set \( v(x) = u(x) + \varepsilon |x|^2 \). We have \( \Delta v = \Delta u + 2\varepsilon n > 0 \) in \( \Omega \). From Step 1 it follows that \( \max_{\Omega} v = \max_{\partial \Omega} v \). Thus

\[
\max_{\Omega} u \leq \max_{\partial \Omega} u + \varepsilon \delta^2,
\]

where \( \delta = \max\{|x|, x \in \Omega\} \). By letting \( \varepsilon \to 0 \) we get \( \max_{\Omega} u \leq \max_{\partial \Omega} u \), and we can conclude.

Corollary 3.3.1 *(Weak Minimum Principle)* Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set and \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be a super-harmonic function. Then

\[
\min_{\Omega} u = \min_{\partial \Omega} u.
\]

Proof. It is enough to observe that \( -u \) is sub-harmonic and \( \max_{\Omega}(-u) = -\min_{\Omega} u \).

Corollary 3.3.2 Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set and \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be a harmonic function. Then

\[
\min_{\Omega} u = \min_{\partial \Omega} u \quad \text{and} \quad \max_{\Omega} u = \max_{\partial \Omega} u.
\]
**Corollary 3.3.3** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set and \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be a harmonic function. Then

\[
\max_{\Omega} |u| = \max_{\partial \Omega} |u|.
\]

**Corollary 3.3.4** *(Uniqueness of the Dirichlet Problem)* Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set and let \( f \in C^2(\overline{\Omega}), \varphi \in C(\overline{\Omega}) \). Then the following problem

\[
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \quad \text{(Poisson Equation)} \\
u &= \varphi \quad \text{in } \partial \Omega
\end{aligned}
\] (3.14)

admits at most one solution.

**Proof.** Let \( u_1, u_2 \) be two solutions of (3.14). We have \( \Delta (u_1 - u_2) = 0 \) and thus \( \max_{\Omega} |u_1 - u_2| = \max_{\partial \Omega} |u_1 - u_2| = 0 \). \( \Box \).

**Exercise 3.3.1** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set, \( v : \Omega \to \mathbb{R} \) harmonic. If \( u : \Omega \to \mathbb{R} \) is sub-harmonic and \( u = v \) on \( \partial \Omega \) then \( u \leq v \) on \( \Omega \) (that is why the name sub-harmonic).

**Remark 3.3.1** The maximum principle is in general false if \( \Omega \) is unbounded. The function

\[
 u(x) = \begin{cases} 
 \log(|x|) & \text{if } n = 2 \\
 1 - |x|^{2-n} & \text{if } n > 2
\end{cases}
\]

is harmonic in \( \Omega = \{ x \in \mathbb{R}^n : |x| > 1 \} \), it is null on \( \partial \Omega \), but \( u(x) > 0 \) inside \( \Omega \).

The maximum principle can be used to get estimates of the solution (which not a priori known) of a Dirichlet Problem for the Poisson equation.
on a certain domain, by comparing it with analogous problems where we explicitely know the solution.

**Example 3.3.2** Let $Q_R = (-R, R)^n, R > 0$ and suppose there is a solution of $u \in C^2(Q_R) \cap C(\overline{Q_R})$ of

$$
\begin{cases}
-\Delta u = 1 & \text{in } Q_R \\
 u = 0 & \text{in } \partial Q_R
\end{cases}
$$

Then $\frac{R^2}{2n} \leq u(0) \leq \frac{R^2}{2}$.

**Solution.** We have $B(0, R) \subseteq Q_R \subseteq B(0, R\sqrt{n})$. Let us consider the problems

$$
\begin{cases}
-\Delta u_1 = 1 & \text{in } B(0, R) \\
u_1 = 0 & \text{in } \partial B(0, R)
\end{cases}
$$

$$
\begin{cases}
-\Delta u_2 = 1 & \text{in } B(0, R\sqrt{n}) \\
u_2 = 0 & \text{in } \partial B(0, R\sqrt{n})
\end{cases}
$$

We have $u_1(x) = \frac{R^2-|x|^2}{2n}$, $u_2(x) = \frac{nR^2-|x|^2}{2n}$, (see Exercise 3.3.2, n.5 ). On $B(0, R)$ we consider the function $v = u - u_1$. We have $\Delta v = 0$ and $v = u$ on $\partial B(0, R)$. Since $u$ is super-harmonic on $Q_R$ and $u = 0$ on $\partial Q_R$ we have $u \geq 0$ on $Q_R$. In particular $v \geq 0$ on $\partial B(0, R)$. Thus $u \geq u_1$ in $B(0, R)$, being $v$ harmonic. In particular $u(0) \geq u_1(0) = \frac{R^2}{2n}$. In the same way one can show that $u(x) \leq u_2(x)$ on $Q_R$ and thus $u(0) \leq u_2(0) = \frac{nR^2}{2n}$. $\square$

**Exercise 3.3.2** 1. Let $\varepsilon > 0$. Show that on $\partial B(0, \varepsilon)$ we have

$$
\frac{\partial \Phi}{\partial \nu} = -\frac{1}{|\partial B(0, \varepsilon)|}.
$$

Thus $\int_{\partial B(0, \varepsilon)} \frac{\partial \Phi}{\partial \nu} dS = -1$.

2. If $f: \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic, then $\Re(f)$ and $\Im(f)$ are harmonic.

3. Suppose that $u \in C^2(\Omega)$ and $x \in \Omega$. Show that

$$
\Delta u(x) = \lim_{r \to 0^+} \frac{2n}{r^2} \left[ \frac{1}{\omega_n} \int_{\partial B(0, 1)} u(x + ry)d\sigma(y) - u(x) \right].
$$
4. Determine a solution of the problem
\[
\begin{cases}
-\Delta u(x) = |x| & |x| < 1 \\
u(x) = 0 & |x| = 1
\end{cases}
\]

**Sol.** \( u(x) = \frac{1-|x|^3}{3(n+1)} \)

5. Let \( R > 0 \). Determine the radial solution of the problem
\[
\begin{cases}
-\Delta u(x) = 1 & |x| < R \\
u(x) = 0 & |x| = R
\end{cases}
\]

**Sol.** \( u(x) = \frac{R^2-|x|^2}{2n} \)

### 3.4 Mean-value formulas

Consider an open set \( \Omega \subseteq \mathbb{R}^n \) and suppose that \( u \) is a harmonic function in \( \Omega \). We next derive the important mean value formulas, which declare that \( u(x) \) equals both the average of \( u \) over the sphere \( \partial B(x, r) \) and the average of \( u \) over the entire ball \( B(x, r) \), provided \( B(x, r) \subset \Omega \).

(We can in general take \( r < \frac{\text{dist}(x, \partial \Omega)}{4} \) in order to be sure that even \( \bar{B}(x, r) \subset \Omega \).)
Definition 3.4.1 Let $\Omega \subseteq \mathbb{R}^n$ and $u$ is continuous function in $\Omega$. Then

(i) $u$ satisfies the mean-value property over spheres if for any $B(x, r) \subset \Omega$

$$u(x) = \int_{\partial B(x, r)} u(y) d\sigma = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x, r)} u(y) d\sigma;$$

(ii) $u$ satisfies the mean-value property over balls if for any $B(x, r) \subset \Omega$

$$u(x) = \int_{B(x, r)} u(y) dy = \frac{n}{\omega_n r^n} \int_{B(x, r)} u(y) dy,$$

where $\omega_n r^{n-1}$ is the surface area of the sphere $\partial B(x, r)$ in $\mathbb{R}^n$ and $\alpha(n) r^n := \frac{\omega_n}{n}$ is the volume of the ball $B(x, r)$ in $\mathbb{R}^n$.

These two versions of the mean-value property are equivalent. In fact we can write (i) as

$$u(x) r^{n-1} = \frac{1}{\omega_n} \int_{\partial B(x, r)} u(y) d\sigma;$$

we can integrate with respect to $r$ and get (ii). If we write (ii) as

$$u(x) r^n = \frac{n}{\omega_n} \int_{B(x, r)} u(y) d\sigma;$$

we can differentiate with respect to $r$ and get (i).
Theorem 3.4.1 (Mean-value formulas for the Laplace’s equation) If \( u \in C^2(\Omega) \) is harmonic, then
\[
u(x) = \int_{\partial B(x,r)} u(y) \, d\sigma(y) = \int_{B(x,r)} u(y) \, dy  \quad (3.18)
\]
for each ball \( B(x, r) \subset \Omega \).

Proof.

1. Set
\[
\phi(r) := \int_{\partial B(x,r)} u(y) \, d\sigma(y) = \int_{\partial B(0,1)} u(x + rz) \, d\sigma(z).
\]
Then
\[
\phi'(r) = \int_{\partial B(0,1)} Du(x + rz) \cdot zd\sigma(z)
\]
\[
= \int_{\partial B(x,r)} Du(y) \cdot \frac{y - x}{r} \, d\sigma(y)
\]
\[
= \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y) \, d\sigma(y)
\]
by Green’s formulas
\[
= \frac{r}{n} \int_{B(x,r)} \Delta u(y) \, dy = 0.
\]

Hence \( \phi \) is constant, and so
\[
\phi(r) = \lim_{t \to 0} \phi(t) = \lim_{t \to 0} \int_{\partial B(x,t)} u(y) \, d\sigma(y) = u(x).
\]

2. Observe that by employing polar coordinates we get
\[
\int_{B(x,r)} u(y) \, dy = \int_0^r \left( \int_{\partial B(x,s)} u(y) \, d\sigma(y) \right) \, ds
\]
\[
= u(x) \int_0^r \omega_n s^{n-1} \, ds = \frac{\omega_n}{n} r^n u(x).
\]

where \( \omega_n \) is the Lebesgue measure of \( \partial B(0,1) \).
Theorem 3.4.2 (Converse to mean-value property) If \( u \in C^2(\Omega) \) satisfies
\[
u(x) = \int_{\partial B(x,r)} udS\]
for each ball \( B(x,r) \subset \Omega \), then it is harmonic.

**Proof.** If \( \Delta u \neq 0 \), there exists some ball \( B(x,r) \subset \Omega \) such that \( \Delta u > 0 \) within \( B(x,r) \). But then for \( \phi \) as above we have
\[
0 = \phi'(r) = \frac{r}{n} \int_{B(x,r)} \Delta u(y) dy > 0,
\]
a contradiction. \( \square \)

As a consequence of the mean value property of the harmonic functions we get the closure of harmonic functions with respect to uniform convergence.

Theorem 3.4.3 Let \( \Omega \subseteq \mathbb{R}^n \), \( \{u_k\}_k \) be a sequence of harmonic functions on \( \Omega \) which uniformly converges to a function \( u \). Then \( u \) is harmonic on \( \Omega \).

**Proof.** Let \( x_0 \in \Omega \) and \( r > 0 \) such that \( B(x_0,r) \subset \Omega \). It holds
\[
u_k(x_0) = \int_{B(x_0,r)} u_k(y) dy, \text{ for every } k \in \mathbb{N}. \tag{3.19}\]
We take the limit for \( k \to +\infty \) in both sides of (3.19) and since \( u_k \) converges uniformly in \( B(x_0,r) \) we can pass to the limit under integral sign. We find
\[
u(x_0) = \int_{B(x_0,r)} u(y) dy.
\]
By arbitrariness of \( x_0 \) we deduce that \( u \) is harmonic. \( \square \)
3.4.1 Properties of harmonic functions

We now present a sequence of interesting deductions about harmonic functions, all based on the mean-value formulas.

a. Strong Maximum Principle

**Theorem 3.4.4 (Strong Maximum Principle)** Let \( \Omega \subseteq \mathbb{R}^n \) be a connected bounded open set and \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be a sub-harmonic (resp. super-harmonic) function. If there exists a point \( x_0 \in \Omega \) such that
\[
u(x_0) = \max_{\Omega} u \quad (\text{resp. } u(x_0) = \min_{\Omega} u)\]
then \( u \) is constant within \( \Omega \).

**Proof.** Let \( u \) be a sub-harmonic function and suppose there is a point \( x_0 \in \Omega \) with \( u(x_0) = M := \max_{\overline{\Omega}} u \). Then for \( 0 < r < \text{dist}(x_0, \partial \Omega) \), the mean value property asserts
\[
M = u(x_0) \leq \frac{1}{\omega_n r^n} \int_{B(x_0, r)} \nu dy \leq M,
\]
(3.20)

This implies that \( u \equiv M \) within \( B(x_0, r) \). Indeed if there is \( y \in B(x_0, r) \) such that \( u(y) < M \) then \( u(z) < M \) for all \( z \in B(y, s) \subseteq B(x_0, r), (s < r) \). Then
\[
M = u(x_0) \leq \frac{1}{\omega_n r^n} \left[ \int_{B(x_0, r) \setminus B(y, s)} \nu dz + \int_{B(y, s)} \nu dz \right] < M.
\]
and we get a contradiction.

Hence the set \( D = \{ x \in \Omega : u(x) = M \} \) is open and relatively closed in \( \Omega \) and thus equals \( \Omega \), being \( \Omega \) connected. \( \square \).
Remark 3.4.1  The strong maximum principle asserts in particular that if $\Omega$ is connected and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
u = g & \text{in } \partial \Omega
\end{cases}
\]

where $g \geq 0$, then is positive everywhere in $\Omega$ if $g$ is positive somewhere on $\partial \Omega$.

b. Regularity  Now we prove that if $u \in C^2$ is harmonic, then necessarily $u \in C^\infty$. Thus harmonic functions are automatically infinitely differentiable. This sort of assertion is called a regularity theorem.

Theorem 3.4.5  (Koebe) If $u \in C(\Omega)$ satisfies the mean-value property (3.4.1) for each $B(x, r) \subset \Omega$, then

\[
u \in C^\infty(\Omega).
\]

Proof. Let $\eta$ be a standard mollifier:

\[
\begin{cases}
C \exp \left( \frac{1}{|x|^2 - 1} \right) & \text{if } |x| < 1 \\
0 & \text{if } |x| \geq 1
\end{cases}
\]

the constant $C > 0$ is selected so that $\int_{\mathbb{R}^n} \eta dx = 1$.

The function $\eta$ is radial. Set $\eta_\varepsilon = \frac{1}{\varepsilon^n} \eta(\frac{x}{\varepsilon})$ and $u_\varepsilon = \eta_\varepsilon * u$ in $\Omega_\varepsilon = \{ x \in \Omega : dist(x, \partial \Omega) > \varepsilon \}$. It is know that $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$.
We show that \( u = u^\varepsilon \) on \( \Omega^\varepsilon \). Indeed if \( x \in \Omega^\varepsilon \) then

\[
 u^\varepsilon(x) = \int_\Omega \eta(x - y)u(y)dy \\
= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta \left( \frac{|x - y|}{\varepsilon} \right) u(y)dy \\
= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta \left( \frac{r}{\varepsilon} \right) \left( \int_{\partial B(x,r)} udS \right) dr \\
= \frac{1}{\varepsilon^n} u(x) \int_0^\varepsilon \eta \left( \frac{r}{\varepsilon} \right) n\alpha(n)r^{n-1}dr \\
= \ u(x) \int_{B(0,\varepsilon)} \eta dy = u(x).
\]

Thus \( u \equiv u^\varepsilon \) on \( \Omega^\varepsilon \) and so \( u \in C^\infty(\Omega^\varepsilon) \) for each \( \varepsilon > 0 \). 

\[ \square \]

**Theorem 3.4.6 (Liouville)**

Let \( u \in C^2(\mathbb{R}^n) \) be a bounded harmonic function in \( \mathbb{R}^n \). Then \( u \) is constant.

**Proof.** In Theorem 3.4.5 we show that \( u \in C^\infty(\mathbb{R}^n) \). We fix \( i \in \{1, \ldots, n\} \), \( u_{x_i} \) is harmonic and thus for every \( x_0 \) and \( r > 0 \) we have

\[
 u_{x_i}(x_0) = \int_{B(x_0,r)} u_{x_i}(\xi) d\xi.
\]

By the Divergence Theorem we have \( \int_{B(x_0,r)} u_{x_i}(\xi) d\xi = \int_{\partial B(x_0,r)} u\nu_i dS \) and thus

\[
 |u_{x_i}(x_0)| \leq \frac{1}{r^n\alpha(n)} \left| \int_{\partial B(x_0,r)} u\nu_i dS \right| \\
\leq \frac{1}{r^n\alpha(n)} \|u\|_{L^\infty} |\partial B(x_0,r)| \\
= \frac{1}{r^n\alpha(n)} \|u\|_{L^\infty} \omega_n r^{n-1} = \frac{n\|u\|_{L^\infty}}{r},
\]

where \( \alpha(n) = \frac{\omega_n}{n} = |B(0,1)| \).
Since \( \lim_{r \to +\infty} \frac{n\|u\|_{L^\infty}}{r} = 0 \), we get that for all \( x_0 \) and for all \( i \in \{1, \ldots, n\} \) we have \( u_{x_i}(x_0) = 0 \). Hence \( u \) is constant in \( \mathbb{R}^n \).

**Theorem 3.4.7** *(Representation formula)* Let \( f \in C_c^2(\mathbb{R}^n) \) and \( n \geq 3 \). Then any bounded solution of

\[-\Delta u = f, \quad \text{in } \mathbb{R}^n\]

has the form

\[ u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy + C, \]

for some constant \( C \in \mathbb{R} \).

**Proof.** We observe that \( \tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy \) is bounded, since \( \Phi(x) \to 0 \) as \( |x| \to +\infty \). Let \( u \in C^2(\mathbb{R}^n) \) be another bounded solution, then Liouville Theorem implies that \( v = u - \tilde{u} \) is constant and we can conclude. \( \square \)

**Theorem 3.4.8** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set and \( u \) harmonic in \( \Omega \). Then for every \( i \in \{1, \ldots, n\} \) we have

\[ |u_{x_i}(x_0)| \leq \frac{2^{n+1}n}{\alpha(n)r^{n+1}} \|u\|_{L^1(B(x_0,r))}, \quad (3.22) \]

for each ball \( B(x_0,r) \subset \Omega \).

**Proof.** Fix \( i \in \{1, \ldots, n\} \). The function \( u_{x_i} \) is harmonic on \( \Omega \), thus

\[ u_{x_i}(x_0) = \frac{2^n}{\alpha(n)r^n} \int_{B(x_0,r/2)} u_{x_i}(y)dy. \]

Thus by the same computations in the proof of Theorem 3.4.6 we get

\[ |u_{x_i}(x_0)| \leq \frac{2^n}{r} \|u\|_{L^\infty(\partial B(x_0,r/2))}. \quad (3.23) \]
If \( x \in \partial B(x_0, r/2) \), then \( B(x, r/2) \subset B(x_0, r) \subset \Omega \) and so

\[
|u(x)| \leq \frac{1}{\alpha(n)} \left( \frac{2}{r} \right)^n \|u\|_{L^1(B(x_0, r))} \tag{3.24}
\]

by the mean value formulas for harmonic functions. By combining (3.23) and (3.24) we get (3.22) and we can conclude. \( \square \)

**Theorem 3.4.9 (Estimates on derivatives of order \( k \geq 0 \))**

If \( u \) is harmonic in \( \Omega \) then one can show by induction the following estimate

\[
|D^\alpha u(x_0)| \leq C_k \frac{\|u\|_{L^1(B(x_0, r))}}{r^{n+k}}, \tag{3.25}
\]

for each ball \( B(x_0, r) \subset \Omega \) and each multiindex \( \alpha \) of order \( |\alpha| = k \).

Here

\[
C_0 = \frac{1}{\alpha(n)}, \quad C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)} \quad (k \geq 1). \tag{3.26}
\]

**Proof.** We establish (3.25) by induction on \( k \). The case \( k = 0 \) is immediate being an immediate consequence of the mean value formula. The case \( k = 1 \) has been proved in Theorem 3.4.8.

Assume \( k \geq 2 \) and that (3.25) and (3.26) are valid for all balls in \( \Omega \) and each multiindex of order less or equal to \( k - 1 \). Fix \( B(x_0, r) \subset \Omega \) and let \( |\alpha| = k \). Then \( D^\alpha u = (D^\beta u)_x \) for some \( i \in \{1, \ldots, n\} \), \( |\beta| = k - 1 \). By calculations similar to those in Theorem 3.4.8, we establish that

\[
|D^\alpha u(x_0)| \leq \frac{nk}{r} \|D^\beta u\|_{L^\infty(\partial B(x_0, r/k))}.
\]

If \( x \in \partial B(x_0, r/k) \) then \( B(x, \frac{k-1}{k}r) \subset B(x_0, r) \subset \Omega \), thus (3.25) and (3.26) for \( k - 1 \) imply

\[
|D^\beta u(x)| \leq \frac{(2^{n+1}n(k - 1))^{k-1}}{\alpha(n)(\frac{k-1}{k}r)^{n+k-1}} \|u\|_{L^1(B(x, \frac{k-1}{k}r))} \tag{3.27}
\]

\[
\leq \frac{(2^{n+1}n(k - 1))^{k-1}}{\alpha(n)(\frac{k-1}{k}r)^{n+k-1}} \|u\|_{L^1(B(x_0, r))} \tag{3.28}
\]

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Combining the previous estimates yields the bound

\[ |D^\alpha u(x_0)| \leq \frac{(2^{n+1}nk)^k}{\alpha(n)(r)^{n+k}}\|u\|_{L^1(B(x_0, r))}. \]

We can conclude. \qed

c) Analyticity.

**Theorem 3.4.10** Assume that \( u \) is harmonic in \( \Omega \). Then \( u \) is analytic.

**Proof.** Let \( x_0 \in \Omega \) and \( r := \frac{1}{4} \text{dist}(x_0, \partial \Omega) \) and \( M = \frac{1}{\alpha(n)r^\alpha}\|u\|_{L^1(B(x_0, 2r))} < +\infty. \)

Since \( B(x, r) \subset B(x_0, 2r) \subset \Omega \) for each \( x \in B(x_0, r) \) we have

\[ \|D^\alpha u\|_{L^\infty(B(x_0, r))} \leq M \left( \frac{2^{n+1}n}{r} \right)^{\alpha} |\alpha|^{\alpha}. \]

From Stirling formula it follows that

\[ |\alpha|^{\alpha} \leq e^{\alpha^2 |\alpha|}. \]

Moreover the Multinomial Theorem implies

\[ n^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!}, \]

whence \( |\alpha|! \leq n^{|\alpha|} |\alpha|! \). It follows that

\[ \|D^\alpha u\|_{L^\infty(B(x_0, r))} \leq M \left( \frac{2^{n+1}n^2e}{r} \right)^{\alpha} |\alpha|! \] \hspace{1cm} (3.29)

The Taylor series for \( u \) at \( x_0 \) is

\[ \sum_{\alpha} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha. \]
Claim. This power series converges provided

$$|x - x_0| < \frac{r}{2^{n+2}n^3e}.$$ 

Proof of the Claim.

The rest of order $N$ of the series is given by

$$R_N(x) = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x - x_0))}{\alpha!} (x - x_0)^\alpha$$

for some $t \in [0, 1]$ depending on $x$. We have

$$|R_N(x)| \leq CM \sum_{|\alpha|=N} \left( \frac{2^{n+1}n^2e}{r} \right)^N \left( \frac{r}{2^{n+2}n^3e} \right)^N$$

$$\leq CMn^N \frac{1}{(2n)^N} \to 0 \text{ as } N \to +\infty. \quad \Box$$

**Theorem 3.4.11 (Harnack Inequality)** Let $\Omega \subseteq \mathbb{R}^n$ be open and $V \subset \Omega$ be a connected open set such that $\overline{V} \subset \Omega$ is compact. Then there is $C > 0$ (depending on $V$) such that

$$\sup_V u \leq C \inf_V u,$$

for every nonnegative harmonic function in $\Omega$.

Theorem 3.4.11 says that the values of a nonnegative harmonic function within $V$ are all comparable. If $u$ is small (large) in some point in $V$ then it is small (large) everywhere.

**Proof of Theorem 3.4.11.** Let $r = \frac{1}{4} \text{dist}(V, \partial \Omega)$. Choose $x, y \in V$, $|x - y| \leq r$. Then

$$u(x) = \int_{B(x,2r)} u(z)dz = \frac{n}{\omega_n 2^{n}r^n} \int_{B(x,2r)} u(z)dz \quad (3.30)$$

$$\geq \frac{n}{\omega_n 2^{n}r^n} \int_{B(y,r)} u(z)dz = \frac{1}{2^n}u(y).$$
On the other hand $B(x,r) \subseteq B(y,2r)$ and therefore

$$u(x) \leq 2^n u(y). \quad (3.31)$$

By combining (3.30) and (3.31) we get $2^{-n} u(y) \leq u(x) \leq 2^n u(y)$ for every $x, y \in V$, $|x - y| \leq r$.

Since $V$ is connected and $\bar{V}$ is compact we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^n$ each of which has radius $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$, $(i = 2, \ldots, N)$. Then $u(x) \geq 2^{-n(N+1)} u(y)$ and we conclude the proof. □

**Exercise 3.4.1** Let $V \subseteq \mathbb{R}^n$ be a connected set and let $U_1, \ldots, U_m$ be nonempty open sets such that $V = \bigcup_{i=1}^m U_i$. Then there exist $i_1 = 1, i_2, \ldots, i_m = m$ in $\{1, \ldots, m\}$ such that $U_{i_{k-1}} \cap U_{i_k} \neq \emptyset$ for $k \in \{2, \ldots, m\}$.

**Solution of Exercise 3.4.1.** We write $i \sim j$ if there exists $i_1 = i, \ldots, i_k = j$ such that $U_i \cap U_{i_{k-1}} \neq \emptyset, \ldots, U_{i_{k-1}} \cap U_{i_k} \neq \emptyset$ and set $I = \{j : j \sim 1\}$. If $I \neq \{1, \ldots, m\}$ the sets $A = \bigcup_{i \in I} U_i$ and $B = \bigcup_{i \notin I} U_i$ are nonempty disjoint open sets of $V$ (if $j \notin I$ and $i \in I$ then $U_j \cap U_i = \emptyset$, otherwise $j \sim i$, $i \sim 1$ implies $j \sim 1$). This contradicts the fact that $V$ is connected. □

**Exercise 3.4.2** Let $U \subseteq \mathbb{R}^n$ be a connected open set, $K \subset U$ be a compact set. Then there exists a connected open set $V$ containing $K$ and such that $\overline{V} \subset U$ is compact.

**Solution of Exercise 3.4.2.** Let us cover $K$ with a finite number of open balls $B_1, \ldots, B_m$ whose closure is contained in $U$. Since $U$ is an open connected set we can connect the balls through continuous curves. Let $\gamma : [0, 1] \to U$ be one of these curves which connect $B_1$ to $B_2$ and let $B(x_1, r), \ldots, B(x_\ell, r)$ be such that $\gamma([0, 1]) \subseteq \bigcup_{i=1}^\ell B(x_i, r) \subseteq \overline{U}$ and $B_1 \cap B(x_1, r) \neq \emptyset$, $B(x_1, r) \cap B(x_2, r) \neq \emptyset$, $\ldots$, $B_2 \cap B(x_\ell, r) \neq \emptyset$. This is possible by Exercise 3.4.1.
It follows that $B_1 \cup B(x_1, r) \cup \ldots B(x_\ell, r) \cup B_2$ is a connected open set. We connect in the same way $B_2$ and $B_3$. We finally take $V$ the union of all the elements of these coverings. □

**Exercise 3.4.3** 1. Show that if $\Omega \subset \mathbb{R}^n$ is a bounded open set then the Strong Maximum Principle implies the Weak Maximum Principle.

2. Let $u \in C^2(\Omega)$ be harmonic in $\Omega$ (connected) and let $x_0 \in \Omega$ such that $|u(x_0)| = \max_{\Omega} |u|$. Show that $u$ is constant.

3. **Reflection Principle.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset which is symmetric with respect to $\mathbb{R}^{n-1} \times \{0\}$, ($(x, -x_n) \in \Omega$ if $(x, x_n) \in \Omega$). Let $\Omega^+$ denote the set $\{x \in \Omega, x_n > 0\}$. Assume that $u \in C^2(\Omega^+) \cap C(\overline{\Omega^+})$ is harmonic in $\Omega^+$, with $u = 0$ on $\Omega^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} 
 u(x_1, \ldots, x_{n-1}, x_n) & \text{if } x_n > 0 \\
 -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0
\end{cases}$$

for $x \in \Omega$.

i) Let $x_0 \in \Omega \cap (\mathbb{R}^{n-1} \times \{0\}), r > 0$ such that $B(x_0, r) \subset \Omega$ and let $w$ be the solution of

$$\begin{cases} 
 \Delta w = 0 & \text{in } B(x_0, r) \\
 w = v(x) & \text{in } \partial B(x_0, r)
\end{cases} \quad (3.32)$$

Show that $w(x, -x_n) = -w(x, x_n)$ in $B(x_0, r)$.

ii) Deduce that $v$ is harmonic.

**Solution of Exercise 3.**

i) We set $\tilde{w}(x, x_n) = -w(x, -x_n)$. We observe that $\tilde{w}_{x_i} x_i (x, x_n) = -w_{x_i} x_i (x, x_n)$, for $i = 1, \ldots, n-1$ and $\tilde{w}_{x_n} x_n (x, x_n) = -w_{x_n} x_n (x, x_n)$, therefore $\Delta \tilde{w} = 0$ in $B(x_0, r)$. It is clear that $\tilde{w} = v$ on $\partial B(x_0, r)$ since $v$ is odd. It follows that $\tilde{w} = w$ by uniqueness of the solution of the Dirichlet problem (3.32), (the existence is given by the Poisson formula).

ii) $v$ is continuous in $\Omega$ and it is harmonic in $\Omega^+$ and $\Omega^-$. It remains to check what happens in a neighborhood of the points $x_0 \in \Omega \cap (\mathbb{R}^{n-1} \times \{0\})$. We take $x_0 \in \Omega$ and $r > 0$ such that $\overline{B(x_0, r)} \subset \Omega$. Let $w$ be the harmonic extension of $v$. Since $w$ is odd we have $w(x, 0) = 0$ on $B(x_0, r) \cap (\mathbb{R}^{n-1} \times \{0\})$. 

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We $\bar{B}^+ = \bar{B}(x_0, r) \cap \Omega^+$ and $\bar{B}^- = \bar{B}(x_0, r) \cap \Omega^-$. We have
\[
\begin{cases}
\Delta w = 0 & \text{in } B^- \\
w = v(x) & \text{in } \partial B^-
\end{cases}
\] (3.33)
and
\[
\begin{cases}
\Delta w = 0 & \text{in } B^+ \\
w = v(x) & \text{in } \partial B^+
\end{cases}
\] (3.34)
By uniqueness $w = v$ in $\bar{B}^+$ and $w = v$ in $\bar{B}^-$. Therefore $w = v$ in $\bar{B}(x_0, r)$, hence $v$ is harmonic.

**An alternative proof of ii).** Clearly $v$ is harmonic in $\Omega^+$ and in $\Omega^-$. Moreover $v \in C(\bar{\Omega})$. Let $x_0 \in \Omega \cap (\mathbb{R}^{n-1} \times \{0\}), r > 0$ such that $\bar{B}(x_0, r) \subset \Omega$. We have
\[
\int_{\bar{B}(x_0, r)} v(y) dy = \int_{\bar{B}^+(x_0, r)} v(y) dy + \int_{\bar{B}^-(x_0, r)} v(y) dy
\]
\[
= \int_{\bar{B}^+(x_0, r)} v(y) dy - \int_{\bar{B}^-(x_0, r)} v(y_1, \ldots, y_{n-1}, -y_n) dy
\]
\[
= \int_{\bar{B}^+(x_0, r)} v(y) dy - \int_{\bar{B}^+} v(y_1, \ldots, y_{n-1}, y_n) dy = 0.
\]
Hence
\[
v(x_0) = 0 = \int_{\bar{B}(x_0, r)} v(y) dy
\]
for every $x_0 \in \Omega \cap (\mathbb{R}^{n-1} \times \{0\}), r > 0$ such that $\bar{B}(x_0, r) \subset \Omega$. Therefore $v$ is harmonic in $\Omega$. □

4. Prove that there exists a constant $C > 0$, depending only on $n$, such that
\[
\max_{B(0,1)} |u| \leq C(\max_{\partial B(0,1)} |u| + \max_{B(0,1)} |f|),
\]
where $u$ is a smooth solution of
\[
\begin{cases}
-\Delta u = f, & \text{in } B(0, 1) \\
u = g, & \text{on } \partial B(0, 1)
\end{cases}
\]
**Hint.** Consider the functions $v_\pm(x) = u \pm C(\max_{B(0,1)} |f(x)|) |x|^2$, with $C > 0$ a suitable constant.
5. **Generalized Liouville Theorem** Suppose that $u$ is harmonic on $\mathbb{R}^n$ and $|u(x)| = O(|x|^m)$, as $|x| \to +\infty$, for some $m \geq 0$. Prove that $u$ is a polynomial of degree at most $p$.

**Solution of Exercise 5.**
Since $u$ is harmonic, it is analytic. Therefore

$$u(x) = \sum_{\alpha} \frac{D^\alpha u(x_0)}{\alpha!}(x - x_0)^\alpha, \quad \forall x \in \mathbb{R}^n$$

where for every $r > 0$ and $|\alpha| = k$ the following estimate holds:

$$|D^\alpha u(x_0)| \leq \frac{(2^{n+1}nk)^k}{\alpha(n)(r)^{n+k}} \|u\|_{L^\infty(\partial B(x_0,r))}.$$  

By assumption $u(x) \leq K|y|^m$ if $x \in \partial B(x_0, r)$ and $r$ is large enough. Therefore if $|\alpha| = k > m$ we get

$$|D^\alpha u(x_0)| \leq \frac{(2^{n+1}nk)^k}{\alpha(n)(r)^{n+k}} r^{n+m} \to 0 \text{ as } r \to +\infty.$$

It follows that $u$ is a polynomial of degree at most $p$.

6. **Harnack Convergence Theorem** Let $\{u_k\}_k$ be an increasing sequence of harmonic functions in an open connected set $\Omega \subseteq \mathbb{R}^n$. Suppose that there is $y \in \Omega$ such that $\lim_{k \to +\infty} u_k(y)$ exists finite. Then $u_k$ converges pointwise to a harmonic function on $\Omega$ and the convergence is uniform on the compact subsets of $\Omega$.

**Solution of Exercise 6.**
Let $K \subset \Omega$ be compact and $V$ an open connected set such that

$$K \cup \{y\} \subseteq V \subseteq \bar{V} \subseteq \Omega$$

(which exists by Exercise 3.4.2). Let $\varepsilon > 0$ be fixed and $N \in \mathbb{N}$ such that $0 \leq u_m(y) - u_\ell(y) \leq \varepsilon$ if $m \geq \ell \geq N$. By Harnack Theorem 3.4.11 applied to $u_m - u_\ell$ there exists $C > 0$ such that

$$\sup_{\bar{V}} u_m - u_\ell \leq C \inf_{\bar{V}} u_m - u_\ell \leq C \varepsilon.$$
Therefore \( \{u_k\}_k \) uniformly converges on \( K \). In particular \( \lim_k u_k(x) \) exists finite for every \( x \in \Omega \). We set \( u(x) = \lim_k u_k(x), \ x \in \Omega \). Since the convergence is uniform on compact sets \( u \) is harmonic in \( \Omega \). \( \Box \)

7. **Removable Singularity Theorem:** Suppose \( u \in C^2(B(0, R) \setminus \{0\}) \cap C(\overline{B(0, R)} \setminus \{0\}) \) is harmonic in \( B(0, R) \setminus \{0\} \) and satisfies

\[
u(x) = \begin{cases} \ o(\log(|x|)), & n = 2 \\ o(|x|^{2-n}), & n \geq 3 \end{cases} \text{ as } |x| \to 0.
\]

Then \( u \) can be defined at \( 0 \) so that it is harmonic in \( B(0, R) \).

**Solution of Exercise 7.** Let \( \tilde{u} \) be the solution of

\[
\begin{align*}
-\Delta \tilde{u} &= 0, & \text{in } & B(0, R) \\
\tilde{u} &= u, & \text{in } & \partial B(0, R).
\end{align*}
\]

Then \( \tilde{u} \) is bounded in \( \overline{B(0, R)} \) (by Maximum Principle \( \sup_{|x| \leq R} |u(x)| \leq \max_{|x|=R} |u| \)) and it is harmonic in \( B(0, R) \).

**Claim:** \( \tilde{u} = u \) in \( B(0, R) \setminus \{0\} \)

**Proof of the Claim.** Let \( w = u - \tilde{u} \). Then \( w \) is harmonic in \( B(0, R) \setminus \{0\} \). Moreover one can easily check that

\[
\lim_{x \to 0} \frac{w(x)}{R^{2-n} - |x|^{2-n}} = 0, \text{ if } n \geq 3, \quad \lim_{x \to 0} \frac{w(x)}{\log(R) - \log(|x|)} = 0 \text{ if } n = 2.
\]

Therefore for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( 0 < |x| \leq \delta \)

\[
|w(x)| \leq \varepsilon (R^{2-n} - |x|^{2-n}) \text{ if } n \geq 3, \quad |w(x)| \leq \varepsilon (\log(R) - \log(|x|)) \text{ if } n = 2.
\]

(3.35)

Now we observe that \( \varepsilon (R^{2-n} - |x|^{2-n}) \) and \( \varepsilon (\log(R) - \log(|x|)) \) are harmonic functions in \( B(0, R) \setminus B(0, \delta) \) (resp. when \( n \geq 3 \) and \( n = 2 \)) and they are zero when \( |x| = R \). The Weak Maximum Principle applied in \( B(0, R) \setminus \overline{B(0, \delta)} \) implies that

\[
|w(x)| \leq \varepsilon (R^{2-n} - |x|^{2-n}) \text{ if } n \geq 3, \quad |w(x)| \leq \varepsilon (\log(R) - \log(|x|)) \text{ if } n = 2,
\]

(3.36)
for all \( \delta \leq |x| \leq R \). By combining (3.36) and (3.37) we get

\[
|w(x)| \leq \varepsilon (R^{2-n} - |x|^{2-n}) \text{ if } n \geq 3, \quad |w(x)| \leq \varepsilon (\log(R) - \log(|x|)) \text{ if } n = 2,
\]

(3.37)
for all \( 0 < |x| \leq R \). By letting \( \varepsilon \to 0 \) we get \( w(x) \equiv 0 \), namely \( \tilde{u}(x) \equiv u(x) \) in \( B(0, R) \setminus \{0\} \). If we redefine \( u(0) := \tilde{u}(0) \) we get the conclusion. \( \Box \)

### 3.5 Green’s Functions

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set of class \( C^1 \). We propose to obtain a general representation formula for the solution of the Poisson’s equation

\[
\begin{aligned}
-\Delta u &= f, \quad \text{in } \Omega \\
\quad u &= g \quad \text{in } \partial \Omega.
\end{aligned}
\]

**Theorem 3.5.1 (Green’s Identity)** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set of class \( C^1 \), \( u \in C^2(\Omega, \mathbb{R}) \), \( x \in \Omega \). Then

\[
\begin{aligned}
u(x) + \int_{\Omega} \Phi(y-x) \Delta u(y) dy &= \int_{\partial \Omega} (\Phi(y-x) \frac{\partial u}{\partial \nu}(y) - \frac{\partial \Phi}{\partial \nu}(y-x)u(y)) d\sigma(y),
\end{aligned}
\]

where \( \frac{\partial \Phi}{\partial \nu}(y-x) = D\Phi(y-x) \cdot \nu(y) \).

**Proof.** We first observe that the function \( y \mapsto \Phi(y-x) \) has a singularity in \( x \), but since \( |\Phi(y-x)| = \frac{c_1}{|x-y|^{n-2}} \) if \( n > 2 \) and \( |\Phi(y-x)| = c_2 |\log(|x-y|)| \) if \( n = 2 \), the function \( y \mapsto \Phi(y-x) \Delta u(y) \) is integrable in \( \Omega \).

We set \( v(y) = \Phi(y-x) \) for \( y \in \Omega \). Given \( \varepsilon > 0 \) such that \( B(x, \varepsilon) \subset \Omega \), the function \( v \) is well defined and harmonic on \( \Omega_\varepsilon = \Omega \setminus B(x, \varepsilon) \).
We apply the Green formula:

\[
\int_{\Omega_\varepsilon} u(y) \Delta \Phi(y - x) - \Phi(y - x) \Delta u(y) dy = \int_{\partial\Omega_\varepsilon} u(y) \frac{\partial \Phi}{\partial \nu}(y - x) d\sigma(y)
\]

\[- \int_{\partial\Omega_\varepsilon} \Phi(y - x)(y) \frac{\partial u}{\partial \nu}(y) d\sigma(y) .
\]

We have \( \Delta \Phi(y - x) = 0 \) in \( \Omega_\varepsilon \) and \( \partial\Omega_\varepsilon = \partial\Omega \cup \partial B(x, \varepsilon) \).

We have

\[
- \int_{\Omega_\varepsilon} \Phi(y - x) \Delta u(y) dy = \int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y - x) d\sigma(y)
\]

\[- \int_{\partial\Omega} \Phi(y - x)(y) \frac{\partial u}{\partial \nu}(y) d\sigma(y) + \int_{\partial B(x, \varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y - x) d\sigma(y)
\]

\[- \int_{\partial B(x, \varepsilon)} \Phi(y - x)(y) \frac{\partial u}{\partial \nu}(y) d\sigma(y) . \quad (3.39)
\]

The following estimate holds

\[
| \int_{\partial B(x, \varepsilon)} \Phi(y - x)(y) \frac{\partial u}{\partial \nu}(y) d\sigma(y) | \leq \left\{ \begin{array}{ll}
  k \log(\varepsilon) \varepsilon & n = 2 \\
  k \varepsilon & n > 2 .
\end{array} \right. \quad (3.40)
\]

Thus \( | \int_{\partial B(x, \varepsilon)} \Phi(y - x)(y) \frac{\partial u}{\partial \nu}(y) d\sigma(y) | \to 0 \) as \( \varepsilon \to 0 \). On the other hand we have

\[
\int_{\partial B(x, \varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y - x) d\sigma(y) = \frac{1}{|\partial B(x, \varepsilon)|} \int_{\partial B(x, \varepsilon)} u(y) d\sigma(y) .
\]

This last estimates justifies the particular choice of the coefficients in \( \Phi \).

Thus \( \int_{\partial B(x, \varepsilon)} u(y) \frac{\partial \Phi}{\partial \nu}(y - x) d\sigma(y) \to u(x) \) as \( \varepsilon \to 0 \).

The Dominated Convergence Theorem permits to conclude that

\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Phi(y - x) \Delta u(y) dy = \int_{\Omega} \Phi(y - x) \Delta u(y) dy .
\]

Thus by passing into the limit as \( \varepsilon \to 0 \) in (3.39) we get (3.38). \( \square \)
3.5.1 Derivation of the Green’s Function

We observe that once we know $\Delta u$ in $\Omega$, $\frac{\partial u}{\partial \nu}$ and $u$ on $\partial \Omega$, then the function $u$ is determined by the formula (3.38).

If $v$ is a harmonic function in $\Omega$ then the Green formula gives

$$
\int_{\Omega} v(y) \Delta u(y) dy = \int_{\partial \Omega} v(y) \frac{\partial u}{\partial \nu} d\sigma - \int_{\partial \Omega} u(y) \frac{\partial v}{\partial \nu} d\sigma .
$$

(3.41)

By subtracting (3.41) to (3.38) (3.41) we get

$$
\begin{align*}
\int_{\Omega} v(y) \Delta u(y) dy &= \int_{\partial \Omega} v(y) \frac{\partial u}{\partial \nu} d\sigma - \int_{\partial \Omega} u(y) \frac{\partial v}{\partial \nu} d\sigma .
\end{align*}
$$

(3.42)

Thus if $v(y) = \Phi(y - x)$ on $\partial \Omega$, we get

$$
\begin{align*}
u(x) &= -\int_{\Omega} (\Phi(y - x) - v(y)) \Delta u(y) dy + \int_{\partial \Omega} (\Phi(y - x) - v(y)) \frac{\partial u}{\partial \nu}(y) d\sigma \\
&+ \int_{\partial \Omega} u(y) \left( \frac{\partial v}{\partial \nu} - \frac{\partial \Phi}{\partial \nu} \right) d\sigma .
\end{align*}
$$

Definition 3.5.1 The Green function for the domain $\Omega$ is the function $G(x, y) = \Phi(y - x) - \Phi^x(y)$ where $\Phi^x(\cdot) \in C^2(\Omega) \cap C(\Omega)$ is the function (if it exists) which solves the problem

$$
\begin{align*}
\Delta \Phi^x &= 0, & \text{in } \Omega \\
\Phi^x(y) &= \Phi(y - x), & \text{on } \partial \Omega
\end{align*}
$$

From the previous computations we get

$$
u(x) = -\int_{\Omega} G(x, y) \Delta u(y) dy - \int_{\partial \Omega} \frac{\partial G(x, y)}{\partial \nu} u(y) dS ,
$$

(3.42)

where

$$
\frac{\partial G}{\partial \nu}(x, y) = D_y G(x, y) \cdot \nu(y)
$$
is the outer normal derivatives of $G$ with respect to the variable $y$. Observe that the term $\frac{\partial u}{\partial \nu}$ does not appear in the equation (3.42): we introduced the corrector precisely to achieve this.

The function $G(\cdot, \cdot)$ depends (if it exists) only on $\Omega$.

Suppose now $u \in C^2(\overline{\Omega})$ solves the boundary-value problem

$$
\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = g & \text{in } \partial \Omega
\end{cases}
$$

for given continuous functions $f, g$. Plugging into (3.42) we obtain

**Theorem 3.5.2 (Representation formula using Green’s function).** If $u \in C^2(\overline{\Omega})$ solves the problem (3.43) then

$$
u(x) = - \int_{\partial \Omega} g(y) \frac{\partial G}{\partial \nu}(x, y) d\sigma(y) + \int_{\Omega} f(y) G(x, y) dy,
$$

$x \in \Omega$.

Here we have a formula for the solution of the boundary-value problem (3.43), provided we can construct Green’s function $G$ for the given domain $\Omega$. This is in general a difficult matter, and can be done only when it has a simple geometry.

**Proposition 3.5.1** Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ for which there exists the Green function $G$ for the Laplacian. Then

i) $G$ is nonnegative.

ii) $G$ is a symmetric function.

**Proof of Proposition 3.5.1.**

i) Let $x \in \Omega$ and $\delta > 0$ be such that $\bar{B}(x, \delta) \subseteq \Omega$. The function $y \mapsto G(x, y)$ is harmonic in $\Omega \setminus \bar{B}(x, \varepsilon)$ for every $\varepsilon < \delta$ and it verifies (for $n > 2$)

$$
\begin{align*}
G(x, y) &= 0 \quad \text{on } \partial \Omega \\
G(x, y) &\geq \frac{1}{(n-2)\omega_n \varepsilon^{n-2}} - M \quad \text{on } \partial B(x, \varepsilon)
\end{align*}
$$

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where
\[ M = \max_{B(x,\delta)} |\Phi(x)|. \]
(The function \( \Phi(x) \) is harmonic in \( \Omega \) and therefore continuous in \( \bar{B}(x,\delta) \)).

Hence \( G(x,y) \geq 0 \) for every \( y \in \partial(\Omega \setminus \bar{B}(x,\varepsilon)) \) if \( \varepsilon \) is small enough. The weak maximum principle implies that \( G(x,y) \geq 0 \) for every \( y \in \Omega \setminus B(x,\varepsilon) \).

Since \( \varepsilon \) is arbitrary \( G(x,y) \geq 0 \) for every \( y \in \bar{\Omega} \setminus \{x\} \). The proof for the case \( n = 2 \) is similar.

ii) Fix \( x,y \in \Omega \). We write \( v(z) = G(x,z), \ w(z) = G(y,z), \ (z \in \Omega) \).

Then \( \Delta v(z) = 0 \), if \( z \neq x \), \( \Delta w(z) = 0 \), if \( z \neq y \) and \( w = v = 0 \) on \( \partial\Omega \).

We apply Green’s identity to \( v,w \) in \( U = \Omega \setminus \{B(x,\varepsilon) \cup B(y,\varepsilon)\} \) for \( \varepsilon > 0 \) small enough:
\[ \int_{\partial B(x,\varepsilon)} \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v d\sigma(z) = \int_{\partial B(y,\varepsilon)} \frac{\partial w}{\partial \nu} v - \frac{\partial v}{\partial \nu} w d\sigma(z), \quad (3.45) \]
where \( \nu \) denotes the inward pointing unit vector field on \( \partial B(x,\varepsilon) \cup \partial B(y,\varepsilon) \).

Now \( w \) is smooth near \( x \); whence \(^{(4)}\)
\[ \left| \int_{\partial B(x,\varepsilon)} \frac{\partial w}{\partial \nu} v d\sigma(z) \right| \leq C \varepsilon^{n-1} \sup_{\partial B(x,\varepsilon)} |v| = o(1), \quad \text{as} \ \varepsilon \to 0. \]

On the other hand, \( v(z) = \Phi(z-x) - \phi^x(z) \) where \( \phi^x \) is smooth in \( \Omega \). Thus
\[ \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial v}{\partial \nu} w d\sigma(z) = \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial \Phi}{\partial \nu}(x-z)w(z) d\sigma(z) \]
\[ - \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \frac{\partial \phi^x}{\partial \nu}(z)w(z) d\sigma(z) = w(x) + 0. \]

Thus the left-hand side of (3.45) converges to \( w(x) \) as \( \varepsilon \to 0 \). Likewise the right hand side converges to \( v(y) \). Consequently
\[ G(y,x) = w(x) = v(y) = G(x,y). \]

\(^{(4)}\) \( v \) behaves on \( \partial B(x,\varepsilon) \) like \( \varepsilon^{2-n} \) (resp. \( |\log(\varepsilon)| \)) if \( n > 2 \) (resp. \( n = 2 \)).
We conclude the proof.

We observe that if \( \Omega \) is a bounded connected open subset of \( \mathbb{R}^n \) then the Green function (if it exists) is strictly positive for every \( x, y \in \Omega, x \neq y \).

### 3.5.2 Green’s function for a ball

We consider the unit ball \( B(0,1) \).

**Definition 3.5.2** If \( x \in \mathbb{R}^n \setminus \{0\} \), the point

\[
\tilde{x} = \frac{x}{|x|^2}
\]

is called the point dual to \( x \) with respect to \( \partial B(0,1) \). The mapping \( x \mapsto \tilde{x} \) is inversion through the unit sphere \( \partial B(0,1) \). If \( x = 0 \), we take \( \tilde{x} = \infty \).

We now employ inversion through the sphere to compute the Green’s function for the unit ball \( \Omega = B(0,1) \). We fix \( x \in B(0,1) \). We must find a *corrector* function \( \Phi^x = \Phi^x(y) \) solving

\[
\begin{align*}
\Delta \Phi^x &= 0, & \text{in } \Omega \\
\Phi^x(y) &= \Phi(y - x) & \text{on } \partial \Omega
\end{align*}
\]

Then Green’s function will be

\[
G(x,y) = \Phi(y - x) - \Phi^x(y).
\]

Assume for the moment that \( n \geq 3 \). Now the mapping \( y \mapsto \Phi(y - \tilde{x}) \) is harmonic for \( y \neq \tilde{x} \). Thus \( y \mapsto |x|^{2-n}\Phi(y - \tilde{x}) \) is harmonic for \( y \neq \tilde{x} \), and so

\[
\Phi^x(y) := \Phi(|x|(y - \tilde{x})) \tag{3.46}
\]

is harmonic in \( B(0,1) \). Furthermore if \( y \in \partial B(0,1) \) and \( x \neq 0 \), then

\[
|x|^2|y - \tilde{x}|^2 = |x|^2 \left( |y|^2 - \frac{2y \cdot x}{|x|^2} + \frac{1}{|x|^2} \right) = |x|^2 - 2y \cdot x + 1 = |y - x|^2.
\]
Thus \(|x||y - \tilde{x}|\)^\(-(n-2) = |x - y|^{-(n-2)}\). Consequently
\[
\Phi^x(y) = \Phi(y - x) \quad y \in \partial B(0, 1).
\] (3.47)

\textbf{Definition 3.5.3} \hspace{1em} The Green’s function for the unit ball is
\[
G(x, y) = \begin{cases} \Phi(y - x) - \Phi(|x|(y - \tilde{x})) & x \neq 0 \\ \Phi(|y|) - \Phi(1) & x = 0 \end{cases} \quad x, y \in B(0, 1), x \neq y.
\] (3.48)

The same formula is valid for \(n = 2\) as well.

Assume now \(u\) solves the boundary-value problem
\[
\begin{align*}
\Delta u &= 0 \quad \text{in } B(0, 1) \\
u &= g \quad \text{in } \partial B(0, 1)
\end{align*}
\] (3.49)

Then using (3.44) we see
\[
u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G}{\partial \nu}(x, y) d\sigma(y). \tag{3.50}
\]

According to formula (3.48)
\[
\frac{\partial G}{\partial y_i}(x, y) = \frac{\partial \Phi}{\partial y_i}(y - x) - \frac{\partial}{\partial y_i} \Phi(|x|(y - \tilde{x})).
\]
But
\[
\frac{\partial \Phi}{\partial y_i}(y - x) = \frac{1}{n\alpha(n)} \frac{x_i - y_i}{|x - y|^n},
\]
furthermore
\[
\frac{\partial}{\partial y_i} \Phi(|x|(y - \tilde{x})) = -\frac{1}{n\alpha(n)} \frac{y_i |x|^2 - x_i}{|x - y|^n}.
\]
if \(y \in \partial B(0, 1)\). Accordingly
\[
\frac{\partial G}{\partial \nu}(x, y) = \sum_{i=1}^{n} y_i \frac{\partial G}{\partial y_i}(x, y) = -\frac{1}{n\alpha(n)} \frac{1 - |x|^2}{|x - y|^n}.
\]
Hence formula (3.50) yields the representation formula

\[ u(x) = \frac{1 - |x|^2}{n \alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n} d\sigma(y). \]

Suppose now \( u \) solves the boundary-value problem

\[
\begin{cases}
\Delta u = 0 & \text{in } B(0,r) \\
u = g & \text{on } \partial B(0,r) 
\end{cases}
\tag{3.51}
\]

for \( r > 0 \). Then \( \tilde{u}(x) = u(rx) \) solves the problem (3.49), with \( \tilde{g}(x) = g(rx) \) replacing \( g \). We change variables to obtain Poisson’s formula

\[ u(x) = \frac{r^2 - |x|^2}{n \alpha(n) r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} d\sigma(y) \tag{3.52} \]

\( x \in B(0,r) \).

The function

\[ K(x, y) = \frac{r^2 - |x|^2}{n \alpha(n) r} \frac{1}{|x-y|^n} \]

\( x \in B(0,r), y \in \partial B(0,r) \) is called Poisson’s kernel for the ball \( B(0,r) \).

We have established (3.52) under the assumption that a smooth solution of (3.51) exists. We next show that this formula in fact gives a solution:

**Theorem 3.5.3 (Poisson Formula for the Ball)** Assume \( g \in C(\partial B(0,r)) \) and define \( u \) by (3.52). Then

(i) \( u \in C^\infty(B(0,r)) \),

(ii) \( \Delta u = 0 \) in \( B(0,r) \),

(iii) \( \lim_{x \to x_0} u(x) = g(x_0) \) for every \( x_0 \in \partial B(0,r) \).

**Proof.** That \( u \) defined by (3.52) is harmonic in \( B(0,r) \) is evident from the fact that \( G \) and hence \( \frac{\partial G}{\partial n} \) are harmonic in \( x \). To establish the continuity
of \( u \) on \( \partial B(0, r) \) we use the Poisson formula (3.52) for the special case \( u = 1 \) to obtain the identity
\[
\int_{\partial B(0, r)} K(x, y) d\sigma(y) = 1
\]
for all \( x \in B(0, r) \).

Now let \( x_0 \in \partial B(0, r) \) and \( \varepsilon > 0 \). Choose \( \delta > 0 \) so that \( |g(x) - g(x_0)| \leq \varepsilon \), if \( |x - x_0| \leq \delta \) and let \( |g| \leq M \) on \( \partial B(0, r) \).

Then if \( |x - x_0| \leq \delta/2 \) we have
\[
|u(x) - g(x_0)| = \left| \int_{\partial B(0, r)} K(x, y)(g(y) - g(x_0)) d\sigma(y) \right|
\leq \int_{|y-x_0| \leq \delta} K(x, y)|g(y) - g(x_0)| d\sigma(y)
+ \int_{|y-x_0| > \delta, y \in \partial B(0, r)} K(x, y)|g(y) - g(x_0)| d\sigma(y)
\leq \varepsilon + \frac{2M(r^2 - |x|^2)r^{n-2}}{(\delta/2)^n}.
\]

If \( |x - x_0| \) is sufficiently small it is clear that \( |u(x) - g(x_0)| < 2\varepsilon \) and hence \( \lim_{x \to x_0, x \in B(0, r)} u(x) = g(x_0) \) for every \( x_0 \in \partial B(0, r) \). \( \square \)

### 3.5.3 Green’s function for a half space

Let us consider the half space
\[
\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}.
\]

**Definition 3.5.4** If \( x \in \mathbb{R}^n_+ \), its reflection in the plane \( \partial \mathbb{R}^n_+ \) is the point
\[
\tilde{x} = (x_1, \ldots, x_{n-1}, -x_n).
\]
In this case a solution of
\[
\begin{align*}
\Delta \Phi^x &= 0, & \text{in } \mathbb{R}^n_+ \\
\Phi^x(y) &= \Phi(y - x) & \text{on } \partial \mathbb{R}^n_+
\end{align*}
\]
is given by
\[
\Phi^x(y) = \Phi(y - \tilde{x}), \quad x, y \in \mathbb{R}^n_+.
\]
The idea is that the corrector \(\Phi^x(y)\) is built from \(\Phi\) by reflecting the singularity from \(x \in \mathbb{R}^n_+\) to \(\tilde{x} \notin \mathbb{R}^n_+\).

**Definition 3.5.5** The Green’s function for the half-space is
\[
G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x}), \quad x, y \in \mathbb{R}^n_+, x \neq y.
\]

Suppose now \(u\) solves the boundary-value problem
\[
\begin{align*}
\Delta u &= 0, & \text{in } \mathbb{R}^n_+ \\
u &= g & \text{on } \partial \mathbb{R}^n_+
\end{align*}
\]
Then
\[
u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x - y|^n} \, d\sigma(y).
\]

The function
\[
K(x, y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}
\]
is the **Poisson’s Kernel** for \(\mathbb{R}^n_+\) and (3.54) is the **Poisson’s formula**.

**Theorem 3.5.4** (Poisson Formula for the Half-Space)
Assume \(g \in C(\mathbb{R}^{n-1}) \cap L_\infty(\mathbb{R}^{n-1})\) and define \(u\) by (3.54). Then
(i) \(u \in C^\infty(\mathbb{R}^n_+) \cap L_\infty(\mathbb{R}^n_+)\),
(ii) \(\Delta u = 0 \text{ in } \mathbb{R}^n_+\),
(iii) \(\lim_{x \to x_0} u(x) = g(x_0) \) for every \(x_0 \in \partial \mathbb{R}^n_+\).

For the Proof we refer to Theorem 14, Chapter 2 in [2].
3.6 Perron’s Method

In this section we solve the Dirichlet problem for the Laplace equation in bounded domains by Perron’s method. The key tools are the maximum principle and the Poisson integral formula. The latter provides the solvability of the Dirichlet problem for the Laplace equation in balls.

Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set.

**Definition 3.6.1** We say that \( u \in C(\overline{\Omega}) \) is sub-harmonic (resp. super-harmonic) in \( \Omega \) if for any ball \( B \subset \Omega \) such that \( \overline{B} \subset \Omega \) and any harmonic \( h \in C^2(B) \cap C(\overline{B}) \) such that \( u \leq h \) (resp. \( u \geq h \)) on \( \partial B \) one has \( u \leq h \) (resp. \( u \geq h \)) in \( B \).

**Proposition 3.6.1** If \( u \in C^2(\Omega) \) then Definition 3.3.1 and Definition 3.6.1 are equivalent.

**Proof.** 1) Let \( u \in C^2(\Omega) \) be such that \( \Delta u \geq 0 \) in \( \Omega \), \( h \) be harmonic in \( B \) and \( u \leq h \) in \( \partial B \). Then \( \Delta(u - h) \geq 0 \). By the maximum principle we have

\[
\max_{\overline{B}}(u - h) = \max_{\partial B}(u - h) \leq 0.
\]

Therefore \( u \leq h \) in \( \overline{B} \).

2) Conversely if \( \Delta u(x_0) < 0 \) at some \( x_0 \in \Omega \), then \( \Delta u(x) < 0 \) in \( \overline{B} := \overline{B}(x_0, r) \subset \Omega \). Let \( w \) solve

\[
\begin{cases}
\Delta w = 0 & \text{in } B \\
w = u & \text{in } \partial B.
\end{cases}
\]

(3.55)

The existence of \( w \) in \( B \) is implied by the Poisson integral formula. We have \( u \leq w \) in \( B \) by assumption.

Next we note that

\[
\begin{cases}
0 = \Delta w > \Delta u & \text{in } B \\
w = u & \text{in } \partial B
\end{cases}
\]

(3.56)
Then by the maximum principle we have \( w \leq u \). Hence \( u = w \) in \( B \), which contradicts the fact that \( \Delta w > \Delta u \). Therefore \( \Delta u \geq 0 \).

The Strong Maximum Principle (resp. Strong Minimum Principle) holds for continuous harmonic sub-harmonic (resp. super-harmonic) functions.

**Proposition 3.6.2 (Strong Maximum/Minimum Principle)**

Let \( \Omega \subseteq \mathbb{R}^n \) be a connected open set and \( u \in C(\overline{\Omega}) \) be sub-harmonic (resp. super-harmonic). Then for all \( x \in \Omega \) we have

\[
u(x) < \max_{\Omega} u \quad \text{or} \quad u(x) \equiv \max_{\Omega} u
\]

(resp. \( u(x) > \min_{\Omega} u \) or \( u \equiv \min_{\Omega} u \)).

**Proof.** Let \( u \) be sub-harmonic and \( x_0 \in \Omega \) be such that \( M = u(x_0) = \max_{\Omega} u(x) \). If \( u \) is not constant, then the set \( \{ x \in \Omega : u(x) = M \} \) is not open. We can suppose that there exists \( r > 0 \) such that \( B(x_0, r) \subseteq \Omega \) and \( u \neq M \) on \( \partial B(x_0, r) \). Let \( w \) solve

\[
\begin{align*}
\Delta w &= 0 & \text{in } B(x_0, r) \\
\quad w &= u & \text{in } \partial B(x_0, r).
\end{align*}
\]

Then \( u \leq w \) on \( B(x_0, r) \). In particular

\[
M = u(x_0) \leq w(x_0) \leq \max_{\partial B(x_0, r)} w \leq M.
\]

By the Strong Maximum Principle for harmonic functions we have \( w \equiv u(x_0) = M \) on \( B(x_0, r) \). Hence \( u \equiv M \) on \( \partial B(x_0, r) \) which is a contradiction. The proof for super-harmonic functions follows by observing that if \( u \) is super-harmonic then \( v = -u \) is sub-harmonic.

**Corollary 3.6.1** Let \( \Omega \subseteq \mathbb{R}^n \) be a connected open set and \( u, v \in C(\overline{\Omega}) \) be resp. sub-harmonic and super-harmonic in \( \Omega \) with \( u \leq v \) on \( \partial \Omega \). Then either \( u < v \) in \( \Omega \) or \( u \equiv v \) in \( \Omega \).
Proof. Exercise.

Next, we describe the Perron’s method.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\varphi \in C(\partial \Omega)$. We will find a function $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ such that

$$
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega \\
u &= \varphi \quad \text{in } \partial \Omega.
\end{align*}
$$

Suppose there exists a solution $u = u_\varphi$. Then for any $v \in C(\overline{\Omega})$ which is sub-harmonic in $\Omega$ with $v \leq \varphi$ on $\partial \Omega$, we obtain $v \leq u_\varphi$ in $\Omega$.

Hence for any $x \in \Omega$

$$u_\varphi(x) = \sup\{v(x) : \quad v \in C(\overline{\Omega}), \text{ is subharmonic in } \Omega \text{ and } v \leq \varphi \text{ on } \partial \Omega\}.$$  

We note that the equality holds since $u_\varphi$ is an element of the set in the right-hand side. In Perron’s method we will show that $u_\varphi$ defined in (3.59) is indeed a solution of (3.58) under appropriate assumptions on the domain.

**Proposition 3.6.3** Let $u$ be sub-harmonic in $\Omega$, $B$ be a ball such that $\overline{B} \subset \Omega$ and $\overline{u}$ be a solution of

$$
\begin{align*}
\Delta \overline{u} &= 0 \quad \text{in } B \\
\overline{u} &= u \quad \text{in } \partial B.
\end{align*}
$$

The harmonic lifting $U$ of $u$ defined by

$$U(x) = \begin{cases} 
\overline{u}(x) & \text{if } x \in B \\
u(x) & \text{if } x \in \Omega \setminus B
\end{cases}$$

is sub-harmonic in $\Omega$ and $u \leq U$ in $\overline{\Omega}$.

**Proof.** Let $B'$ be a ball such that $\overline{B'} \subset \Omega$ and $h$ be harmonic in $B'$, $U \leq h$ on $\partial B'$. We observe that $U = u$ on $B' \setminus B$. Since $u = U$ on $\partial B$ one has $u \leq U$ in $B$. Therefore $u \leq h$ on $\partial B'$ and $u \leq h$ in $B'$. In particular $u = U \leq h$ on $B' \setminus B$.

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Now we consider the set $B \cap B'$. On $\partial B' \cap B$ we have $U \leq h$ and on $\partial B \cap B'$ we have $U = \bar{u} = u \leq h$. Both $U$ and $h$ are harmonic in $B \cap B'$ and $U \leq h$ on $\partial(B \cap B')$. By the maximum principle we have $U \leq h$ in $B \cap B'$. Hence $U \leq h$ in $B'$. Therefore $U$ is sub-harmonic in $\Omega$.

Proposition 3.6.3 asserts that we obtain greater sub-harmonic functions if we preserve the values of sub-harmonic functions outside the balls and extend them inside the balls by the Poisson integral formula.

**Proposition 3.6.4** Let $u_1, \ldots, u_N$ be sub-harmonic in $\Omega$. Then $u(x) = \max\{u_1(x), \ldots, u_N(x)\}$ is sub-harmonic in $\Omega$.

**Proof.** Exercise.

Let $\varphi \in L^\infty(\partial \Omega)$. We set

$$S_\varphi = \{u \in C(\overline{\Omega}), u \text{ sub-harmonic}, u \leq \varphi \text{ on } \partial \Omega\}$$

We observe that $S_\varphi \neq \emptyset$ since $v(x) \equiv \inf_{\partial \Omega} \varphi$ is sub-harmonic and $v \leq \varphi$ in $\partial \Omega$.

We prove next the Perron’s Theorem.

**Theorem 3.6.1** [Perron’s Theorem] For every $x \in \Omega$ we have $\sup\{v(x) : v \in S_\varphi\} < +\infty$ and the function $u(x) = \sup\{v(x) : v \in S_\varphi\}$ is harmonic in $\Omega$.

**Proof.** If $v \in S_\varphi$ then $v \leq \sup_{\partial \Omega} \varphi$ in $\partial \Omega$. Hence $v \leq \sup_{\partial \Omega} \varphi$ in $\Omega$ and $u(x) \leq \sup_{\partial \Omega} \varphi$ for every $x \in \Omega$ as well. Now we fix $y \in \Omega$, let $(v_n)_n$ be a sequence in $S_\varphi$ such that $v_n(y) \to u(y)$. We may suppose without loss of generality that $v_n$ is increasing. Let $B = B(y, r) \subset \Omega$ and $V_n$ be the harmonic lifting of $v_n$ on $B$. Then $v_n \leq V_n$ in $B$ and $V_n \leq \varphi$ on $\partial \Omega$. Therefore $V_n \in S_\varphi$, $V_n \leq u$ in $B$ and $v_n(y) \leq V_n(y) \leq u(y)$. It follows that $\lim_{n \to +\infty} V_n(y) = u(y)$. We observe that $(V_n)_n$ is monotone: $V_n = v_n \leq v_{n+1} = V_{n+1}$ on $\partial B$, hence $V_n \leq V_{n+1}$ on $B$. 

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By Harnack convergence theorem (see Exercise 3.4.1 8), the sequence uniformly converges to a harmonic function \( V \) in \( B \). It is clear that \( V \leq u \) in \( B \) and \( V(y) = u(y) \). If \( V(\zeta) < u(\zeta) \) for some \( \zeta \in B \) then there exists \( \tilde{u} \in S_\varphi \) such that \( V(\zeta) < \tilde{u}(\zeta) \). We set \( w_k = \max\{V_k, \tilde{u}\} \) and denote by the \( W_k \) the harmonic lifting of \( w_k \). \( W_k \) uniformly converges to a harmonic function \( W \) in \( B \). We have \( \tilde{u} \leq W \) and \( V \leq W \leq u \) in \( B \). Thus \( V(y) = W(y) = u(y) \).

The function \( V - W \) is harmonic in \( B \), \( V - W \leq 0 \) and \( V(y) - W(y) = 0 \), by the Strong Maximum Principle \( V = W \) in \( B \) in particular \( V(\zeta) = W(\zeta) \geq \tilde{u}(\zeta) \) which is a contradiction. Therefore \( V(\zeta) = u(\zeta) \) for every \( \zeta \in B \) and \( u \) is harmonic in \( \Omega \).

The fact that \( u = \varphi \) on \( \partial \Omega \) is related to geometric properties of \( \partial \Omega \).

**Definition 3.6.2 (Barrier)** Let \( \xi \in \partial \Omega \). A function \( w \in C(\overline{\Omega}) \) is called barrier function at \( \xi \) relative to \( \Omega \) if

i) \( w \) is super-harmonic in \( \Omega \),

ii) \( w > 0 \) in \( \Omega \setminus \{\xi\} \) and \( w(\xi) = 0 \).

We say that a point \( \xi \in \partial \Omega \) is regular for the laplacian if there is a barrier function in \( \xi \) relative to \( \Omega \).

**Proposition 3.6.5** Let \( \Omega \subseteq \mathbb{R}^n \) be open. We suppose that the Dirichlet problem (3.58) has a solution for every \( \varphi \in C(\partial \Omega) \). Then for every \( \xi \in \partial \Omega \) there exists a barrier in \( \xi \) relatively to \( \Omega \).

**Proof.** Let \( \xi \in \partial \Omega \) and set \( \psi(x) = |x - \xi| \) for \( x \in \partial \Omega \). Let \( U \) be a harmonic function in \( \Omega \) such that \( U = \psi \) on \( \partial \Omega \).

**Claim 1:** \( \Psi(x) = |x - \xi| \) is sub-harmonic in \( \Omega \).

**Proof of the Claim 1.** \( \Delta \Psi(x) = \frac{n-1}{|x - \xi|} > 0 \) in \( \Omega \).

We observe that \( \Psi - U = 0 \) on \( \partial \Omega \). By maximum principle \( \Psi - U \leq 0 \) in \( \Omega \). Since \( U(x) \geq \Psi(x) > 0 \) in \( \Omega \setminus \{\xi\} \) we deduce that \( U \) is a barrier function. \( \square \)
Conversely it holds:

**Theorem 3.6.2** Let \( \varphi \in L^\infty(\partial \Omega) \) and \( u \) be the harmonic function defined by the Perron’s method. If \( \xi \in \partial \Omega \) is a regular point for the laplacian and \( \varphi \) is continuous in \( \xi \) then

\[
\lim_{x \to \xi} u(x) = \varphi(\xi).
\]

**Proof.** Let \( w \) be a barrier function in \( \xi \). Let \( \varepsilon > 0 \) be an arbitrary constant. By the continuity of \( \varphi \) in \( \xi \) there is \( \delta > 0 \) such that

\[
|\varphi(x) - \varphi(\xi)| \leq \varepsilon,
\]

for any \( x \in \partial \Omega \cap B(\xi, \delta) \). We set \( \alpha = \min_{|x-\xi|\geq \delta} w(x) \) and \( M = \sup_{\partial \Omega} |\varphi| \).

Observe that the function \( \psi(x) = \varphi(\xi) - \varepsilon - \beta w(x) \) is sub-harmonic and \( \psi(x) \leq \varphi(x) \) for every \( x \in \partial \Omega \) as soon as \( \beta w(x) \geq \varphi(\xi) - \varepsilon - \varphi(x) \) (this holds for every \( \beta > 0 \), if \( |x - \xi| \leq \delta \) and for \( \beta w(x) > 2M \) and \( \varepsilon < M \) if \( |x - \xi| > \delta \)).

Hence if \( \beta \geq \frac{2M}{\alpha} \) we have \( \psi \in S_\varphi \).

In a similar way one can prove that if \( \beta \geq \frac{2M}{\alpha} \) the function \( \zeta(x) = \varphi(\xi) + \varepsilon + \beta w(x) \) is super-harmonic with \( \zeta(x) \geq \varphi(x) \) in \( \partial \Omega \). Since \( u \) is harmonic we have

\[
\psi(x) \leq u(x) \leq \zeta(x), \quad \forall x \in \Omega
\]

and therefore

\[
|u(x) - \varphi(\xi)| \leq \varepsilon + \beta w(x) \quad \forall x \in \Omega.
\]

Since \( w(x) \to 0 \) as \( x \to \xi \) we get \( u(x) \to \varphi(\xi) \) as \( x \to \xi \) and we can conclude. \( \square \)

**Corollary 3.6.2** The Dirichlet Problem (3.58) has a solution for every \( \varphi \in C(\partial \Omega) \) if and only if every point \( \xi \in \partial \Omega \) is regular for the laplacian.

**Proof.** Exercise.
Remark 3.6.1 Barrier functions can be constructed for a large class of domains \( \Omega \). Take for example the case where \( \Omega \) satisfies the **exterior sphere condition** at \( \xi \in \partial \Omega \) in the sense that there exists a ball \( B(y_0, r) \) such that

\[
\Omega \cap B(y_0, r) = \emptyset, \quad \overline{\Omega} \cap \overline{B}(y_0, r) = \{\xi\}.
\]

To construct a barrier function at \( \xi \) we set

\[
w_\xi(y) = \Phi(\xi - y_0) - \Phi(y - y_0), \quad \text{for any } y \in \Omega,
\]

where \( \Phi \) is the fundamental solution.

It is easy to show that \( w_\xi \) is a barrier function. We note that the exterior sphere condition always hold for \( C^2 \) domains.

In particular if every point of \( \partial \Omega \) is regular then there exists the Green function for \( \Omega \).

In summary Perron’s method yields a solution of the Dirichlet problem for the Laplacian equation. This method depends essentially on the maximum principle and the solvability of the Dirichlet problem in balls. An important feature here is that the interior existence problem is separated from the boundary behavior of solutions, which is determined by the local geometry of domains.

### 3.7 Poisson Equations: Classical solutions

In this section we discuss briefly the Poisson equation

\[
-\Delta u = f, \quad \text{in } \Omega, \quad (3.61)
\]

where \( \Omega \subseteq \mathbb{R}^n \) and \( f \in C(\Omega) \).

If \( u \) is a smooth solution of (3.61) in \( \Omega \), then obviously \( f \) is smooth. Conversely, we ask whether \( u \) is smooth if \( f \) is smooth. We note that \( \Delta u \) is just a linear combination of second derivatives of \( u \). To proceed we define
where $\Phi$ is the fundamental solution of the Laplace operator. The function $w_f$ is called the **Newtonian potential** of $f$ in $\Omega$. It is well-defined if $\Omega$ is bounded and $f \in L^\infty(\Omega)$. We recall that

$$|D\Phi(x-y)| \sim \frac{1}{|x-y|^{n-1}} \quad \text{and} \quad |D^2\Phi(x-y)| \sim \frac{1}{|x-y|^n},$$

as $y \to x$.

By differentiating under integral sign formally we have

$$\partial_{x_i} w_f(x) = \int_\Omega \partial_{x_i} \Phi(x-y) f(y) dy$$

for any $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$. The right hand side is a well-defined integral and defines a continuous function of $x$, (Exercise).

We need extra conditions to infer that $w_f \in C^2$ and $-\Delta w_f = f$.

**Lemma 3.7.1** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, $f \in L^\infty(\Omega)$. Assume that $f \in C^{k-1}(\Omega)$ for some integer $k \geq 2$. Then $w_f \in C^k(\Omega)$ and $-\Delta w_f = f$ in $\Omega$. Moreover if $f$ is smooth in $\Omega$ then $w_f$ is smooth in $\Omega$.

**Proof.** For simplicity we write $w = w_f$.

**Step 1.** Suppose $f$ has compact support in $\Omega$.

By the change of variables $z = y - x$ we have

$$w(x) = \int_{\mathbb{R}^n} \Phi(z) f(z + x) dz.$$

Since $f$ is at least $C^1$ by a simple differentiation under integral sign and an integration by parts we obtain

$$w_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(z) f_{x_i}(z + x) dz = \int_{\mathbb{R}^n} \Phi(z) f_{z_i}(z + x) dz$$

$$= -\int_{\mathbb{R}^n} \Phi_{z_i}(z) f(z + x) dz.$$
For \( f \in C^{k-1}(\Omega), \; k \geq 2 \) we can differentiate under the integral sign to get

\[
\partial^\alpha w_{x_i}(x) = \int_{\mathbb{R}^n} \Phi_{z_i}(z) \partial^\alpha f(z + x) \, dz,
\]

for any multiindex \( \alpha \) with \( |\alpha| \leq k - 1 \). Hence \( w \in C^k \) in \( \Omega \). Moreover if \( f \) is smooth in \( \Omega \) then \( w \) is smooth in \( \Omega \).

Next we calculate \( \Delta w \) if \( f \) is at least \( C^1 \). For any \( x \) we have

\[
\Delta w = -\int_{\mathbb{R}^n} \sum_{i=1}^n \Phi_{z_i}(z) f_{z_i}(z + x) \, dz
\]

\[
= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \sum_{i=1}^n \Phi_{z_i}(z) f_{z_i}(z + x) \, dz
\]

\[
= -\lim_{\varepsilon \to 0} \int_{\partial B(0, \varepsilon)} \frac{\partial \Phi}{\partial \nu}(z) f(z + x) \, d\sigma(z)
\]

\[
= -\lim_{\varepsilon \to 0} \frac{1}{\omega_n \varepsilon^{n-1}} \int_{\partial B(0, \varepsilon)} f(z + x) \, d\sigma(z) = -f(x),
\]

where \( \nu \) is the unit exterior normal to the boundary \( \partial B(0, \varepsilon) \) of the set \( \mathbb{R}^n \setminus B(0, \varepsilon) \).

**Step 2.** For every \( x_0 \in \Omega \) we prove that \( w \in C^k \) and \( -\Delta w = f \) in a neighborhood of \( x_0 \). To this end, we take \( r < \text{dist}(x_0, \partial \Omega) \) and a function \( \eta \in C_c^\infty(\mathbb{R}^n) \) with \( \text{supp}(\eta) \subset B(x_0, r) \), \( \eta \equiv 1 \) in \( B(x_0, r/2) \). Then we write

\[
w(x) = \int_{\Omega} \Phi(x - z)(1 - \eta(z)) f(z) \, dz + \int_{\Omega} \Phi(x - z) \eta(z) f(z) \, dz
\]

\[
= w_1(x) + w_2(x).
\]

The first integral is actually over \( \Omega \setminus B(x_0, r/2) \) since \( \eta \equiv 1 \) in \( B(x_0, r/2) \). Therefore \( w_1(x) \) is smooth in \( B(x_0, r/4) \) and \( \Delta w_1(x) = 0 \) in \( B(x_0, r/4) \).

For the second integral, \( \eta(z) f(z) \) is a \( C^{k-1} \) function with compact support in \( \Omega \). From Step 1 it follows that \( w_2 \in C^k(\Omega) \) and \( -\Delta w_2 = \eta f \).

Therefore \( w \in C^k(B(x_0, r/4)) \) and

\[
-\Delta w(x) = \eta(x) f(x) = f(x)
\]

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for any \( x \in B(x_0, r/4) \). Moreover if \( f \) is smooth in \( \Omega \) so are \( w_2 \) and \( w \) is smooth in \( \Omega \). We can conclude. \( \square \)

We now prove a regularity for general solutions of (3.2.1).

**Theorem 3.7.1** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set, \( f \in C(\Omega) \). Suppose \( u \in C^2(\Omega) \) satisfies (3.61) in \( \Omega \). If \( f \in C^{k-1}(\Omega) \) for some integer \( k \geq 2 \) then \( u \in C^k(\Omega) \). Moreover if \( f \) is smooth in \( \Omega \) then \( u \) is smooth in \( \Omega \).

**Proof.** Take an arbitrary bounded subset \( \Omega' \) of \( \Omega \) and let \( w = w_{f, \Omega} \) be the Newtonian potential of \( f \) in \( \Omega' \). By Lemma 3.7.1 \( w \in C^k \) and \( -\Delta w = f \) in \( \Omega' \). Now we set \( v = u - w \). Since \( u \in C^2 \), so is \( v \) and \( \Delta v = 0 \). It follows that \( v \) is smooth in \( \Omega' \). Therefore \( u = v + w \in C^k(\Omega') \). It is obvious that \( f \) is smooth in \( \Omega \) then \( w \) and hence \( u \) are smooth in \( \Omega \). We can conclude. \( \square \)

Theorem 3.7.1 is an optimal result concerning the smoothness.

Even though \( \Delta u \) is just one particular combination of second derivatives of \( u \), the smoothness of \( \Delta u \) implies the smoothness of all second derivatives.

Next, we solve the Dirichlet problem for the Poisson equation.

**Theorem 3.7.2** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set satisfying the exterior sphere condition at every boundary point, \( f \in C^1(\Omega) \cap L^\infty(\Omega) \) and \( \varphi \in C(\partial \Omega) \). Then there exists a solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) of the Dirichlet Problem

\[
\begin{align*}
-\Delta u &= f \quad \text{in } \Omega \\
\varphi &= u \quad \text{in } \partial \Omega.
\end{align*}
\]

Moreover if \( f \) is smooth in \( \Omega \) then \( u \) is smooth in \( \Omega \).

**Proof.** Let \( w \) be the Newtonian potential of \( f \) in \( \Omega \). By Lemma 3.7.1 for \( k = 2 \) \( w \in C^2(\Omega) \cap C(\overline{\Omega}) \) and \( \Delta w = f \) in \( \Omega \). Now consider the Dirichlet problem

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\[
\begin{aligned}
\Delta v &= 0 \quad \text{in } \Omega \\
v &= \varphi - w \quad \text{in } \partial \Omega.
\end{aligned}
\]  

(3.63)

Corollary 3.6.2 implies the existence of a solution \( v \in C^\infty(\Omega) \cap C(\overline{\Omega}) \) of (3.63). Then \( u = v + w \) is the desired solution of (3.62). If \( f \) is smooth in \( \Omega \) then \( u \) is smooth in \( \Omega \) by Theorem 3.7.1 and we conclude. \( \square \)

It is natural to ask whether the equation (3.61) admits any \( C^2 \)-solutions if \( f \) is continuous. The answer turns out to be negative.

**Example 3.7.1** Let \( f : B(0,1) \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) be defined by \( f(0) = 0 \) and

\[
f(x) = \frac{x_2^2 - x_1^2}{|x|^2} \left[ \frac{2}{(-\log |x|)^{1/2}} + \frac{1}{4(-\log |x|)^{3/2}} \right]
\]

for \( x \neq 0 \). \( f \) is continuous in \( B(0,1) \). Define the function

\[
u(x) = \begin{cases} 
(x_1^2 - x_2^2)(-\log |x|)^{1/2} & x \in B(0,1) \setminus \{0\} \\
0 & x = 0
\end{cases}
\]

One can verify that

\[-\Delta u = f \quad \text{in } x \in B(0,1) \setminus \{0\}\]

(3.64)

and

\[
\lim_{x \to 0} u_{x_1x_1} = \infty.
\]

Hence \( u \) is not in \( C^2(B_1) \). Actually the equation (3.64) has not \( C^2 \) solutions.

Better adapted to the Poisson equation are Hölder spaces. The study of the elliptic differential equations in Hölder spaces is known as the Schauder theory. In its simplest form, it asserts that all second derivatives of \( u \) are Hölder continuous if \( \Delta u \) is, (see e.g [3]).
Exercise 3.7.1 Let $\Omega \subset \mathbb{R}^n$ be an open set of class $C^2$, then it satisfies the exterior sphere condition in every $\xi \in \partial \Omega$.

Solution.

Step 1. We show that if there is $c > 0$ such that $U = \bar{\Omega}^c = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t > \frac{c}{2}|x|^2\}$, then there exists $r > 0$ such that $B = B((0,r),r) \subseteq U$.

We observe that $(x,t) \in B$ if and only if $t \in (r - \sqrt{r^2 - |x|^2}, r + \sqrt{r^2 - |x|^2})$ and $|x| < r$. The inequality $t > \frac{c}{2}|x|^2$ is satisfied for $(x,t) \in B$ if $r - \sqrt{r^2 - |x|^2} > \frac{c}{2}|x|^2$ and this holds if we take $r < \frac{1}{c}$.

Step 2. In the general case let $\xi \in \partial \Omega$ be of class $C^2$. We choose an orthonormal coordinates system so that the point lies at 0 and the tangent hyperplane to $\partial \Omega$ at 0 is the hyperplane $t = 0$. By definition of point of class $C^2$ there exists an $\epsilon$-neighborhood $V$ of the origin on which $U$ is defined by $t > f(x)$ where $f$ is a $C^2$ function, $f(0) = f'(0) = 0$. The Hessian matrix of $f$ at 0 is symmetric and hence diagonalizable, with real eigenvalues $\lambda_1, \ldots, \lambda_n$. (These are the principal curvatures of $\partial \Omega$ at the origin). Choose any $c > \max\{\lambda_1, \ldots, \lambda_n\}$. Then there is $\delta > 0$ such that $p(x) = \frac{c}{2}|x|^2 \geq f(x)$ for $|x| < \delta$. Therefore by applying Step 1 if $r < \min\{\frac{1}{c}, \delta, \frac{c}{2}\}$ it holds:

$$B((0,r),r) \subseteq \{(x,t) : |x| < \delta, 0 < p(x) < t < \epsilon\} \subseteq U \cap V.$$

Moreover $B((0,r),r) \cap \partial \Omega = \{0\}$.

Note: If $\Omega$ is of class $C^\alpha$ with $0 < \alpha < 2$ then the exterior sphere condition may not be satisfied. If we take for instance the function $f(x) = |x|^{3/2}$ it is clear that a ball $B((0,r),r)$ cannot be contained in the epigraph $t > |x|^{3/2}$, (see Fig. 1 and Fig.2 ).
3.8 Energy Methods

In this section we illustrate some “energy methods”, which is to say techniques involving the $L^2$ norms of various expressions.

**a. Uniqueness.**

Consider first the boundary-value problem

$$\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = g & \text{in } \partial \Omega
\end{cases} \quad (3.65)$$

We set forth a simple alternative proof of the uniqueness. Assume $\Omega \subset \mathbb{R}^n$ bounded and $\partial \Omega$ is $C^1$.

**Theorem 3.8.1 (Uniqueness)** There exists at most one solution $u \in C^2(\Omega)$ of (3.65).

**Proof.** Assume $\hat{u}$ is another solution and set $w = u - \hat{u}$. Then $\Delta w = 0$ in $\Omega$ and so integration by parts shows

$$0 = -\int_{\Omega} w \Delta w \, dx = \int_{\Omega} |Dw|^2 \, dx.$$ 

Thus $Dw = 0$ in $\Omega$ and since $w = 0$ on $\partial \Omega$, we deduce $w = 0$ in $\Omega$. \qed

**b. Dirichlet’s principle**

Next we show that a solution of (3.65) can be characterized as the minimizer of an appropriate functional. For this, we define the energy functional

$$I[w] := \int_{\Omega} \frac{1}{2} |Dw|^2 - w f \, dx,$$
$w$ belonging to the admissible set

$$ \mathcal{A} := \{ w \in C^2(\bar{\Omega}) \mid w = g \text{ on } \partial \Omega \}. $$

**Theorem 3.8.2 (Dirichlet Principle)**

Assume $u \in C^2(\bar{\Omega})$ solves (3.65). Then

$$ I[u] = \min_{w \in \mathcal{A}} I[w]. \quad (3.66) $$

Conversely, if $u \in \mathcal{A}$ satisfies (3.66), then $u$ solves the problem (3.65).

In other words if $u \in \mathcal{A}$, the PDE $-\Delta u = f$ is equivalent to the statement that $u$ minimizes the energy $I[\cdot]$.

**Proof.** 1. Choose $w \in \mathcal{A}$. Then (3.65) implies

$$ 0 = \int_{\Omega} (-\Delta u - f)(u - w) \, dx. $$

An integration by parts yields

$$ 0 = \int_{\Omega} Du \cdot D(u - w) - f(u - w) \, dx, $$

and there is no boundary term since $u - w = 0$ on $\partial \Omega$. Hence

$$ \int_{\Omega} |Du|^2 - uf \, dx = \int_{\Omega} Du \cdot Dw - wf \, dx $$

$$ \int_{\Omega} |Du|^2 \, dx - uf \, dx \leq \int_{\Omega} \frac{1}{2} |Dw|^2 + \int_{\Omega} \frac{1}{2} |Du|^2 - wf \, dx. $$

Rearranging we conclude that

$$ I[u] \leq I[w] \quad (w \in \mathcal{A}). \quad (3.67) $$

Since $u \in \mathcal{A}$ then (3.66) follows from (3.67).
2. Now conversely, suppose (3.66) holds. Fix any $v \in C^2_c(\Omega)$ and write

$$i(\tau) := I[u + \tau v], \ (\tau \in \mathbb{R}).$$

Since $u + \tau v \in \mathcal{A}$ for each $\tau$, the scalar function $i(\cdot)$ has a minimum at zero, and thus

$$i'(0) = 0,$$

provided this derivative exists. But

$$i(\tau) = \int_{\Omega} \frac{1}{2} |Du + \tau Dv|^2 - (u + \tau v) f dx = \int_{\Omega} \frac{1}{2} |Du|^2 + \tau Du \cdot Dv + \frac{1}{2} |\tau Dv|^2 - (u + \tau v) f dx.$$

Consequently

$$0 = i'(0) = \int_{\Omega} Du \cdot Dv - vf = \int_{\Omega} (-\Delta u - f)v dx.$$

This identity is valid for each function $v \in C^2_c(\Omega)$ and so $-\Delta u = f$ in $\Omega$. $\square$
Chapter 4

Heat equation

4.1 Motivations

Let us begin with some motivation. The heat equation is the simplest physical model for the spread of mass or temperature by diffusion. Think for example of the transport of smoke in still air. We think of smoke particles as being much larger than the air molecules, yet sufficiently small that they feel the random kicks of collisions with many atoms. The average displacement of any particle is zero, yet there are fluctuations about this mean that increase with time. The macroscopic manifestation of these fluctuations is diffusion. This microscopic picture of diffusion underlies a classical theory of diffusion derived by Euler (though the name usually attach to the heat equation is that of Fourier).

There are two steps in the modeling process. Let \( u(x, t) \) denote the density of smoke at a position in space. If the smoke only gets transported (and not created or destroyed) then we may use conservation of mass to write

\[
  u_t + \text{div}(F) = 0
\]

where \( F \) denotes the flux of particles at any point in space. Since there are two unknowns, we need another equation. This is a constitutive relation usually called Fick’s law. The flux is related to the density (or temperature) \( u \) through \( F = -kD u \). The experimental basis for Fick’s law is that heat flows in the direction of steepest descent.
The nonhomogeneous heat equation or diffusion equation is

\[ u_t - k \Delta u = f \text{ in } \Omega \times (0, +\infty), \tag{4.1} \]

where \( u = u(x, t) \) is the unknown, \( k > 0, \Omega \subset \mathbb{R}^n \) and \( f \in C(\Omega \times (0, +\infty)) \) is a given function. If \( f \equiv 0 \) then the equation (4.1) is called homogeneous.

A solution of (4.1) is a function \( u: \Omega \times (0, +\infty) \to \mathbb{R} \) which is of class \( C^2 \) with respect the variable \( x \in \Omega \) and of class \( C^1 \) with respect to \( t \in (0, +\infty) \) which satisfies the equation (4.1) pointwise.

Since the heat equation can be considered as the evolution equation associated to the Poisson equation, a good strategy to study the heat equation is to reproduce the arguments and the results obtained for the Poisson’s equation.

It will not be restrictive to study the equation (4.1) with \( k = 1 \). Indeed \( u(x, t) \) is a solution of (4.1) if and only if \( v(x, t) = u(x, \frac{t}{k}) \) is a solution of

\[ u_t - \Delta u = \frac{1}{k} f(x, \frac{t}{k}) \text{ in } \Omega \times (0, +\infty), \tag{4.2} \]

### 4.2 Derivation of the Fundamental solution

We consider the Heat Equation

\[ u_t - \Delta u = 0 \text{ in } \Omega \times (0, +\infty), \tag{4.3} \]

We observe that the equation (4.3) is invariant by dilation with respect to \( x \) and \( t \) but with different scaling. Namely for every \( \lambda > 0, u^\lambda(x, t) = u(\lambda x, \lambda^2 x) \) is still a solution of (4.3). We observe also that

\[
\int_{\mathbb{R}^n} u^\lambda(x, t) dx = \int_{\mathbb{R}^n} u(\lambda x, \lambda^2 x) dx
\]

\[ = \lambda^{-n} \int_{\mathbb{R}^n} u(y, \lambda^2 t) dy. \]

If \( \lambda = t^{-1/2} \) then

\[
\frac{1}{t^{n/2}} \int_{\mathbb{R}^n} u\left( \frac{x}{\sqrt{t}}, 1 \right) dx = \int_{\mathbb{R}^n} u(y, 1) dy.
\]
We look for a radial symmetric solution with respect to $x$ which satisfies the normalizing condition

$$
\int_{\mathbb{R}^n} u(x, t) \, dx = 1, \quad \forall t > 0.
$$

(4.4)

Thus it should be

$$
\frac{1}{t^{n/2}} \int_{\mathbb{R}^n} u\left(\frac{x}{\sqrt{t}}, 1\right) \, dx = \int_{\mathbb{R}^n} u(x, t) \, dx, \quad \forall t > 0.
$$

The condition (4.4) and the radial symmetry suggest that we look for a solution of the form

$$
u(x, t) = \frac{1}{t^{n/2}} v\left(\frac{|x|}{\sqrt{t}}\right).
$$

We construct the equation for $v$. We compute

$$
u_t = -\frac{n}{2} \frac{v}{t^{n/2+1}} - \frac{1}{2} \frac{|x| v'}{t^{n/2+3/2}},
$$

$$
u_{x_i x_i} = \frac{1}{t^{n/2+1}} \left( v' - \frac{x_i^2 v'}{|x|^3} + \frac{x_i^2 v''}{|x|^2 \sqrt{t}} \right).
$$

Thus $u$ verifies the equation 4.3 if and only if $v$ satisfies

$$
v'' + \left( \frac{n - 1}{s} + \frac{s}{2} \right) v' + \frac{n}{2} v = 0 \text{ in } (0, +\infty)
$$

(4.5)

the normalizing condition with respect the function $v$ is given by

$$
\int_{\mathbb{R}^n} v(|x|) \, dx = 1.
$$

We can write the equation (4.5) as follows

$$
(2s^{n-1} v' + s^n v)' = 0
$$

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thus
\[ 2s^{n-1}v' + s^n v = C \]
for some constant \( C \in \mathbb{R} \). The easiest choice is to take \( C = 0 \).

Actually this is a necessary choice in order that the normalizing condition is satisfied (To check as exercise). It remains to solve the equation
\[ v' = -\frac{s}{2}v \]
whose general solution is
\[ v(s) = Ae^{-s^2/4}. \]

The constant \( A \) is determined by imposing the normalizing condition
\[ 1 = A \int_{\mathbb{R}^n} e^{-|x|^2/4} dx = A(2\sqrt{\pi})^{n/2}, \]
and thus \( A = \frac{1}{(4\pi)^{n/2}} \).

To conclude we find
\[ u(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad \forall (x,t) \in \mathbb{R}^n \times (0, +\infty). \]

We define the fundamental solution of the heat equation the function
\[
\Phi(x,t) = \begin{cases} 
\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{for } x \in \mathbb{R}^n, t > 0 \\
0 & \text{for } x \in \mathbb{R}^n, t < 0.
\end{cases}
\]

We observe that \( \Phi \) has a singularity in \((0,0)\).

We list the main propierties of the fundamental solution:

1. \( \Phi \in C^\infty \) on its domain and it satisfies the heat equation (4.3)
2. \( \Phi \) satisfies the normalization condition \( \int_{\mathbb{R}^n} \Phi(x,t) dx = 1 \) for every \( t > 0 \);
3. \( \Phi \) is radially symmetric with respect to \( x \);
4. \( \Phi > 0 \) in \( \mathbb{R}^n \times (0, +\infty) \);

5. \( \Phi(\cdot, t) \to 0 \) uniformly in \( \mathbb{R}^n \) as \( t \to +\infty \);

6. for every \( \varepsilon > 0 \) we have \( \Phi(\cdot, t) \to 0 \) uniformly in \( \mathbb{R}^n \setminus B(0, \varepsilon) \) as \( t \to 0^+ \);

4.3 Homogenous Cauchy Problem

We study the Cauchy problem

\[
\begin{aligned}
\{ u_t - \Delta u &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0, +\infty) \\
 u(x, 0) &= g(x) \quad \text{in} \quad \mathbb{R}^n 
\end{aligned}
\tag{4.6}
\]

where the function \( g: \mathbb{R}^n \to \mathbb{R} \) is given.

\[\text{Theorem 4.3.1} \quad \text{Given a function } g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \text{ and let } y \text{ be defined by}
\]

\[
u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t)g(y)dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x - y|^2}{4t}}g(y)dy
\tag{4.7}
\]

for \( x \in \mathbb{R}^n, t > 0 \). Then

(i) \( u \in C^\infty(\mathbb{R}^n \times (0, +\infty)) \);

(ii) \( u \) verifies the equation (4.3);

(iii) \( \lim_{(x, t) \to (x_0, 0^+)} u(x, t) = g(x_0) \) for every \( x_0 \in \mathbb{R}^n \);

(iv) \( \|u(\cdot, t)\|_\infty \leq \|g\|_\infty \) for every \( t > 0 \).

**Proof.** 1. The function \( \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} \) is infinitely differentiable and the partial derivatives are in \( L^1(\mathbb{R}^n \times [\delta, +\infty)) \), for every \( \delta > 0 \).

---

\( ^{(1)} \) One may apply the following result on the convolution of two functions: Let \( f \in C^k(\mathbb{R}^n) \) and \( g: \mathbb{R}^n \to \mathbb{R} \) be measurable such that \( f \ast g \) exists in a neighborhood of \( x_0 \). Suppose that there exists a neighborhood \( U \) of \( x_0 \) and for every \( |\alpha| \leq k \) some functions \( \gamma_\alpha \in L^1(\mathbb{R}^n) \) satisfying \( |D^\alpha f(x - y)g(y)| \leq \gamma_\alpha(y), \forall x \in U \) and for a.e. \( y \in \mathbb{R}^n \). Then \( D^\alpha(f \ast g)(x_0) = (D^\alpha f) \ast g(x_0) \).
This implies that \( u \in C^\infty(\mathbb{R}^n \times (0, +\infty)) \) and
\[
 u_t - \Delta u = \int_{\mathbb{R}^n} [\Phi_t(x - y, t) - \Delta \Phi(x - y, t)] g(y) dy = 0
\]

2. Fix \( x_0 \in \mathbb{R}^n, \varepsilon > 0 \). Choose \( \delta > 0 \) such that
\[
 |g(y) - g(x_0)| < \frac{\varepsilon}{2} \quad \text{if} \quad |y - x_0| < \delta.
\]
Consider \( |x - x_0| < \frac{\delta}{2} \).

\[
 |u(x, t) - g(x_0)| = |\int_{\mathbb{R}^n} \Phi(x - y, t)(g(y) - g(x_0)) dy| 
\]
\[
 \leq \int_{B(x_0, \delta)} |\Phi(x - y, t)(g(y) - g(x_0))| dy + \int_{B^c(x_0, \delta)} |\Phi(x - y, t)(g(y) - g(x_0))| dy 
\]
\[
 \leq \frac{\varepsilon}{2} + 2||g||_{\infty} \int_{|y - x_0| \geq \delta} \left( \frac{1}{4\pi t} \right)^{n/2} e^{-\frac{|x - y|^2}{4t}} dy 
\]
\[
 \leq \frac{\varepsilon}{2} + 2||g||_{\infty} \int_{|y - x_0| \geq \delta} \left( \frac{1}{4\pi t} \right)^{n/2} e^{-\frac{|x_0 - y|^2}{4t}} dy 
\]
by setting \( z = \frac{y - x_0}{4\sqrt{t}} \)
\[
 \leq \frac{\varepsilon}{2} + 2||g||_{\infty} \left( \frac{4}{\pi} \right)^{n/2} \int_{|z| \geq \frac{\delta}{4\sqrt{t}}} e^{-|z|^2} dz. 
\]

Since \( \int_{\mathbb{R}^n} e^{-|z|^2} dz < +\infty \), there exists \( t_0 > 0 \) such that for all \( t < t_0 \)
\[
 2||g||_{\infty} \left( \frac{4}{\pi} \right)^{n/2} \int_{|z| \geq \frac{\delta}{4\sqrt{t}}} e^{-|z|^2} dz \leq \frac{\varepsilon}{2}. 
\]
Finally if \( |(x, t) - (x_0, 0)| \leq \min(\delta/2, t_0) \), then
\[
 |u(x, t) - g(x_0)| \leq \varepsilon. 
\]

3. For every \( (x, t) \in \mathbb{R}^n \times (0, +\infty) \) we have
\[
 |u(x, t)| \leq \int_{\mathbb{R}^n} |\Phi(x - y, t)g(y)| dy \leq ||g||_{\infty} \int_{\mathbb{R}^n} |\Phi(x - y, t)| dy 
\]
\[
 = ||g||_{\infty} 
\]
since Φ is positive and by the normalization condition.

Remark 4.3.1 1. The properties (i), (ii), (iv) of Theorem 4.3.1 still hold under the assumption that $g \in L^\infty$.

2. If $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $g \geq 0$ and $g \not\equiv 0$ then $u(x, t)$ given by (4.7) is in fact positive for all $x \in \mathbb{R}^n$ and $t > 0$. We interpret this fact by saying that the heat equation forces infinite propagation speed for disturbances.

3. Even if the initial data $g$ is discontinuous, the solution of the Cauchy problem given by Theorem 4.3.1 is $C^\infty$ for all $t > 0$ We say that the heat equation has a regularizing effect.

4. If $g \in L^1(\mathbb{R}^n)$ then $u(\cdot, t) \to 0$ uniformly in $\mathbb{R}^n$ as $t \to +\infty$. Indeed

$$|u(x, t)| \leq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} |g(y)|dy$$

and therefore

$$\sup_{x \in \mathbb{R}^n} |u(x, t)| \leq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |g(y)|dy.$$ 

We observe that by Fubini Theorem and the fact that

$$\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} dx = 1$$

we have

$$\int_{\mathbb{R}^n} u(x, t) dx = \int_{\mathbb{R}^n} g(x) dx, \ \forall t > 0.$$ 

The conservation of the integral of $u(x, t)$ with respect to $x$ means that the total amount of heat is conserved.
Exercise 4.3.1 For any $x \geq 0$ define the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.$$ 

Consider the function

$$g(x) = \begin{cases} 
1 & |x| \leq 1/2 \\
0 & \text{otherwise}
\end{cases}$$

Show that the solution of (4.6) is given by

$$u(x, t) = -\frac{1}{2} \text{erf} \left( \frac{x - 1/2}{2\sqrt{t}} \right) + \frac{1}{2} \text{erf} \left( \frac{x + 1/2}{2\sqrt{t}} \right)$$

for all $(x, t) \in \mathbb{R}^n \times (0, +\infty)$. Which is the limit of $u(x, t)$ as $t \to +\infty$?

4.3.1 Tychonov’s counterexample

In general the solution of the Cauchy Problem

$$\begin{cases} 
    u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\
    u(x, 0) = f(x) & \text{in } \mathbb{R}^n
\end{cases}$$

with $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is not unique in the class of function $C(\mathbb{R}^n \times [0, +\infty)) \cap C^2(\mathbb{R}^n \times (0, +\infty))$.

In dimension 1, let us consider the problem

$$\begin{cases} 
    u_t - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\
    u(x, 0) = 0 & \text{in } \mathbb{R}
\end{cases}$$

(4.9)

The function $u \equiv 0$ is clearly a solution. We are going to construct a nontrivial solution of (4.9).

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be the function

$$\varphi(z) = \begin{cases} 
    e^{-\frac{|z|^2}{4}} & z \neq 0 \\
    0 & z = 0
\end{cases}$$

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The function $\varphi$ is holomorphic in $\mathbb{C} \setminus \{0\}$. Moreover the function $t \to \varphi(t)$, $t \in \mathbb{R}$ is of class $C^\infty(\mathbb{R})$ and $\varphi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Let us consider the series of functions

$$u(x, t) = \sum_{n=0}^{\infty} \varphi^{(n)}(t) \frac{x^{2n}}{(2n)!}, \quad t \geq 0, \quad x \in \mathbb{R}^n.$$ 

Claim 1. The sum defining $u$ and the series of the derivatives of any order converge uniformly on any set of the form $[-R, R] \times [T, +\infty)$.

Claim 2. $u$ is a continuous function up to the boundary in the half-space $t \geq 0$.

We observe that from the second claim it follows that $u$ attains the initial datum 0 at the time $t = 0$. By the first claim we can interchange sum and partial derivatives. Then we can compute:

$$u_{xx}(x, t) = \sum_{n=1}^{\infty} \varphi^{(n)}(t) \frac{x^{2n-2}}{(2n-2)!} = \sum_{n=0}^{\infty} \varphi^{(n+1)}(t) \frac{x^{2n}}{(2n)!} = \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \varphi^{(n)}(t) \frac{x^{2n}}{(2n)!} = u_t(x, t).$$

Proof of Claim 1 For fixed $t > 0$, by the Cauchy formula for holomorphic functions we obtain

$$\varphi^{(n)}(t) = \frac{n!}{2\pi i} \int_{|z-t| = \frac{t}{2}} \frac{\varphi(z)}{(z-t)^{n+1}} \frac{dz}{z-t}.$$ 

On the circle $|z-t| = \frac{t}{2}$ we have $|\varphi(z)| \leq e^{-R(1/z^2)} \leq e^{-4/t^2}$. Thus

$$|\varphi^{(n)}(t)| \leq \frac{n!}{2\pi} \int_{0}^{2\pi} \frac{e^{-4/t^2}}{(t/2)^{n+1}} \frac{t}{2} \frac{d\theta}{2} = n! 2^n e^{-4/t^2} \frac{t}{t^n}.$$ 

The following inequality holds for every $n \in \mathbb{N}$:

$$\frac{n! 2^n}{(2n)!} \leq \frac{1}{n!}$$
Hence we get

\[
|u(x, t)| \leq \sum_{n=0}^{\infty} |\varphi^{(n)}(t)| \frac{|x|^{2n}}{(2n)!} \leq \sum_{n=0}^{\infty} n!2^n e^{-4/t^2} |x|^{2n} \frac{t^n}{(2n)!} \\
\leq e^{-4/t^2} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{|x|^2}{t} \right)^n = e^{-4/t^2 + |x|^2/t},
\]

where the last term converges uniformly for \( t \geq T > 0 \) and \( |x| \leq R < +\infty \).

By **Weierstrass’ criterion**, the sum defining \( u \) converges uniformly on the same set. In particular we get

\[
limit_{t \to 0} |u(x, t)| \leq \lim_{t \to 0} e^{-4/t^2 + |x|^2/t} = 0,
\]

with uniform convergence for \( |x| \leq R \). This proves **claim 2**.

The study of convergence of the series of derivatives is analogous and is left as an exercises to the reader. \( \square \)

### 4.4 Nonhomogeneous Cauchy Problem

We consider now the problem

\[
\begin{aligned}
&u_t - \Delta u = f(x, t) \quad \text{in } \mathbb{R}^n \times (0, +\infty) \\
u(x, 0) = 0 \quad \text{in } \mathbb{R}^n
\end{aligned}
\]  

(4.10)

where the function \( f: (\mathbb{R}^n \times (0, +\infty)) \to \mathbb{R} \) is given.

We observe that for every \( y \in \mathbb{R}^n \) and \( 0 < s < t \) the function \((x, t) \mapsto \Phi(x - y, t - s)\) is a solution of the heat equation.

For fixed \( s \), the function

\[
u(x, t; s) = \int_{\mathbb{R}^n} \Phi(x - y, t - s)f(y, s)dy
\]

solves

\[
\begin{aligned}
&u_t(\cdot; s) - \Delta(\cdot, s) = 0 \quad \text{in } \mathbb{R}^n \times (s, +\infty) \\
u(\cdot, s; s) = f(\cdot, s) \quad \text{in } \mathbb{R}^n \times \{t = s\}
\end{aligned}
\]  

(4.11)
which is just an initial-value problem with the starting time \( t = 0 \) replaced by \( t = s \) and \( g \) replaced by \( f(\cdot, s) \). Duhamel’s principle asserts that we can build a solution of (4.10) by integrating \( u(x, t; s) \) with respect to \( s \). The idea is to consider

\[
u(x, t) = \int_0^t u(x, t; s) ds \quad (x \in \mathbb{R}^n, \ t \geq 0).
\]

Rewriting we have

\[
u(x, t) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) dy ds,
\]

for \( x \in \mathbb{R}^n, t > 0 \).

\[\text{Theorem 4.4.1 (Solution of nonhomogeneous problem)} \]

Assume that \( f \in C^{2,1}(\mathbb{R}^n \times (0, +\infty)) \) with compact support and define \( u \) by (4.12). Then

(i) \( u \in C^{2,1}(\mathbb{R}^n \times (0, +\infty)) \);

(ii) \( u_t(x, t) - \Delta u(x, t) = f(x, t) \) in \( \mathbb{R}^n \times (0, +\infty) \);

(iii) \( \lim_{(x,t) \to (x_0,0+)} u(x, t) = 0 \) for each point \( x_0 \in \mathbb{R}^n \).

\[\text{Proof.} \quad \text{Since } \Phi \text{ has a singularity we cannot directly differentiate under the integral sign. We first make a change of variable:}
\]

\[
u(x, t) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi s)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4s}} f(x-y, t-s) dy ds.
\]

As \( f \in C^{2,1}(\mathbb{R}^n \times (0, +\infty)) \) with compact support and \( \Phi \) is smooth near \( s = t > 0 \) we get

\[
u_t(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) f_t(x-y, t-s) dy ds + \int_{\mathbb{R}^n} \Phi(y, t) f(x-y, 0) dy,
\]
and 
\[ u_{x_ix_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)f_{x_ix_j}(x-y,t-s)dyds. \]

Therefore \( u_t, u_{x_ix_j}, u, u_{x_i} \in C(\mathbb{R}^n \times (0, +\infty)) \). (To check!)

\[
\begin{align*}
  u_t(x,t) - \Delta u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)(f_t - \Delta f)(x-y,t-s)dyds + \int_{\mathbb{R}^n} \Phi(y,t)f(x-y,0)dy \\
  &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)(-f_s - \Delta f)(x-y,t-s)dyds + \int_{\mathbb{R}^n} \Phi(y,t)f(x-y,0)dy \\
  &= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)(-f_s - \Delta f)(x-y,t-s)dyds \\
  &\quad + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)f(x-y,0)dy \\
  &\quad \quad =: J_\varepsilon \\
  &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)(-f_s - \Delta f)(x-y,t-s)dyds \\
  &\quad + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)f(x-y,0)dy \\
  &\quad \quad =: I_\varepsilon \\
  &= \int_{\mathbb{R}^n} \Phi(y,t)f(x-y,0)dy \\
  &\quad \quad =: K.
\end{align*}
\]

We estimate the three last terms:
\[
|J_\varepsilon| \leq (\|f_t\|_{L^\infty} + \|D^2f\|_{L^\infty}) \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)dyds \leq C\varepsilon.
\]

By estimating \( I_\varepsilon \) we perform an integration by parts:

\[
I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,\varepsilon)f(x-y,0)dy \\
\quad - \int_{\mathbb{R}^n} \Phi(y,t)f(x-y,0)dy.
\]

Hence
\[
u_t(x,t) - \Delta u(x,t) = J_\varepsilon + \int_{\mathbb{R}^n} \Phi(y,\varepsilon)f(x-y,0)dy.
\]
By letting $\varepsilon \to 0$ we get
\[ u_t(x,t) - \Delta u(x,t) = f(x,t) \]

We finally note that
\[ |u(x,t)| \leq t \|f\|_{L^\infty} \]

Therefore $\lim_{t \to 0} u(x,t) = 0$ and the convergence is uniform in $x \in \mathbb{R}^n$. □

\begin{remark}
We can combine Theorems 4.3.1 and 4.4.1 to discover that
\[ u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)dyds + \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy \]

is, under the hypotheses on $g$ and $f$ as above, a solution of
\[
\begin{cases}
  u_t - \Delta u = f(x,t) & \text{in } \mathbb{R}^n \times (0,+\infty) \\
  u(x,0) = g(x) & \text{in } \mathbb{R}^n
\end{cases}
\]
(4.13)
\end{remark}

### 4.5 Maximum Principle

Given a domain $\Omega \subset \mathbb{R}^n$, for every $T > 0$ we introduce the following subsets of $\mathbb{R}^n \times [0, +\infty)$
\[
\Omega_T = \Omega \times (0,T] = \{(x,t) | x \in \Omega, 0 < t \leq T\}
\]
\[
\Gamma_T = (\Omega \times \{0\}) \cup (\partial \Omega \times [0,T]).
\]
Thus $\Omega_T$ is a cylinder in $\mathbb{R}^{n+1}$, which is called \textit{parabolic cylinder}. The set $\Gamma_T$ is called \textit{parabolic boundary}.

We interpret $\Omega_T$ as being the \textit{parabolic interior} of $\overline{\Omega} \times [0,T]$; note carefully that $\Omega_T$ includes the top $\Omega \times \{t = T\}$. The parabolic boundary $\Gamma_T$ comprises the bottom and vertical sides of $\Omega \times [0,T]$ but not the top.
Theorem 4.5.1 (Weak Maximum Principle) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u \in C^{2,1}(\Omega_T) \cap C(\Omega_T \cup \Gamma_T)$ be a solution of the heat equation in $\Omega_T$. Then

$$\max_{\Omega_T \cup \Gamma_T} u = \max_{\Gamma_T} u.$$ 

Proof. For every $\varepsilon > 0$ we define

$$u^\varepsilon(x, t) = u(x, t) + \varepsilon|x|^2, \quad \forall (x, t) \in \Omega_T \cup \Gamma_T.$$ 

We have

$$u_t^\varepsilon - \Delta u^\varepsilon = u_t - \Delta u - 2n\varepsilon < 0, \quad \text{in } \Omega_T.$$ 

This implies that the maximum of $u^\varepsilon$ cannot be achieved in $\Omega_T$. Indeed if $(x, t) \in \Omega_T$ would be a maximum point for $u^\varepsilon$ then $u_t^\varepsilon(x, t) \geq 0$ and $D^2u^\varepsilon(x, t) \leq 0$. In particular $\Delta u^\varepsilon(x, t) \leq 0$ and thus $u_t^\varepsilon(x, t) - \Delta u^\varepsilon(x, t) \geq 0$ which is a contradiction. Thus

$$\max_{\Omega_T \cup \Gamma_T} u^\varepsilon = \max_{\Gamma_T} u^\varepsilon.$$ 

It follows that there exists $(x^\varepsilon, t^\varepsilon) \in \Gamma_T$ such that

$$u^\varepsilon(x, t) \leq u^\varepsilon(x^\varepsilon, t^\varepsilon), \quad \forall (x, t) \in \Omega_T \cup \Gamma_T.$$ 

Then

$$\max_{\Omega_T \cup \Gamma_T} u \leq u(x^\varepsilon, t^\varepsilon) + \varepsilon|x^\varepsilon|^2.$$ 

Since $\Gamma_T$ is compact, there exists a sequence $\varepsilon_k \to 0$ and a point $(\bar{x}, \bar{t})$ such that $(x^{\varepsilon_k}, t^{\varepsilon_k}) \to (\bar{x}, \bar{t})$. Hence

$$\max_{\Omega_T \cup \Gamma_T} u \leq u(\bar{x}, \bar{t}) \leq \max_{\Gamma_T} u.$$ 

Obviously $\max_{\Gamma_T} u \leq \max_{\Omega_T \cup \Gamma_T} u$ and this conclude the proof. \qed
### 4.5.1 Uniqueness of bounded solutions

**Theorem 4.5.2** Let \( g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). Then the Cauchy problem

\[
\begin{align*}
   u_t - \Delta u &= 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty) \\
   u(x, 0) &= g(x) \quad \text{in } \mathbb{R}^n
\end{align*}
\]  

(4.14)

has a unique bounded solution.

**Proof.**

**Existence:** it has already been proved in Theorem 4.3.1.

**Uniqueness:**

**Step 1.** We first show that if \( u \) is a bounded solution of (4.14) then

\[
\sup_{\mathbb{R}^n \times (0, +\infty)} u(x, t) = \sup_{\mathbb{R}^n} g(x) .
\]

We set \( M = \sup_{\mathbb{R}^n \times (0, +\infty)} u(x, t) \) and \( M_0 = \sup_{\mathbb{R}^n} g(x) \) and we define

\[
u^\varepsilon(x, t) = u(x, t) + \varepsilon(-2nt - |x|^2), \quad \forall (x, t) \in \mathbb{R}^n \times [0, +\infty).
\]

We have \( u_t^\varepsilon - \Delta u^\varepsilon = 0 \) and

\[
u^\varepsilon(x, 0) \leq g(x) - \varepsilon|x|^2 \leq M_0, \quad \forall x \in \mathbb{R}^n.
\]

Moreover for \( R > 0 \) large enough we have

\[
\sup_{(x, t) \in \partial B(0, R) \times [0, R]} u^\varepsilon(x, t) \leq \sup_{(x, t) \in \partial B(0, R) \times [0, R]} u(x, t) - \varepsilon R^2
\]

\[
\leq M - \varepsilon R^2 < M_0.
\]

Then by the maximum principle

\[
\sup_{(x, t) \in B(0, R) \times (0, R]} u^\varepsilon(x, t) \leq M_0
\]

for all \( R > 0 \) large enough and thus

\[
u^\varepsilon(x, t) \leq M_0, \quad \forall (x, t) \mathbb{R}^n \times [0, +\infty).
\]
By letting $\varepsilon \to 0$ we get the desired estimate.

**Step 2.** Suppose now that $u$ and $\tilde{u}$ be two bounded solution of (4.14). Then $v = u - \tilde{u}$ is a bounded solution of the Cauchy problem

\[
\begin{cases}
u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\
u(x, 0) = 0 & \text{in } \mathbb{R}^n
\end{cases}
\] (4.15)

By what it has been proved in Step 1 we get $u \leq \tilde{u}$ in $\mathbb{R}^n \times (0, +\infty)$. By arguing in the same way for the function $w = \tilde{u} - u$ we find that $u \leq \tilde{u}$ in $\mathbb{R}^n \times (0, +\infty)$. Therefore $u$ and $\tilde{u}$ coincide.

---

**Remark 4.5.1** The uniqueness for the Cauchy problem holds in the class of bounded functions. Actually one can prove the uniqueness in the class of unbounded functions satisfying the growth condition $|u(x,t)| \leq Ae^{a|x|^2}$, with $A, a > 0$. This growth condition is necessary. For instance the function (Tychonov’s Example)

\[u(x, t) = \sum_{k=0}^{+\infty} \frac{x^{2k}}{(2k)!} \frac{d^k}{dt^k} e^{-1/t^2}, \quad \forall (x, t) \in \mathbb{R} \times (0, +\infty)\]

is solution of the heat equation and $u(x, t) \to 0$ as $t \to 0^+$. This solution diverges as $|x| \to +\infty$ much more rapidly as $e^{a|x|^2}$ for every $a > 0$. Since $v(x, t) = u(x + a, t)$ is a solution for every $a \in \mathbb{R}$ we have that the Cauchy problem

\[
\begin{cases}
u_t - \Delta u = 0 & \text{in } \mathbb{R} \times (0, +\infty) \\
u(x, 0) = 0 & \text{in } \mathbb{R}
\end{cases}
\] (4.16)

has infinite solutions.

---

**4.6 Mean-value formula**

We want to derive a kind of analogous to the mean-value property for harmonic functions. Let us observe that for fixed $x$ the spheres $\partial B(x, r)$
are the level sets of fundamental solution $\Phi(x - y)$ for Laplace’s equation. This suggests that for fixed $(x, t)$ the level sets of fundamental solution $\Phi(x - y, t - s)$ for the heat equation may be relevant.

**Definition 4.6.1** For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r > 0$, we define

$$E(x, t, r) := \{(y, s) \in \mathbb{R}^{n+1} \mid s \leq t, \Phi(x - y, t - s) \geq \frac{1}{r^n}\}$$

This is a region in space-time, the boundary of which is a level set of $\Phi(x - y, t - s)$. Note that the point $(x, t)$ is at the center of the top. $E(x, t; r)$ is sometimes called a heat ball.

**Theorem 4.6.1** \(\text{(A mean-value property for the heat equation)}\)

Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in C^{2,1}(\Omega_T)$ solve the heat equation. Then

$$u(x, t) = \frac{1}{4r^n} \int \int_{E(x, t; r)} u(y, s) \frac{|x - y|^2}{(t - s)^2} dy ds,$$

Formula 4.17 is a sort of analogue for the heat equation of the mean-value formulas for Laplace’s equation. Observe that the right hand side involves only $u(y, s)$ for times $s \leq t$. This is reasonable, as the value $u(x, t)$ should not depend upon future times.

**Proof.** Up to changing coordinates we may suppose that $(x, t) = (0, 0)$ . We write $E(r) = E(0, 0; r)$ We set

$$\phi(r) := \frac{1}{r^n} \int \int_{E(r)} u(y, s) \frac{|y|^2}{s^2} dy ds$$

$$= \int \int_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} dy ds.$$
We compute
\[
\phi'(r) = \int \int_{E(1)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^{2}}{s^2} + 2ru_s \frac{|y|^{2}}{s} dy ds \\
= \frac{1}{r^{n+1}} \int \int_{E(r)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^{2}}{s^2} + 2ru_s \frac{|y|^{2}}{s} dy ds \\
= \quad A + B
\]
Let us introduce the function
\[
\psi := -\frac{n}{2} \log(-4\pi s) + \frac{|y|^{2}}{4s} + n \log r, \tag{4.18}
\]
and observe \(\psi = 0\) on \(\partial E(r)\), since \(\Phi(y, -s) = r^{-n}\) on \(\partial E(r)\). We use (4.18) to write
\[
B = \frac{1}{r^{n+1}} \int \int_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i} dy ds \\
= -\frac{1}{r^{n+1}} \int \int_{E(r)} 4nu_s \psi + 4 \sum_{i=1}^{n} u_{y_i} y_i \psi dy ds;
\]
there is no boundary term since \(\psi = 0\) on \(\partial E(r)\). Integrating by parts with respect to \(s\), we discover that
\[
B = \frac{1}{r^{n+1}} \int \int_{E(r)} -4nu_s \psi + 4 \sum_{i=1}^{n} u_{y_i} y_i \psi_{y_i} dy ds \\
= \frac{1}{r^{n+1}} \int \int_{E(r)} -4nu_s \psi + 4 \sum_{i=1}^{n} u_{y_i} y_i (-\frac{n}{2s} - \frac{|y|^{2}}{4s^2}) dy ds \\
= \frac{1}{r^{n+1}} \int \int_{E(r)} -4nu_s \psi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dy ds - A
\]
Consequently, since \(u\) solves the heat equation,
\[
\phi'(r) = \frac{1}{r^{n+1}} \int \int_{E(r)} -4n \Delta u \psi - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dy ds \\
= \frac{1}{r^{n+1}} \int \int_{E(r)} 4nu_{y_i} \psi_{y_i} - \frac{2n}{s} \sum_{i=1}^{n} u_{y_i} y_i dy ds = 0
\]
Thus $\phi$ is constant and therefore

$$\phi(r) = \lim_{t \to 0} \phi(t) = u(0, 0) \lim_{t \to 0} \frac{1}{t^n} \int \int_{E(t)} \frac{|y|^2}{s^2} dy ds = 4u(0, 0).$$

Above we use the fact

$$\frac{1}{t^n} \int \int_{E(t)} \frac{|y|^2}{s^2} dy ds = 4 \quad \text{EXERCISE}$$

4.6.1 Strong Maximum Principle for the Heat Equation

**Theorem 4.6.2** Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. Assume that $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ is a solution of the Heat equation such that

$$u(x_0, t_0) = \max_{\Omega_T} u.$$ 

Then $u$ is constant in $\Omega_{t_0}$.

Similar assertions are valid with “min” replacing “max”.

**Remark 4.6.1** If $u$ attains its maximum (or minimum) at an interior point, then $u$ is constant at earlier times. This accords with our strong intuitive interpretation of the variable $t$ as denoting time: the solution will be constant on the interval $[0, t_0]$ provided the initial and the boundary conditions are constant. The solution may change at times $t > t_0$, provided the boundary conditions alter after $t_0$. The solution will however not respond to changes in boundary conditions until these changes happen.
1. Suppose there exists a point \((x_0, t_0) \in \Omega_T\) with \(u(x_0, t_0) = M =: \max_{\Omega_T} u\). Then for sufficiently small \(r > 0\), \(E(x_0, t_0; r) \subset \Omega_T\). We employ the mean-value property to deduce

\[
M = u(x_0, t_0) = \frac{1}{4r^n} \int \int_{E(x_0, t_0; r)} u(y, s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dy ds \leq M.
\]

Equality holds only if \(u\) is identically equal to \(M\) within \(E(x_0, t_0; r)\). Consequently \(u(y, s) = M\) for all \((y, s) \in E(x_0, t_0; r)\).

Draw any line segment \(L\) in \(\Omega\) connecting \((x_0, t_0)\) with some other points
(y_0, s_0) \in \Omega_T \text{ with } s < t. \text{ Consider}

\[ r_0 := \min \{ s \geq s : u(x, t) = M, \; \forall (x, t) \in L, s \leq t \leq t_0 \}. \]

Since \( u \) is continuous, the minimum is attained. Then \( u(z_0, r_0) = M \) for some \((z_0, r_0) \in L \cap \Omega_T\) and so \( u \equiv M \) on \( E(z_0, r_0; r) \) for all sufficiently small \( r > 0 \). Since \( E(z_0, r_0; r) \) contains \( L \cap \{ r_0 - \sigma \leq t \leq r_0 \} \) for some small \( \sigma > 0 \), we have a contradiction. Thus \( r_0 = s_0 \) and \( u \equiv M \) on \( L \).

2. Now fix a point \((x, t) \in \Omega_{t_0}\). There exist points \( \{x_0, x_1, \ldots, x_m = x\} \) such that the line segments in \( \mathbb{R}^m \) connecting \( x_{i-1} \) to \( x_i \) lie in \( \Omega \) for every \( i \). Select times \( t_0 > t_1 > \ldots t_m = t \). Then the line segments in \( \mathbb{R}^{n+1} \) connecting \((x_{i-1}, t_{i-1})\) to \((x_i, t_i)\) lie in \( \Omega_T \). According to Step 1, \( u \equiv M \) on each segment and so \( u(x, t) = M \).

3. By continuity of the solution \( u(x, t_0) = M \) for all \( x \in \overline{\Omega} \). \( \square \)

4.6.2 Regularity

Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and \( T > 0 \). We next show that solutions of the homogeneous heat equation in \( \Omega_T \) are smooth.

**Theorem 4.6.3** Suppose \( u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T}) \) is a solution of the heat equation in \( \Omega_T \). Then \( u \in C^\infty(\Omega_T) \).

For the proof we refer the reader to Theorem 8 Section 2.3.3 in [2].

Given \( r > 0 \) and \((x, t) \in \mathbb{R}^{n+1}\), we define the closed parabolic cylinder as follows:

\[ C_r(x, t) := \{(y, s) : |y - x|^2 \leq r, \; t - r^2 \leq s \leq t\} \]
Theorem 4.6.4. For each pair of nonnegative integer numbers \( \ell, k \) there is a positive constant \( C_{\ell,k} \) such that

\[
\max_{C_{r/2}(x,t)} |D_{x}^\ell D_{t}^k u(y,s)| \leq C_{\ell,k} \|u\|_{L^1(C_{r}(x,t))}
\]

for all cylinder \( C_{r/2}(x,t) \subset C_{r}(x,t) \subset \Omega_T \) and for all solutions \( u \) of the heat equation in \( \Omega_T \).

For the proof we refer the reader to Theorem 9 Section 2.3.3 in [2].

4.7 Energy Methods

a. Uniqueness.

Let us consider the initial/boundary-value problem

\[
\begin{align*}
\left\{
\begin{array}{ll}
  u_t - \Delta u = f, & \text{in } \Omega_T \\
  u = g & \text{in } \Gamma_T
\end{array}
\right.
\end{align*}
\]

(4.19)

We assume that \( \Omega \subset \mathbb{R}^n \) is bounded, open and of class \( C^1 \). The terminal time is given.

Theorem 4.7.1 (Uniqueness) There exists at most one solution \( u \in C^{2,1}(\Omega_T) \) of (4.19).

Proof. 1. If \( \tilde{u} \) is another solution, \( w = u - \tilde{u} \) solves

\[
\begin{align*}
\left\{
\begin{array}{ll}
  w_t - \Delta w = 0, & \text{in } \Omega_T \\
  w = 0 & \text{in } \Gamma_T
\end{array}
\right.
\end{align*}
\]

(4.20)

2. Set

\[
e(t) := \int_{\Omega} w^2(x,t)dx \quad 0 \leq t \leq T.
\]

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Then
\[ e'(t) = 2 \int_{\Omega} w w_t \, dx \]
\[ = 2 \int_{\Omega} w \Delta w \, dx \]
\[ = -2 \int_{\Omega} |Dw|^2 \, dx \leq 0. \]

Thus
\[ e(t) \leq e(0) = 0, \quad 0 \leq t \leq T. \]

Consequently \( w = u - \tilde{u} = 0. \)

**b. Backwards uniqueness** A rather more subtle question concerns uniqueness *backward in time* for the heat equation. For this suppose that \( u \) and \( \tilde{u} \) are both smooth solutions of the heat equation in \( \Omega_T \), with the same boundary conditions on \( \partial\Omega \):

\[
\begin{cases}
  w_t - \Delta w = 0, & \text{in } \Omega_T \\
  \tilde{w} = g & \text{in } \partial\Omega \times [0, T]
\end{cases}
\]

for some function \( g \). Note carefully that we are not supposing \( u = \tilde{u} \) at time \( t = 0 \).

**Theorem 4.7.2 (Backwards uniqueness)** Let \( g \in C(\partial\Omega \times [0, T]) \). Let \( u_1, u_2 \in C^{1,2}(\bar{\Omega}_T) \) be two solutions

\[
\begin{cases}
  \tilde{w}_t - \Delta \tilde{w} = 0, & \text{in } \Omega_T \\
  \tilde{w} = g & \text{in } \partial\Omega \times [0, T]
\end{cases}
\]

If \( u_1(x, T) = u_2(x, T) \) for every \( x \in \Omega \) then \( u_1 = u_2 \) in \( \bar{\Omega}_T \).

**Proof.** We set \( w = u_1 - u_2 \). We have \( w \in C^{1,2}(\bar{\Omega}_T) \) is a solution

\[
\begin{cases}
  w_t - \Delta w = 0, & \text{in } \Omega_T \\
  w = 0 & \text{in } \partial\Omega \times [0, T] \\
  w(x, T) = 0 & \text{in } \Omega
\end{cases}
\]
We set 
\[ e(t) = \int_{\Omega} w^2(x,t)dx \quad \forall t \in [0,T]. \]
We have \( e(t) \geq 0 \) for every \( t \in [0,T] \), \( e(T) = 0 \) and as we have already seen
\[ e'(t) = -2 \int_{\Omega} |\nabla w|^2 dx. \]
We compute
\[ e''(t) = -2 \int_{\Omega} \partial_t(|\nabla w|^2)dx = -4 \int_{\Omega} \nabla w \cdot \nabla w_t dx \]
\[ = 4 \int_{\Omega} \Delta w \cdot w_t dx - 4 \int_{\partial \Omega} \frac{\partial w}{\partial \nu} \cdot w_t d\sigma(x) \]
\[ = 4 \int_{\Omega} |\Delta w|^2 dx. \]
Moreover
\[ (e'(t))^2 = 4 \left( \int_{\Omega} w \Delta w dx \right)^2 \leq e(t)e''(t). \]
Suppose now that there exits \( t_1 \in [0,T] \) such that \( e(t_1) > 0 \). Since \( e(T) = 0 \) it must be \( t_1 < T \) and there exits \( t_2 \in (t_1,T] \) such that \( e(t_2) = 0 \) and \( e(t) > 0 \) for all \( t \in [t_1,t_2) \). For \( t \in [t_1,t_2) \) we set \( f(t) = \log(e(t)) \) and we observe that \( f''(t) \geq 0 \). Thus \( f \) is convex in \([t_1,t_2)\). But \( f(t) \to -\infty \) as \( t \to t_2^- \) which is contradiction. Thus \( e(t) = 0 \) for every \( t \in [0,T], \) namely \( u_1 = u_2 \). \( \square \)
Appendix A

Convolution product

Let \( f, g \) be two measurable functions in \( \mathbb{R}^n \). The convolution between \( f \) and \( g \) is the function \( f \ast g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy \) which is defined for all \( x \) for which \( y \mapsto f(x-y)g(y) \) is summable. For instance \( f \ast g \) is well defined if \( f \in C_c(\mathbb{R}^n) \) and \( g \in L^1_{loc}(\mathbb{R}^n) \).

A.1 Properties of the Convolution

We recall the following result

**Proposition A.1.1** Let \( f \in L^1(\mathbb{R}^n) \), \( g \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq +\infty \). Then for a.e. \( x \in \mathbb{R}^n \) the function \( y \mapsto f(x-y)g(y) \) is summable, \( f \ast g \in L^p(\mathbb{R}^n) \) and

\[
\| f \ast g \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^1(\mathbb{R}^n)} \| g \|_{L^p(\mathbb{R}^n)}.
\]  

For the proof of Proposition A.1.1 we refer the reader e.g to [1].

One of the most important properties of the convolution is the regularizing effect. More precisely it holds the following result.
Theorem A.1.1 Let $f \in C^k(\mathbb{R}^n)$ and $g: \mathbb{R}^n \to \mathbb{R}$ be measurable. Suppose that $f \ast g$ exists in a neighborhood $U$ of $x_0$. If for $x_0 \in \mathbb{R}^n$ there exist a neighborhood $U$ of $x_0$ and for all $|\alpha| \leq k$ some functions $\varphi_\alpha \in L^1(\mathbb{R}^n)$ such that $|D^\alpha f(x-y)g(y)| \leq \varphi_\alpha(y)$ for a.e $x \in U$, then $D^\alpha(f \ast g)(x_0) = (D^\alpha f \ast g)(x_0)$.

Proof. The proof is an application of the derivation property under integral sign. □

Next we see some conditions under which the hypotheses of Theorem A.1.1 are verified.

Proposition A.1.2 The hypotheses of Theorem A.1.1 are verified in every $x_0 \in \mathbb{R}^n$ if it holds:

i) $f \in C^k(\mathbb{R}^n)$, $D^\alpha f \in L^\infty(\mathbb{R}^n)$ for every $|\alpha| \leq k$ and $g \in L^1(\mathbb{R}^n)$.

ii) $f \in C^k(\mathbb{R}^n)$, $|D^\alpha f(x)| \leq \frac{C_\alpha}{(1+|x|^2)^{m_\alpha}}$, $m_\alpha > \frac{n}{2}$ and $g \in L^\infty(\mathbb{R}^n)$.

iii) $f \in C^k_c(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$.

Proof of Proposition A.1.2.

i) We observe that $|D^\alpha f(x-y)g(y)| \leq \|D^\alpha f\|_{L^\infty} g(y) \in L^1(\mathbb{R}^n)$ for every $x \in \mathbb{R}^n$.

ii) We fix $x_0 \in \mathbb{R}^n$. For all $x, y \in \mathbb{R}^n$ it holds $|y-x_0| \leq |x-y| + |x-x_0|$. Hence if $|x-x_0| \leq \frac{1}{2}$ and $|y-x_0| \geq 1$ we get $|y-x_0| \leq 2|y-x|$. In particular for $|x-x_0| \leq \frac{1}{2}$ we get

$$|D^\alpha f(x-y)g(y)| \leq \varphi_\alpha(y) = \begin{cases} \frac{C_\alpha\|g\|_{L^\infty}}{(1+|y-x_0|^2)^{m_\alpha}} & \text{if } |y-x_0| \geq 1 \\ C_\alpha\|g\|_{L^\infty} & \text{if } |y-x_0| \leq 1 \end{cases}$$
Since \( m_\alpha > \frac{a}{2} \), \( \varphi_\alpha(y) \in L^1(\mathbb{R}^n) \).

iii) We observe that \( D^\alpha f(x - y) = 0 \) if \( x - y \notin \text{spt}(f) \), namely \( y \notin x - \text{spt}(f) \). Let \( U \) be a compact neighborhood of \( x_0 \) and set \( K = U - \text{spt}(f) \). Then \( D^\alpha f(x - y) = 0 \) if \( x \in U \) and \( y \notin K \). Hence

\[
|D^\alpha f(x - y)g(y)| \leq \varphi_\alpha(y) = \|D^\alpha f|_{L^\infty}|g(y)|\mathbb{1}_K(y) \in L^1(\mathbb{R}^n).
\]

We can conclude. \( \Box \)
Bibliography


