MATHEMATIK III-PARTIELLE DIFFERENTIALGLEICHUNGEN, D-CHEM

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Abstract

These notes are based on the lectures the author gave for the course *Mathematik III- Partielle Differentialgleichungen* during the Fall-Semesters 2008-2015 at D-CHEM, ETH, Zürich. They are not supposed to replace the several books in literature on Partial Differential Equations. They simply aim at offering the students a support for the preparation to the exam.

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Chapter 1

Preliminaries

This course is an introduction to Partial Differential Equations (in short PDEs). PDEs is a subject about differential equations for unknown functions of several variables, the derivatives involved are partial derivatives.

We first list some notations we will use throughout these notes.

1.1 Some notations

i) A *multindex* is a vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each component α_i is a nonnegative integer. The order of α is the number $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

ii) Given a multiindex α, $u\colon \Omega \subset I\!R^n \to I\!R$ and $x = (x_1, \ldots, x_n)$ define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}.$$

iii) If k is nonnegative integer,

$$D^k u(x) := \{D^\alpha u(x) \ : |\alpha| = k\}.$$

Special cases

If $k = 0$, $D^0 u = u$.

If $k = 1$, we regard the elements of Du as being arranged in a vector:

$$Du = (u_{x_1}, \cdots, u_{x_n}) = \text{gradient vector}.$$

If $k = 2$ we regard the elements of D^2u as being arranged in a matrix:

$$D^2u = \begin{pmatrix} u_{x_1x_1} & \cdots & u_{x_1x_n} \\ \vdots & \ddots & \vdots \\ u_{x_nx_1} & \cdots & u_{x_nx_n} \end{pmatrix}$$

The Laplacian of u is defined by

$$\Delta u = \text{trace}(D^2u) = \sum_{i=1}^{n} u_{x_ix_i}.$$

iv) Given a vector field $\mathbf{F}\colon \Omega \subset I\!R^n \to I\!R^n$,

$$F(x_1, \ldots, x_n) = (F^1(x_1, \ldots, x_n), \ldots, F^n(x_1, \ldots, x_n))$$

the divergence of $\mathbf{F}$ is defined by

$$\text{div}\mathbf{F} := \sum_{i=1}^{n} \partial_i F^i.$$

Observe that $\Delta u = \text{div}(\nabla u)$.

v) If f is function of one variable x, we will denote by $f'(x)$, $\dot{f}(x)$, $\dfrac{df}{dx}(x)$ its derivative with respect to x.

vi) We sometimes employ a subscript attached to the symbols D, D^2 etc. to denote the variables being differentiated. For example if $u = u(x, y)$, $x \in I\!R^n, y \in I\!R^m$ then $D_xu = (u_{x_1}, \cdots, u_{x_n})$ and $D_yu = (u_{y_1}, \cdots, u_{y_m})$.

vii) Given $x \in I\!R^n$, $r > 0$ we denote by $B(x, r) = \{y : \ |y - x| < r\}$ and $\bar{B}(x, r) := \{y : \ |y - x| \leq r\}$, and $\partial B(x, r) := \{y : \ |y - x| = r\}$.

viii) Given an open connected set Ω of $I\!R^n$ we denote by $\bar{\Omega}$ the closure of Ω and by $\partial\Omega$ or $bdy(\Omega)$ the boundary of Ω.

1.2 What is a PDE?

A PDE describes a relation between an unknown function and its partial derivatives. PDEs appear frequently in all area of physics and engineering. Moreover, in recent years we have seen a dramatic increase in the use of PDEs in areas such as biology, chemistry, computer science (particularly in relation to image processing and graphics) and economics (finance). In each area where there is an interaction between a number of independent variables, we attempt to define functions in these variables and to model a variety of processes by constructing equations for these functions.

The general form of a PDE for a function $u(x_1, \ldots, x_n)$ is

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_{11}}, \ldots) = 0 \tag{1.1}$$

where $F\colon \Omega \times I\!R \times I\!R^n \times \ldots \to I\!R$ is a given function.

We recall that in an ordinary differential equation (in short ODE) model the unknown u is a function of a single independent variable (the time t). In general an ODE model has the form

$$\frac{du}{dt} = f(u,t), \quad t > 0\,,$$

where f is a given functional relation between t and u.

A PDE model differs from an ODE model in that the unknown depends on more that one independent variable.

Whereas an ODE models the evolution of a system in time, and observations are made in time, a PDE models the evolution of a system in both time and space. PDE models may also be independent of time, but depend on several spatial variables.

A *solution* of a PDE is a function $u(x_1, \ldots, x_n)$ that satisfies the equation identically.

The analysis of PDEs has many facets. The classical approach that dominated the 19*th* century was to develop methods for finding explicits solutions. These notes aim at describing some of these methods.

Some examples of PDEs (all of which occur in physical theory) are:

1) $u_x + yu_y = 0$, (transport equation).

2) $u_x + uu_y = 0$, (Burger's equation).

3) $u_{xx} + u_{yy} = 0$, (Laplace's equation).

4) $u_{tt} - u_{xx} + u^3 = 0$, (wave with interaction).

5) $u_t - iu_{xx} = 0$, (quantum mechanics).

6) $u_t + uu_x + u_{xxx} = 0$, (dispersive wave).

7) $u_{tt} + u_{xxxx} = 0$, (vibrating bar).

1.3 Classification

The PDEs are often classified into different types.

1. The order of an equation

   The first classification is according the order of the equation. The order is defined to be the order of the highest derivative in the equation. If the highest derivative is of order k, the equation is said to be of order k. Thus, for example,the equation $u_{tt} - u_{xx} = f(x,t)$ is called a second-order equation, while $u_t - u_{xxxx} = 0$ is called a fourth-order equation.

2. Linear Equations Another classification is in two groups: linear and nonlinear equations. *Linearity* means the following. Write the equation in the form $Lu = 0$ where L is an *operator*. That is, if v is any function, Lv is a new function. For instance $L = \frac{\partial}{\partial x}$ is the operator that takes v into its partial derivative v_x. The definition we want for linearity is

   $$L(u+v) = L(u) + L(v), \quad L(cu) = cL(u) \tag{1.2}$$

   for any functions u, v and any constant c. Whenever (1.2) holds (for all choices of u, v and c), L is called *linear operator*.

   The equation

   $$Lu = 0 \tag{1.3}$$

is called *linear* if L is a linear operator. Equation (1.3) is called *homogeneous.* The equation

$$Lu = g \tag{1.4}$$

where $g \neq 0$ is a given function of the independent variables, is called an *nonhomogeneous linear equation.* Thus for example the equation $x^6 u_x + e^{xy} u_y + \sin(x^2 + y^2)u = \log(x)$ is a nonhomogeneous linear equation. The following three second order equations are nonlinear: $u_t + uu_{xx} = 0$, $u_{tt} - u_x + \sin(u) = 0$, and $u_{tt} - u_{xx} + u^3 = 0$, the first because of the product uu_{xx}, the second because the unknown u is tied up in the nonlinear sine function and the third because of the cubic term :

$$(u + v)^3 = u^3 + 3u^2 v + 3uv^2 + v^3 \neq u^3 + v^3.$$

The advantage of linearity for the equation $Lu = 0$ is that if $u_1, \ldots, u_n$ are all solutions, so is any linear combination

$$\sum_{j=1}^{n} c_j u_j(x), \quad c_j \in \mathbb{R}.$$

This is called the Superposition Principle. Another consequence of linearity is that if you add a homogeneous solution (a solution of (1.3)) to a nonhomogeneous solution (a solution of (1.4)), you get a nonhomogeneous solution.

1.4 Some Examples

Example 1.4.1 (Simple Transport) Consider a fluid, water, say, flowing at a constant rate c along a horizontal pipe of fixed cross section in the positive x direction. A substance, say, a pollutant, is suspended in the water. Let $u(x, t)$ be its concentration in grams/centimeter at time t. Then

$$\boxed{u_t + cu_x = 0.}$$

That is, the rate of change u_t of concentration is proportional to the gradient u_x. Solving this equation, we find that the concentration is a function of $x - ct$ only. This means that the substance is transported to the right at a fixed speed c. Each individual particle moves to the right at speed c.

Example 1.4.2 (Vibration of a string) Consider a thin string of length L, which undergoes relatively small transverse vibrations (think of a string of a musical instrument, say a violin string). Let ρ be the linear density of the string, measured in units of mass per unit of length. We will assume that the string is made of homogeneous material and its density is constant along the entire length of the string. The displacement of the string from its equilibrium state at time t and position x will be denoted by $u(t, x)$. We will position the string on the x-axis with its left endpoint coinciding with the origin of the xu coordinate system. At a given instant t, the string might look as shown in Figure (1.1).

We make the following *assumptions:*

1) When the string vibrates, the motion takes place in the x, u -plane (i.e. the motion is planar) and each point on the string moves in a direction perpendicular to the x-axis (i.e., we have transverse vibrations).

2) The string is perfectly flexible (or elastic).

3) The tension acts tangentially to the string at each point.

4) The tension is large compared to the force of gravity and no other external forces act on the string.

Two implications of these assumptions are as follows:

1′) From 1) , the net horizontal force on any section of the string is zero.

2′) From 2), there is no resistance to bending, thus, the shape of the string does not produce any non-tangential force on the string.

Let $u(x, t)$ be the displacement from the x-axis at time t. Consider a segment of the string from a point x to a point $x + h, h > 0$ (see Figure

(1.2)). Let $T(x)$ be the magnitude of the tension at x (and similarly for $T(x+h)$) and let $\theta(x)$ be the angle between $T(x)$ and the x-axis (and similarly for $\theta(x+h)$).

From assumption 1′), the net horizontal force on the section $[x, x+h]$ is

$$T(x+h)\cos(\theta(x+h)) - T(x)\cos(\theta(x)) = 0.$$

Thus

$$T(x+h)\cos(\theta(x+h)) = T(x)\cos(\theta(x)) = H. \tag{1.5}$$

where H is a constant, since x and $x+h$ are arbitrary points on the string.

Let $\rho(x)$ be the linear density (mass per unit length) of the string at equilibrium. The mass, m of the section $[x, x+h]$ is $m = h\rho$. We apply Newton's second law in the vertical direction

Force= Mass × (average) Acceleration

$$T(x+h)\sin(\theta(x+h)) - T(x)\sin\theta(x)) = h\rho u_{tt}(x+h_1) \tag{1.6}$$

for some $0 < h_1 < h$. We divide equation (1.6) by the expression in (1.5):

$$\frac{T(x+h)\sin(\theta(x+h))}{T(x+h)\cos(\theta(x+h))} - \frac{T(x)\sin\theta(x))}{T(x)\cos\theta(x))} = \frac{h\rho u_{tt}(x+h_1)}{H}$$

giving

$$\tan(\theta(x+h)) - \tan(\theta(x)) = \frac{h\rho u_{tt}(x+h_1)}{H}$$

Since $\tan(\theta(x)) = u_x(x,t)$, dividing by h we get

$$\frac{u_x(x+h,t) - u(x,t)}{h} = \frac{\rho u_{tt}(x+h_1)}{H}$$

Let $h \to 0$:

$$u_{xx}(x,t) = \frac{\rho}{H} u_{tt}(x,t).$$

Let $c = \sqrt{\frac{H}{\rho}}$ which has the unit of a speed, the previous partial differential equation becomes:

$$\boxed{c^2 u_{xx} = u_{tt}}$$

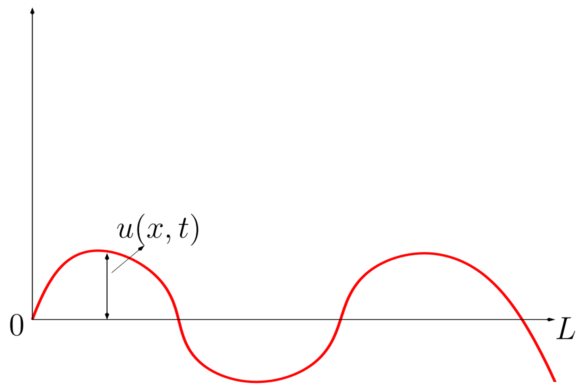

Figure 1.1: Vibrating string

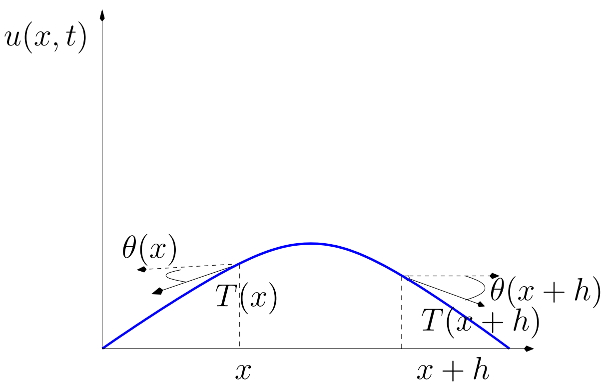

Figure 1.2: Tension on a segment $[x, x + h]$ of the string

and this is our one-dimensional wave equation for the vibrating string.

There are many variations of this equation:

- If significant air resistance r is present, we have an extra term proportional to the speed u_t, thus:

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \quad \text{where } r > 0.$$

- With transverse elastic force: force is proportional to the displacement u

$$u_{tt} - c^2 u_{xx} + k u = 0, \quad \text{where } k > 0.$$

- If there is an externally applied force, it appears as an extra term, thus:

$$u_{tt} - c^2 u_{xx} = f(x, t),$$

  which makes the equation nonhomogeneous.

The same wave equation or a variation of it describes many other wavelike phenomena, such as the vibrations of an elastic bar, the sound waves in a pipe, and the long water waves in a straight canal.

Example 1.4.3 (Diffusion) Let us consider a motionless liquid filling a straight tube and a chemical substance, say a dye, which is diffusing through the liquid. Simple diffusion is characterized by the following law. [it is not to be confused with convection (transport), which refers to currents in the liquid]. The dye moves from regions of higher concentration to regions of lower concentration . The rate motion is proportional to the concentration gradient (Fick's law of diffusion). Let $u(x, t)$ be the concentration (mass per unit length) of the dye at position x of the pipe at time t.

In the section of pipe from x_0 to x_1, the mass of dye is

$$M(t) = \int_{x_0}^{x_1} u(x, t) dx, \quad \text{so} \quad \frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x, t) dx.$$

The mass of the pipe cannot change except by flowing in or out of its ends. By Fick's law

$$\frac{dM}{dt} = \text{flow in} - \text{flow out} = ku_x(x_1, t) - ku_x(x_0, t),$$

where k is a proportionality constant. Therefore, theses two expressions are equal:

$$\int_{x_0}^{x_1} u_t(x,t)dx = ku_x(x_1,t) - ku_x(x_0,t).$$

Differentiating with respect to x_1, we get

$$\boxed{u_t = ku_{xx}}$$

This is the *diffusion squation.*

In three dimensions we have

$$\iiint_D u_t\, dxdydz = \iint_{\partial D} k(n \cdot \nabla u)d\sigma,$$

where D is any solid domain and ∂D is its bounding surface. By the divergence theorem we get the three-dimensional diffusion equation

$$\boxed{u_t = k(u_{xx} + u_{yy} + u_{zz}) = k\Delta u.}$$

Example 1.4.4 (Heat Flow) Let $u(x, y, z, t)$ be the temperature and let $H(t)$ be the amount of heat contained in a region $D \subset I\!R^3$. Then

$$H(t) = \iiint_D c\rho u dxdydz,$$

where c is the specific heat of the material and ρ is its density (mass per unit volume). The change in heat is

$$\frac{dH}{dt}(t) = \iiint_D c\rho u_t dxdydz.$$

Fourier's law says that heat flows from hot to cold regions proportionately to the temperature gradient. But heat cannot be lost from D except by leaving it through the boundary. This is the law of *conservation of energy.* Therefore, the change of heat energy in D also equals the heat flux across the boundary,

$$\frac{dH}{dt}(t) = \iint_{\partial D} \kappa(n \cdot \nabla u) d\sigma,$$

where κ is the *heat conductivity.* By the divergence theorem

$$\iiint_D c\rho u_t dxdydz = \iiint_D \mathrm{div}(\kappa \nabla u) dxdydz$$

and we get the *heat equation*

$$\boxed{c\rho u_t = \mathrm{div}(\kappa \nabla u)}$$

If c, ρ and κ are constants, it is exactly as the diffusion equation.

Example 1.4.5 (Laplace Equation) Some physical systems do not depend upon time, but rather only upon spatial variables. Such models are called static or equilibrium models. For example if Ω represents a bounded, three-dimensional spatial region in which no charges are present, and on the boundary $\partial\Omega$ of the region there is a given, time-independent electric potential (in electrostatics, the gradient of the potential is the electric field vector), then it is known that the electric potential $u = u(x, y, z)$ inside Ω satisfies the Laplace equation, a PDE having the form

$$u_{xx} + u_{yy} + u_{zz} = 0, \quad (x, y, z) \in \Omega. \tag{1.7}$$

If we denote the given boundary potential by $f(x, y, z)$, $(x, y, z) \in \partial\Omega$ then the equation (1.7) along with the boundary condition

$$u(x, y, z) = f(x, y, z), \quad (x, y, z) \in \partial\Omega \tag{1.8}$$

is an equilibrium model for electrostatics. Solving Laplace's equation in a region Ω subject to a the boundary condition (1.8) on the boundary is called Dirichlet Problem.

Example 1.4.6 (The Schrödinger equation) One of the fundamental equations of quantum mechanics, derived in 1926 by Erwin Schrd̈ingier, governs the evolution of the wave function u of a particle in a potential field V:

$$i\hbar u_t = -\frac{\hbar}{2m}\Delta u + Vu.$$

Here V is a known function (potential), m is the particle's mass, and $\hbar$ is Planck's constant divided by 2π.

1.5 Associated conditions

PDEs have in general infinitely many solutions In order to obtain a unique solution one must supplement the equation with additional conditions. The kind of conditions are motivated by the physics and come in two varieties, initial conditions and boundary conditions. An *initial condition* specifies the physical star at a particular time t_0. For the diffusion equation the initial condition is

$$u(x, t_0) = \phi(x),$$

where $\phi(x)$ is a given function. For a diffusing substance, Φ is the initial concentration. For heat flow, $\phi(x)$ is the initial temperature. For the wave equation there is a pair of initial conditions

$$u(x, t_0) = \phi(x), \quad u_t(x, t_0) = \psi(x),$$

where ϕ is the initial position and ψ is the initial velocity. Both of them must be specified in order to determine the position (x, t) at later times.

In each physical problem there is a domain D in which the PDE is valid. For the vibrating string, D, is the interval $0 < x < L$, so the boundary of D consist only of two points $x = 0$ and $x = L$. For the diffusing chemical substance, D is the container holding the liquid, so its boundary is a surface $S = bdyD$. In these cases it is necessary to specify some *boundary condition* if the solution is to be determined. The three most important kinds of boundary conditions are:

(D) u is specified ("Dirichlet condition")

(N) the normal derivative $\frac{\partial u}{\partial n}$ is specified ("Neumann condition")

(R) $\frac{\partial u}{\partial n} + au$ is specified ("Robin conditions") where a is given function.

Each of these types of boundary conditions is to hold for all t and for all x belonging to $bdyD$.

In one-dimensional problems where D is just an interval $0 < x < L$, the boundary consists of just the two endpoints, and theses boundary conditions take the simple forms

$$(D)u(0,t) = g(t), \quad \text{and} \quad u(L,t) = h(t)$$

$$(N)\frac{\partial u}{\partial n}(0,t) = g(t), \quad \text{and} \quad \frac{\partial u}{\partial n}(L,t) = h(t)$$

and similarly for the Robin condition.

1.6 Types of second-order equations

In this section we consider second-order linear equations for functions in two independents variables x, y. Such an equation has the form

$$L(u) = au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \tag{1.9}$$

where $a, b, \ldots, f, g$ are given functions of x, y and $u(x,y)$ is the unknown function. We introduce the factor 2 in front of the coefficient b for convenience. The operator

$$L_0(u) = au_{xx} + 2bu_{xy} + cu_{yy}$$

that consists of the second-order terms of the operator L is called the *principal part* of L. It turns out that many fundamental properties of the solutions of (1.9) are determined by its principal part, and, more precisely by the *sign* of the discriminant $\delta(L) := b^2 - ac$ of the equation. We classify the equation according to the sign of $\delta(L)$.

Definition 1.6.1 *Equation (1.9) is said to be* hyperbolic *at a point* (x, y) *if*

$$\delta(L)(x,y) = b(x,y)^2 - a(x,y)c(x,y) > 0,$$

it is said to be parabolic *at* (x,y) *if* $\delta(L)(x,y) = 0$ *and it is said to be* elliptic *at* (x,y) *if* $\delta(L)(x,y) < 0$.

Let Ω *be an open connected set of* $I\!R^2$*. The equation is hyperbolic (resp., parabolic, elliptic) in* Ω*, if it is hyperbolic (resp., parabolic, elliptic) at all points* $(x,y) \in \Omega$*.*

The three (so called) *fundamental equations of mathematical physics*, the heat equation, the wave equation. the Laplace equation are linear second-order equations. One can easily verify that the wave equation is hyperbolic, the heat equation is parabolic and the Laplace equation is elliptic.

Exercise 1.6.1 Consider the equation

$$u_{xx} + 2u_{xy} + [1 - q(y)]u_{yy} = 0,$$

where

$$q(y) = \begin{cases} -1 & y < -1, \\ 0 & |y| \leq 1, \\ 1, & y > 1. \end{cases}$$

Find the domains where the equation is hyperbolic, parabolic, and elliptic.

Exercise 1.6.2 For each of the following equations state the order and whether it is nonlinear, linear homogeneous, linear nonhomogeneous; provide reasons.

a) $u_t - u_{xx} + 1 = 0$.nonhomogeneous

b) $u_t - u_{xx} + xu = 0$.

c) $u_t - u_{xxt} + uu_x = 0$.

d) $u_{tt} - u_{xx} + x^2 = 0$.

e) $u_x(1 + u_x^2)^{-1/2} + u_y(1 + u_y^2)^{-1/2} = 0$.

f) $u_t + u_{xxxx} + \sqrt{1+u} = 0$.

Remark 1.6.1 To understand these notes you should have, first, a good command of the concepts in the calculus of several variables. Here are a few things to keep in mind.

- Derivatives are *local.* For instance, to calculate the derivatives $\frac{\partial u}{\partial x}(x_0, y_0)$ at a particular point, you need to know just the values of the function $u(x, y_0)$ for x *near* x_0.
- Mixed derivatives are equal: $u_{xy} = u_{yx}$. (We assume throughout these notes that all derivatives exist and are continuous).
- The chain rule is used frequently in PDEs; for instance,

$$\frac{\partial}{\partial x}[f(g(x,y))] = f'(g(x,y))\,\frac{\partial g}{\partial x}(x,y).$$

- Jacobian
- Divergence Theorem
- Infinite series of functions and their differentiation.
- We will often reduce PDEs to ODEs , so we must know how to solve simple ODEs. But we will not need to know anything about tricky ODEs.

Chapter 2

The One Dimensional Wave Equation

In this chapter we study the one-dimensional wave equation on the whole real line $-\infty < x < +\infty$. Real physical situations are usually on finite intervals. We are justified in taking x on the whole real line for two reasons. Physically speaking, if you are sitting far away from the boundary, it will take a certain time for the boundary to have a substantial effect on you, and until that time the solutions we obtain in this Chapter are valid. Mathematically speaking, the absence of a boundary is a big simplification. We shall derive simple explicit formulas for solutions and discuss some important properties of the solutions.

2.1 General Solutions

The homogeneous wave equation in one (spatial) dimension has the form

$$u_{tt} - c^2 u_{xx} = 0, \quad \text{in } -\infty < x < +\infty, \ t > 0, \tag{2.1}$$

where $c \in \mathbb{R}$ is called the wave speed. We introduce the new variables

$$\xi = x + ct \quad \text{and} \quad \eta = x - ct,$$

and we set $w(\xi,\eta) := u(x(\xi,\eta),t(\xi,\eta))$. Using the chain rule for the function $u(x,t) = w(x+ct, x-ct)$, we obtain

$$\begin{aligned} u_t &= w_\xi \xi_t + w_\eta \eta_t = c(w_\xi - w_\eta) \\ u_x &= w_\xi \xi_x + w_\eta \eta_x = w_\xi + w_\eta \\ u_{tt} &= c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}) \\ u_{xx} &= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}. \end{aligned}$$

Hence

$$u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta} = 0.$$

Since $(w_\xi)_\eta = 0$, it follows that $w_\xi = f(\xi)$ and then

$$w = \int f(\xi) d\xi + G(\eta).$$

Therefore the general solution of the equation $w_{\xi\eta} = 0$ has the form

$$w(\xi,\eta) = F(\xi) + G(\eta),$$

where $F, G \in C^2(I\!R)^{(1)}$ are two arbitrary functions. Thus, in the original variables, the general solution of the wave equation is

$$u(x,t) = F(x+ct) + G(x-ct). \tag{2.2}$$

In other words, if u is a solution of (2.1), then there exist two real functions $F, G \in C^2$ such that (2.2) holds. Conversely any two functions $F, G \in C^2$ give a solution of the wave equation via the formula (2.2).

For a fixed $t_0 > 0$, the graph of the function $G(x - ct_0)$ has the same shape as the graph of the function G, except that it is shifted to the right by a distance ct_0. Therefore, the function $G(x-ct)$ represents a wave moving to the right with velocity c and it is called a forward wave. The function $F(x+ct)$ is a wave traveling to the left with the same speed, and it is called a backward wave. Indeed c can be called the wave speed.

Equation (2.2) demonstrates that any solution of the wave equation is the sum of two such traveling waves.

$^{(1)}C^2(I\!R) := \{f : I\!R \to I\!R : f' \text{ and } f'' \text{ exist and are continuous in } I\!R\}$

2.2 The Cauchy problem and d'Alembert's formula

The *Cauchy problem* for the one-dimensional wave equation is given by

$$u_{tt} - c^2 u_{xx} = 0, \;\; x \in I\!R, \;\; t > 0 \tag{2.3}$$

$$u(x,0) = f(x), \tag{2.4}$$

$$u_t(x,0) = g(x) \quad . \tag{2.5}$$

A solution of this problem can be interpreted as the amplitude of the vibration of an infinite (ideal) string. The initial conditions f, g are given functions that represent respectively the amplitude u, and the velocity u_t of the string at time $t = 0$.

A *classical solution* of the Cauchy problem is a function u that is continuously twice differentiable for all $t > 0$, such that u, u_t are continuous in the half-space $t \geq 0$ and such that (2.4)-(2.5) are satisfied.

We recall that the general solution of the wave equation is of the form (2.2). Our aim is to find F and G such that the initial conditions (2.4)-(2.5) are satisfied. Substituting $t = 0$ into (2.2) we obtain

$$u(x,0) = F(x) + G(x) = f(x). \tag{2.6}$$

Differentiating (2.2) with respect to t and substituting $t = 0$ we have

$$u_t(x,0) = cF'(x) - cG'(x) = g(x). \tag{2.7}$$

Integration of (2.7) over the interval $[0, x]$ yields

$$F(x) - G(x) = \frac{1}{c}\int_0^x g(s)ds + C, \tag{2.8}$$

where $C = F(0) - G(0)$. Equations (2.6) and (2.8) are two linear algebraic equations for $F(x)$ and $G(x)$. The solution of the system of equations is given by

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds + \frac{C}{2}, \tag{2.9}$$

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds - \frac{C}{2}. \tag{2.10}$$

By substituting these expressions for F and G into the general solution (2.2) we obtain the formula

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, \qquad (2.11)$$

which is called d'Alembert's formula. Formula (2.11) illustrates the *finite speed of propagation* property associated to the wave equation. More precisely, the value of the solution at (t,x) is only influenced by the $[x-ct, x+ct]$; changes to the initial data outside of this interval have no effect on the solution at (t,x). We will reexamine this property later.

Example 2.2.1 Let $u(x,t)$ be the solution of the following Cauchy problem

$$u_{tt} - 9u_{xx} = 0, \quad x \in I\!R,\ t > 0\,,$$
$$u(x,0) = f(x) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2, \end{cases}$$
$$u_t(x,0) = g(x) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2, \end{cases}$$

a) Find $u(0,\frac{1}{6})$.

b) Discuss the large behavior of the solution.

c) Find the maximal value of $u(x,t)$ and the points where this maximum is achieved.

Proof. a) Since

$$u(x,t) = \frac{f(x+3t)+f(x-3t)}{2} + \frac{1}{6}\int_{x-3t}^{x+3t} g(s)ds,$$

it follows that for $x=0$ and $t=\frac{1}{6}$ we have

$$u(0,\frac{1}{6}) = \frac{f(\frac{1}{2})+f(-\frac{1}{2})}{2} + \frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(s)ds = \frac{7}{6}.$$

b) Fix $\xi \in I\!R$ and compute $\lim_{t\to+\infty} u(\xi,t)$. Clearly we have

$$\lim_{t\to+\infty} f(\xi+3t) = 0, \quad \lim_{t\to+\infty} f(\xi-3t) = 0,$$

and

$$\lim_{t\to+\infty}\int_{\xi-3t}^{\xi+3t} g(s)ds = \int_{-2}^{2} ds = 4.$$

Therefore $\lim_{t\to+\infty} u(\xi,t) = \frac{2}{3}$.

c) It is left as an exercise. (Sol: The solution attains its maximum at the point $(0,\frac{2}{3})$ where $u(0,\frac{2}{3}) = \frac{5}{3}$).

2.3 Domain of dependence and region of influence

Consider again the Cauchy problem (2.3)-(2.5), and examine what is the information that actually determines the solution at a fixed point (x_0,t_0). Consider the plane (x,t) and the lines passing through the point (x_0,t_0)

$$x - ct = x_0 - ct_0, \quad x + ct = x_0 + ct_0.$$

These two straight lines intersect the x-axis at the points $(x_0 - ct_0, 0)$ and $(x_0+ct_0, 0)$ respectively. The triangle formed by these lines and the interval $[x_0 - ct_0, x_0 + ct_0]$ is called characteristic triangle (see Fig. 2.2). By the d'Alembert formula

$$u(x_0,t_0) = \frac{f(x_0+ct_0)+f(x_0-ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)ds.$$

Therefore the value of u at the point (x_0,t_0) is determined by the values of f at the vertices of the interval $[x_0 - ct_0, x_0 + ct_0]$ and by the value of g along this interval. Thus, $u(x_0,t_0)$ depends only on the part of the initial data that is given on the interval $[x_0 - ct_0, x_0 + ct_0]$. This interval is called domain of dependence of u at the point (x_0,t_0). If we change the initial data at points outside this interval, the value of the solution u at the point (x_0,t_0) will not change.

We may ask now the opposite question: which are the points on the half-plane $t > 0$ that are influenced by the initial data on a fixed interval $[a,b]$? The set of all such points is called the region of influence of the interval $[a,b]$. It follows that this interval influences the value of the solution u at

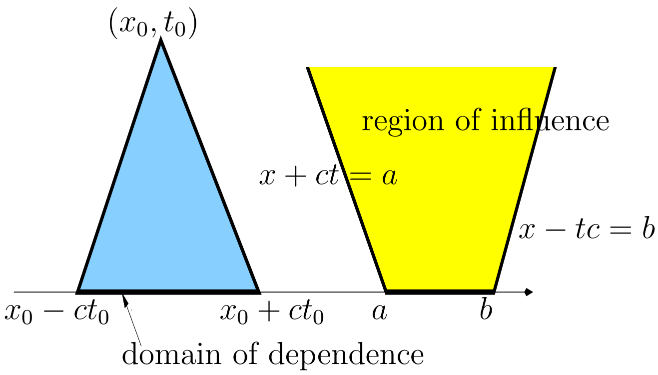

Figure 2.1: Region of influence and domain of dependence

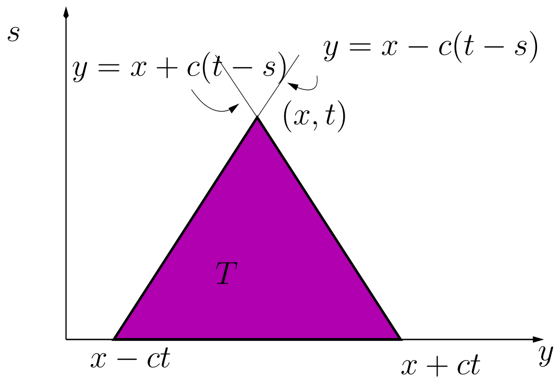

Figure 2.2: Characteristic Triangle

a point (x_0, t_0) if and only if $[x_0 - ct_0, x_0 + ct_0] \cap [a, b] \neq \emptyset$. Hence the initial data along the interval $[a, b]$ influence only points (x, t) satisfying $x - ct \leq b$ and $x + ct \geq a$.

Assume, for instance, that the initial data f and g vanish outside the interval $[a, b]$. Then the amplitude of the vibrating string is zero at every point outside the influence region of the interval. On the other hand for a fixed point x_0 on the string, the effect of the perturbation (from the zero data) along the interval $[a, b]$ will be felt after a time $t_0 \geq 0$ and eventually, for t large enough the solution takes the constant value $u(x_0, t) = \frac{1}{2c} \int_a^b g(s) ds$. This occurs precisely at points (x_0, t) that are inside the cone $x_0 - ct \leq a$ and $x_0 + ct \geq b$.

2.4 The Duhamel's Method

The Duhamel's principle is a general method for obtaining solutions to nonhomogeneous linear evolution equations like the heat equation or wave equation. We consider the following nonhomogeneous problem

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad x \in I\!R,\ t > 0 \tag{2.12}$$

$$u(x,0) = \varphi(x), \tag{2.13}$$

$$u_t(x,0) = \psi(x) \qquad . \tag{2.14}$$

This problem models, for example, the vibrations of a very long string in the presence of an external force F.

In order to solve it we consider for every initial time $s \geq 0$ the following family of problems:

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & x \in I\!R,\ t > s\,, \\ v(x,s) = 0\,, & x \in I\!R\,, \\ v_t(x,s) = F(x,s) & x \in I\!R. \end{cases} \tag{2.15}$$

We observe that v is a solution of (2.15) if and only if the function $w(x,t) := v(x,t+s)$ solves

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & x \in I\!R,\ t > 0\,, \\ w(x,0) = 0\,, & x \in I\!R\,, \\ w_t(x,0) = F(x,s) & x \in I\!R. \end{cases} \tag{2.16}$$

Conversely if $w = w(x,t)$ is a solution of (2.16), then $v(x,t) := w(x,t-s)$ is a solution of (2.15). By the d'Alembert Formula we have

$$w(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} F(\xi,s)d\xi\,,$$

and we get a solution of (2.15) for $t \geq s \geq 0$:

$$v(x,t) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} F(\xi,s)d\xi =: u(x,t;s).$$

Theorem 2.4.1 *A solution of the problem*

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t), & x \in I\!R,\ t > 0\,, \\ u(x,0) = 0\,, & x \in I\!R\,, \\ u_t(x,0) = 0 & x \in I\!R \end{cases} \tag{2.17}$$

is given by

$$u(x,t) := \int_0^t u(x,t;s)ds, \quad x \in I\!R, t \geq 0\,, \tag{2.18}$$

with

$$u(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} F(\xi,s)d\xi.$$

Intuitively, one can think of the nonhomogeneous problem as a set of homogeneous problems each starting afresh at a different time slice $t = s$. By linearity, one can add up (integrate) the resulting solutions through time s and obtain the solution for the nonhomogeneous problem. This is the essence of Duhamel's principle.

Now we can write a solution of (2.12) as $u = u_1 + u_2$ (by the Superposition principle) where u_1 is a solution of (2.17) and u_2 is a solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in I\!R,\ t > 0\,, \\ u(x,0) = \varphi(x) & x \in I\!R\,, \\ u_t(x,0) = \psi(x) & x \in I\!R. \end{cases} \tag{2.19}$$

By combining the d'Alembert's Formula and (2.18) we obtain the explicit formula of (2.12) :

$$
\begin{aligned}
u(x,t) &= \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds \\
&+ \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy\right) ds \qquad (2.20)\\
&= \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds + \frac{1}{2c}\iint_T F(y,s)dyds\,,
\end{aligned}
$$

where

$$T = \{(y,s) \in I\!R^2 : \ 0 \le s \le t, \ |y-x| \le c(t-s)\}$$

is the characteristic triangle through (x,t), (see Fig. 2.2).

Remark 2.4.1 In many cases it is possible to reduce a nonhomogeneous problem to a homogeneous problem if we find a particular solution v of the given nonhomogeneous equation. The technique is particularly useful when F has a simple form, for example, where $F = F(x)$ or $F = F(t)$. Suppose that such a particular solution v is found and consider the function $w = u - v$. By the *superposition principle*, w should solve the following Cauchy problem

$$
\begin{aligned}
w_{tt} - c^2 w_{xx} &= 0, \quad x \in I\!R, \ t > 0 \\
w(x,0) &= f(x) - v(x,0), \\
w_t(x,0) &= g(x) - v_t(x,0) \qquad .
\end{aligned}
$$

Hence w can be found by using d'Alembert formula for the homogeneous equation. Then $u = w + v$ is the solution of the original problem.

Example 2.4.1 Solve the problem

$$
\begin{aligned}
u_{tt} - u_{xx} &= t^7, \quad x \in I\!R, \ t > 0 \\
u(x,0) &= 2x + \sin(x), \qquad x \in I\!R \\
u_t(x,0) &= 0 \qquad x \in I\!R.
\end{aligned}
$$

We look for a particular solution of the form $v = v(t)$. It can be easily verified that $v(x,t) = \frac{1}{72}t^9$ is such a solution.

Now we need to solve the homogeneous problem

$$\begin{aligned} w_{tt} - w_{xx} &= 0, \quad x \in I\!R, \ t > 0 \\ w(x,0) = f(x) - v(x,0) &= 2x + \sin(x), \qquad x \in I\!R \\ w_t(x,0) = g(x) - v_t(x,0) &= 0 \qquad x \in I\!R. \end{aligned}$$

Using d'Alembert formula for the homogeneous equation, we have

$$w(x,t) = 2x + \frac{1}{2}\sin(x+t) + \frac{1}{2}\sin(x-t)$$

and the solution of the original problem is given by $u(x,t) = 2x + \sin(x)\cos(t) + \frac{1}{72}t^9$.

Example 2.4.2 Solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= xt, \ \ x \in I\!R, \ t > 0 \\ u(x,0) &= 0, \qquad x \in I\!R \\ u_t(x,0) &= 1 \qquad x \in I\!R. \end{aligned}$$

For each (x,t) the characteristic triangle is given by the picture Fig. 2.2.

We apply the formula (2.20):

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} ds + \frac{1}{2}\iint_T ys\, dyds \\
&= t + \frac{1}{2}\int_0^t ds \int_{s+x-t}^{-s+x+t} ysdy \\
&= t + \frac{1}{2}\int_0^t sds \left(\frac{y^2}{2}\bigg|_{s+x-t}^{-s+x+t} \right) \\
&= t + \frac{1}{2}\int_0^t s\frac{(x+t-s)^2-(x-t+s)^2}{2} ds \\
&= t + \frac{1}{2}\int_0^t s\frac{2x(2t-2s)}{2} ds \\
&= t + \int_0^t sx(t-s)ds \\
&= t + \int_0^t (xts - s^2x)ds = t + (xt\frac{s^2}{2} - \frac{s^3}{3}x)\mid_0^t \\
&= t + \frac{xt^3}{6}.
\end{aligned}$$

Chapter 3

Fourier Series

Fourier series are series of cosine and sine terms and arise in the important practical task of representing general periodic functions. They constitute a very important tool in solving problems that involve ordinary and partial differential equations. The theory of Fourier series is rather complicated, but the application of these series is simple. Fourier series are more universal than Taylor series, because many discontinuous periodic functions of practical interest can be developed in Fourier series but, of course, do not have Taylor series representation.

The French mathematician and physicist Jean Baptist Joseph Fourier (see Fig. 3.1) introduced the Fourier series in his main work "Théorie analytique de la chaleur" (1822), where is developed the theory of heat conduction.

3.1 Periodic Function

A function $f(x)$ is called **periodic** if it is defined for all real x (except perhaps for certain isolated points) and if there is some positive real number p such that

$$f(x+p) = f(x), \quad \forall x \in \mathbb{R}. \tag{3.1}$$

This number p is called a period of $f(x)$. The graph of such function is obtained by periodic repetition of its graph in any interval of length p, see Fig. 3.11.

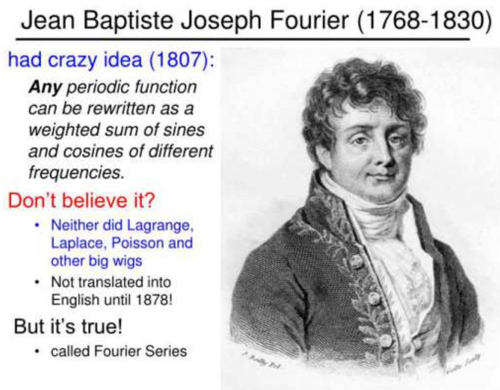

Figure 3.1: Jean Baptiste Joseph Fourier (1768-1830)

Familiar periodic functions are the sine and cosine functions. We note that the function $f \equiv c = const$ is also periodic in the sense of definition. Example of functions that are not periodic are $x, x^2, x^3, e^x, \ln(x)$. We notice that for any integer m we have

$$f(x + mp) = f(x), \quad \forall x \in I\!R.$$

If a periodic function $f(x)$ has a smallest period $p > 0$, this is often called **the fundamental period** of $f(x)$. For $\cos(x)$ and $\sin(x)$ the fundamental period is 2π, (see Fig. 3.3), for $\cos(2x)$ and $\sin(2x)$ it is π (see Fig. 3.4) and so on. A function without fundamental period is $f = const$.

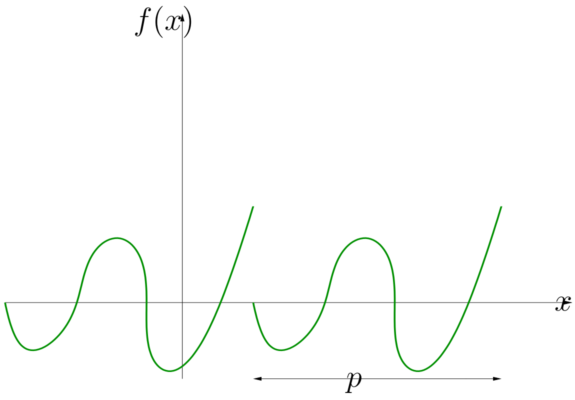

Figure 3.2: Periodic function

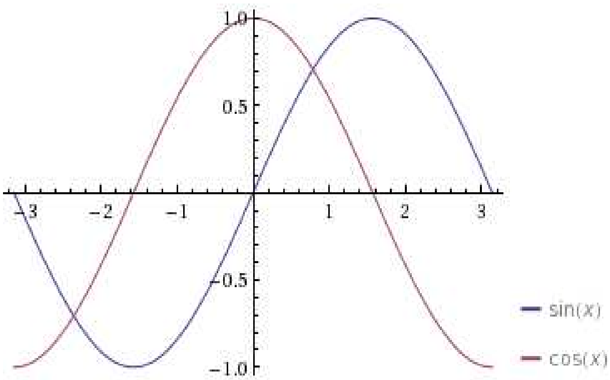

Figure 3.3: The graph of $sin(x)$ and $cos(x)$

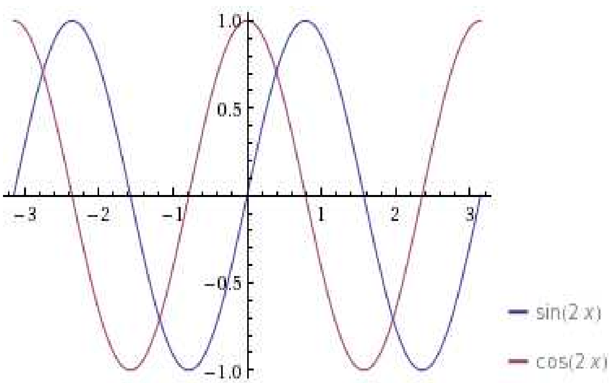

Figure 3.4: The graph of $sin(2x)$ and $cos(2x)$

3.2 Trigonometric Series

Our problem is the representation of various functions of period $p = 2\pi$ in terms of simple functions

$$1,\ \cos(x),\ \sin(x),\ \cos(2x),\ \sin(2x), \ldots, \cos(nx),\ \sin(nx), \ldots$$

The series that will arise in this connection will be of the form

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)). \tag{3.2}$$

where a_n, b_n are real constants. The series (3.2) is called **trigonometric series**, $a_n, b_n \in I\!R$ are called **coefficients** of the series.

We see that each term of the series (3.2) has period 2π. Hence if the series (3.2) converges, its sum will be a function of period 2π. The point is that trigonometric series can be used for representing any important periodic function f of any period p.

Exercise 3.2.1 1. Find the fundamental period of $\cos(2\pi x)$, $\sin(\frac{2\pi x}{k})$, $\tan(\frac{\pi x}{k})$, $\sin(nx)$ $(n, k > 0)$.

2. If p is a period of $f(x)$, show that np with $n = 2, 3, \ldots$ is a period of $f(x)$.

3. If $f(x)$ is a periodic function of period p, show that $f(ax)$, $a > 0$ is a periodic function of period $\frac{p}{a}$.

4. Sketch the graph of the following functions which are assumed to be periodic with period 2π and for $-\pi < x < \pi$ are given by the formulas: $f(x) = \pi - |x|$, $f(x) = |\sin(x)|$, $f(x) = e^{|x|}$.

3.3 Fourier Series

Fourier series arise from the practical task of representing a given periodic function $f(x)$ in terms of cosine and sine functions. These series are trigonometric series.

3.3.1 Euler Formulas for the Fourier coefficients

Let us assume that $f(x)$ is a periodic function of period 2π and is integrable over a period. Let us further assume that $f(x)$ can be represented by a trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{3.3}$$

Our aim is to determine the coefficients a_n, b_n of (3.3). We first recall the following result.

Lemma 3.3.1 *Let m, n be two integers. Then we have the following* **orthogonality relations***:*

$$\int_{-\pi}^{\pi} \sin(nx)dx = \int_{-\pi}^{\pi} cos(nx)dx = 0, \quad n \neq 0 \tag{3.4}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0 \tag{3.5}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \begin{cases} \pi & \textit{if } n = m \neq 0 \\ 0 & \textit{otherwise}, \end{cases} \tag{3.6}$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} \pi & \textit{if } n = m \\ 0 & \textit{otherwise}. \end{cases} \tag{3.7}$$

Proof We just prove the formula (3.6), being the proof of other similar formulas.

We apply Werner's formula:

$$\sin(nx)\sin(mx) = \frac{1}{2}[\cos((n-m)x) - \cos((n+m)x)].$$

Therefore if $n \neq m$ we have

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos((n-m)x) - \cos((n+m)x)]dx$$
$$= \frac{1}{2}\left[\frac{\sin(n-m)x}{n-m}\right]\Bigg|_{-\pi}^{\pi} - \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m}\right]\Bigg|_{-\pi}^{\pi}$$
$$= 0.$$

If $n = m$ then

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos((n-m)x) - \cos((n+m)x)]dx$$
$$= \frac{1}{2}\int_{-\pi}^{\pi} dx = \pi. \quad \square$$

1. Determination of a_0 : Integrating both sides of (3.3) from $-\pi$ to π we get

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}[a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))]dx. \tag{3.8}$$

If term-by-term integration is allowed we obtain

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nx)dx + b_n\int_{-\pi}^{\pi}\sin(nx)dx\right).$$

By applying (3.4) we get

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx. \tag{3.9}$$

2. Determination of a_n of the cosine terms : We multiply (3.3) by $\cos(mx)$, where m is a fixed positive integer and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \int_{-\pi}^{\pi}[a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))]\cos(mx)dx. \tag{3.10}$$

Integrating term-by-term, the right hand side becomes

$$a_0 \int_{-\pi}^{\pi} \cos(mx)dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx + b_n \int_{-\pi}^{\pi} \cos(mx)dx \sin(nx))dx \right]. \tag{3.11}$$

By applying Lemma 3.3.1 we finally get

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(mx)dx\,, \quad m = 1, 2, 3, \ldots \tag{3.12}$$

3. Determination of b_n of the sine terms : We multiply (3.3) by $\sin(mx)$, where m is a fixed positive integer and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))] \sin(mx)dx. \tag{3.13}$$

Integrating term-by-term the right hand side becomes

$$a_0 \int_{-\pi}^{\pi} \sin(mx)dx + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx + b_n \int_{-\pi}^{\pi} \sin(mx)dx \sin(nx))dx \right]. \tag{3.14}$$

By applying Lemma 3.3.1 we finally get

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(mx)dx\,, \quad m = 1, 2, 3, \ldots \tag{3.15}$$

We have obtained the so-called Euler formulas:

$$\begin{aligned} a_0 &= a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)dx\,, \quad n = 1, 2, 3, \ldots \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)dx\,, \quad n = 1, 2, 3, \ldots \end{aligned} \tag{3.16}$$

The number in (3.16) are called the Fourier coefficients of $f(x)$. The trigonometric series (3.2) with the coefficients given by (3.16) is called the Fourier series associated to $f(x)$.

Exercise 3.3.1 Find the Fourier coefficients of the periodic function f(x) in Fig. 3.5. The formula is

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 \le x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x).$$

Solution : It is easy to see that $a_0 = 0$. we compute a_n and b_n

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos(nx)dx + \int_{0}^{\pi} k\cos(nx)dx\right] \\ &= \frac{1}{\pi}\left[-k\frac{\sin(nx)}{n}\bigg|_{-\pi}^{0} + k\frac{\sin(nx)}{n}\bigg|_{0}^{\pi}\right] = 0. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin(nx)dx + \int_{0}^{\pi} k\sin(nx)dx\right] \\ &= \frac{1}{\pi}\left[k\frac{\cos(nx)}{n}\bigg|_{-\pi}^{0} + -k\frac{\cos(nx)}{n}\bigg|_{0}^{\pi}\right]. \end{aligned}$$

Since $\cos(-x) = \cos(x)$ and $\cos(0) = 1$ we get

$$b_n = \frac{2k}{n\pi}(1 - \cos(n\pi)) = \frac{2k}{n\pi}(1 - (-1)^n).$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin(x), \quad S_2 = \frac{4k}{\pi}(\sin(x) + \frac{1}{3}\sin(3x)), ...$$

at $x = 0$ the point of discontinuity of $f(x)$, all partial sums have the value zero, the arithmetic mean of the values of $-k$ and k of our function $(\frac{k-k}{2})$.

Furthermore,assuming that $f(x)$ is the sum of the series and setting $x = \frac{\pi}{2}$, we have

$$f(\frac{\pi}{2}) = k = \frac{4k}{\pi}(1 - \frac{1}{3} + \frac{1}{5} + \ldots),$$

thus

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \ldots = \frac{\pi}{4}.$$

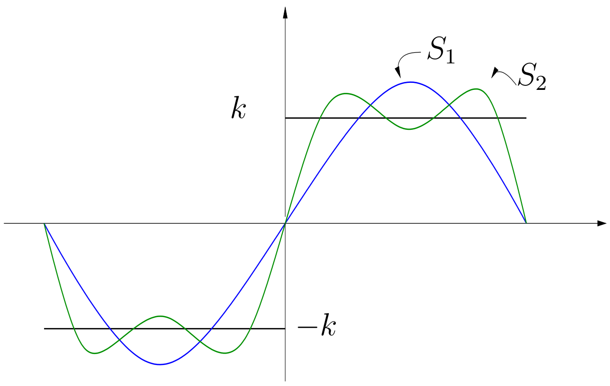

Figure 3.5: The square wave function and the first 2 partial sums of its Fourier Series

3.3.2 Convergence and sum of Fourier series

Suppose that $f(x)$ is any given periodic function of period 2π for which the integrals in (3.16) exist, for instance, $f(x)$ is continuous or merely piecewise continuous (continuous except for finitely many finite jumps in the interval of integration). Then we can compute the Fourier coefficients (3.16) of $f(x)$ and use them to form the Fourier series of $f(x)$. It would be nice if the series thus obtained converged and had the sum $f(x)$. Most functions appearing in applications are such that this is true (except at jumps of $f(x)$, which we discuss below). In this case, in which the Fourier series of $f(x)$ does represent $f(x)$, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

with an equality sign. If the Fourier series of $f(x)$ does not have the sum $f(x)$ or does not converge, one still writes

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where the symbol $\sim$ indicates that the trigonometric series on the right has the Fourier coefficients of $f(x)$ as its coefficients, so it is the Fourier series of $f(x)$.

The class of functions that can represented by Fourier series is surprisingly large.

Theorem 3.3.1 (Representation by a Fourier Series) *If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \leq x \leq \pi$ and has a left-hand derivative and right-hand derivative at each point of that interval, then the Fourier series of $f(x)$ is convergent. Its sum is $f(x)$ at each point x_0 at which $f(x)$ is continuous and the sum of the series is the average of the left- and right-limits of $f(x)$ at each point x_0 at which $f(x)$ has a jump.*[1]

3.3.3 Functions of any period $p = 2L$

The functions considered so far had period 2π. Of course, in applications periodic functions will generally have other periods. But the transition from period $p = 2\pi$ to period $p = 2L$ is simple. It ammounts to a contraction of scale on the axis.

If a function $f(x)$ of period $p = 2L$ has Fourier series, then this series is

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x) \tag{3.17}$$

with the Fourier coefficients of $f(x)$ given by the Euler formulas

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)dx\,, \quad n = 1, 2, 3, \ldots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx\,, \quad n = 1, 2, 3, \ldots \end{aligned} \tag{3.18}$$

Example 3.3.1 Find the Fourier series of the π periodic function $f(x) = \sin^2(x)$.

Solution: In this case we can use the following trigonometric formula:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x).$$

Hence we have

$$\sin^2 x = \frac{1 - \cos(2x)}{2}. \tag{3.19}$$

In this case $2L = \pi$ and the Fourier series of $\sin^2 x$ has the form:

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(2nx) + b_n \sin(2nx)).$$

By comparing with (3.19) we find that $b_n = 0$ for every $n \geq 1$, $a_0 = -1/2$, $a_2 = -1/2$ and $a_n = 0$, for every $n \geq 2$.

Exercise 3.3.2 Find the Fourier series of the periodic function $f(x)$ of period $p = L$.

1. $f(x) = -1$ if $-1 < x < 0$, $f(x) = 1$, if $0 \leq x < 1$, $p = 2L = 2$.
2. $f(x) = |x|$, if $-2 < x < 2$, $p = 2L = 4$.
3. $f(x) = 1 - x^2$, if $-1 < x < 1$, $p = 2L = 2$.

3.3.4 Even and odd functions.

Definition 3.3.1 *a) A function $f(x)$ is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. The graph of $f(x)$ is symmetric with respect to the $y-axis$.*

b) A function $f(x)$ is odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. The graph of $f(x)$ is symmetric with respect to the origin.

Key facts :

1. If $f(x)$ is an even function then

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx. \tag{3.20}$$

2. If $f(x)$ is an odd function then

$$\int_{-L}^{L} f(x)dx = 0. \tag{3.21}$$

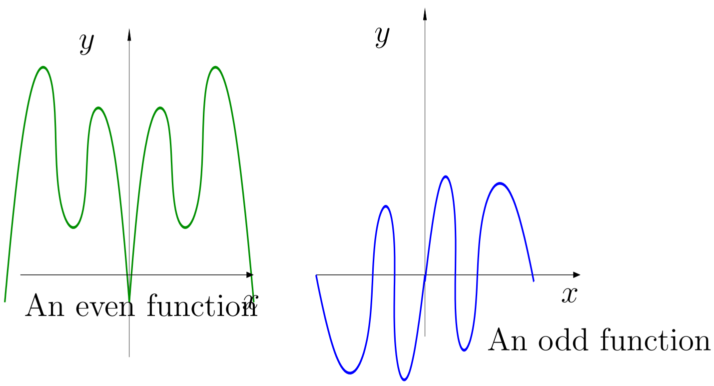

Figure 3.6: Even and Odd functions

3. The product of an even and an odd function is odd.
4. If $f(x)$ is even then $f(x)\sin(\frac{n\pi}{L}x)$ is odd and thus $b_n = 0$.
5. If $f(x)$ is odd then $f(x)\cos(\frac{n\pi}{L}x)$ is odd and thus $a_n = 0$
6. The Fourier series of an even function of a period $2L$ is a Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x)$$

with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi}{L}x)dx\,, \quad n = 1, 2, 3, \ldots$$

7. The Fourier series of an odd function of a period $2L$ is a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)dx\,, \quad n = 1, 2, 3, \ldots$$

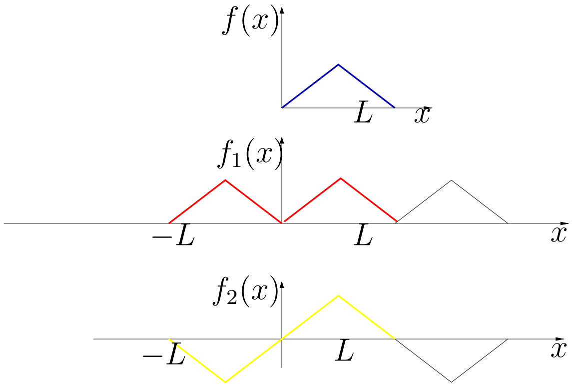

Figure 3.7: Even and odd periodic extension

3.3.5 Half-range expansions

In applications we often want to employ a Fourier series for a function f that is given on a some interval , say $0 \leq x \leq L$. This function can be the displacement of a violin string of length L or the temperature in a metal bar of lenght L. For our function f we can consider either a Fourier cosine series which represents the even periodic extension f_1 of f or a Fourier sine series which represents the odd periodic extension f_2 of f. This motivates the name half-range expansion: f is given only on half the range, half the interval of periodicity of lenght $2L$.

Example 3.3.2 Triangle and its half-range expansion (Fig. 3.7) Find the two half-range expansions of the function

$$f(x) = \begin{cases} \frac{2}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2}{L}(L-x) & \text{if } \frac{L}{2} < x < L. \end{cases}$$

Solution. (a) Even periodic extension. We obtain

$$a_0 = \frac{1}{L}\left[\frac{2}{L}\int_0^{L/2} x dx + \frac{2}{L}\int_{L/2}^{L}(L-x)dx\right] = \frac{1}{2}$$

$$a_n = \frac{2}{L}\left[\frac{2}{L}\int_0^{L/2} x\cos(\frac{n\pi}{L}x)dx + \frac{2}{L}\int_{L/2}^{L}(L-x)\cos(\frac{n\pi}{L}x)dx\right]$$

$$= \frac{4}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus

$$a_n = \begin{cases} 0 & \text{if } n = 2k+1 \\ 0 & \text{if } n = 4k \\ -\frac{16}{n^2\pi^2} & \text{if } n = 4k+2. \end{cases}$$

Hence the first half-range expansion of $f(x)$ is

$$f(x) = \frac{1}{2} - \frac{16}{\pi^2}\left(\frac{1}{2^2}\cos(\frac{2\pi}{L}x) + \frac{1}{6^2}\cos(\frac{6\pi}{L}x) + \dots\right).$$

The Fourier cosine series represents the even periodic extension of the given function $F(x)$ of period $2L$.

(b) Odd periodic extension. Similarly we obtain

$$b_n = \frac{8}{n^2\pi^2}\sin(\frac{n\pi}{2}) = \begin{cases} 0 & \text{if } n = 2k \\ \frac{8}{n^2\pi^2} & \text{if } n = 4k+1 \\ -\frac{8}{n^2\pi^2} & \text{if } n = 4k+3. \end{cases}$$

Hence the other half-range expansion of $f(x)$ is

$$f(x) = \frac{8}{\pi^2}\left(\frac{1}{1^2}\sin(\frac{\pi}{L}x) - \frac{1}{3^2}\sin(\frac{3\pi}{L}x) + \dots\right).$$

This series represents the odd periodic extension of $f(x)$ of period $2L$.

Exercise 3.3.3 Find the two half-range expansions of the function $f(x) = \sin^2(x)$, $x \in (0, \pi/2)$. Draw also a picture of the even and odd periodic extensions of $\sin^2(x)$.

3.4 Complex Fourier Series

In this section we show that the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \tag{3.22}$$

can be written in complex form, which sometimes simplifies calculations. We recall that

$$e^{ix} = \cos(x) + i\sin(x) \tag{3.23}$$

$$e^{-ix} = \cos(x) - i\sin(x) \tag{3.24}$$

From (3.23) and (3.24) it follows that

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}) \tag{3.25}$$

$$\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix}) \tag{3.26}$$

Thus for all n integer

$$\begin{aligned} a_n \cos(nx) + b_n \sin(nx) &= \frac{1}{2}a_n(e^{inx} + e^{-inx}) + \frac{1}{2i}b_n(e^{inx} - e^{-inx}) \\ &= \frac{1}{2}e^{inx}(a_n - ib_n) + \frac{1}{2}e^{-inx}(a_n + ib_n). \end{aligned}$$

Writing $a_0 = c_0$, $\frac{a_n - ib_n}{2} = c_n$ and $\frac{a_n + ib_n}{2} = k_n$, then (3.22) becomes

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n e^{-inx}). \tag{3.27}$$

We get

$$\begin{aligned} c_n &= \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(cos(nx) - i\sin(nx))dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ k_n &= \frac{a_n + ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(cos(nx) + i\sin(nx))dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx. \end{aligned}$$

By writing $k_n = c_{-n}$ we finally obtain

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with coefficients

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx\,, \quad n = 0, \pm1, \pm2, \pm3, \ldots$$

This is the so-called **complex form of the Fourier series**. The c_n are called the **complex Fourier coefficients** of $f(x)$.

For a function of period $2L$ our reasoning gives the complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}$$

with coefficients

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i\frac{n\pi}{L}x}dx\,, \quad n = 0, \pm1, \pm2, \pm3, \ldots$$

Exercise 3.4.1 Find the complex Fourier Series of $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ and obtain from it the usual Fourier series.

Solution.

Since $\sin n\pi = 0$ we have $e^{in\pi} = \cos n\pi = (-1)^n$. With this by integration we get

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx = \frac{1}{2\pi}\frac{1}{1-in}e^{x-inx}\,\Big|_{x=-\pi}^{\pi} \\ &= \frac{1}{2\pi}\frac{1}{1-in}(e^{\pi} - e^{-\pi})(-1)^n. \end{aligned}$$

On the right,

$$\frac{1}{1-in} = \frac{1+in}{1+n^2}, \quad e^{\pi} - e^{-\pi} = 2\sinh\pi.$$

Hence the complex Fourier series is

$$e^x = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{1+in}{1+n^2} e^{inx} \quad (-\pi < x < \pi). \tag{3.28}$$

From this we derive the real Fourier series. We have

$$(1+in)e^{inx} = (\cos nx - n \sin nx) + i(n \cos nx + \sin nx). \tag{3.29}$$

Now (3.28) also we have a corresponding term with $-n$ instead of n. Since $\cos(-nx) = \cos(nx)$ and $\sin(-nx) = -\sin(nx)$ we obtain in this term

$$(1-in)e^{-inx} = (\cos nx - n \sin nx) - i(n \cos nx + \sin nx). \tag{3.30}$$

If we add (3.29) and (3.30), the imaginary part cancel. Hence their sum is

$$2(\cos nx - n \sin nx), \quad n = 1, 2, \ldots$$

For $n = 0$ we get 1 because there is only one term. Hence the real Fourier series is

$$e^x = \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (\cos(nx) - n \sin(nx))$$

where $-\pi < x < \pi$.

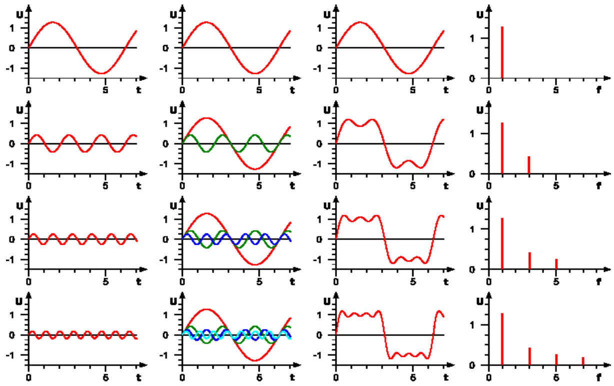

Figure 3.8: The Fourier Series of the Square Signal

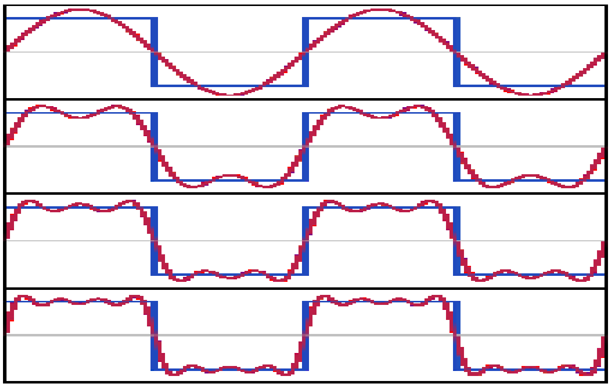

Figure 3.9: The first four partial sums of the Fourier series for the Square Signal

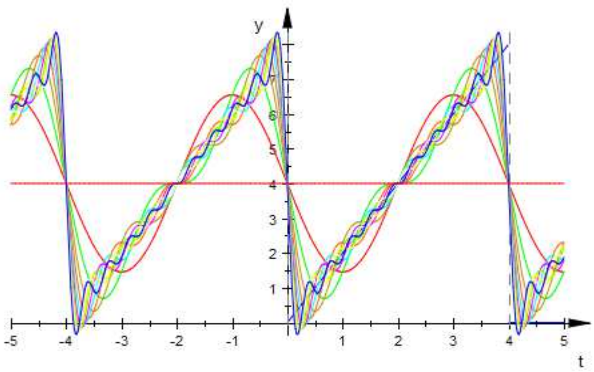

Figure 3.10: The Fourier Series of the Sawtooth Wave

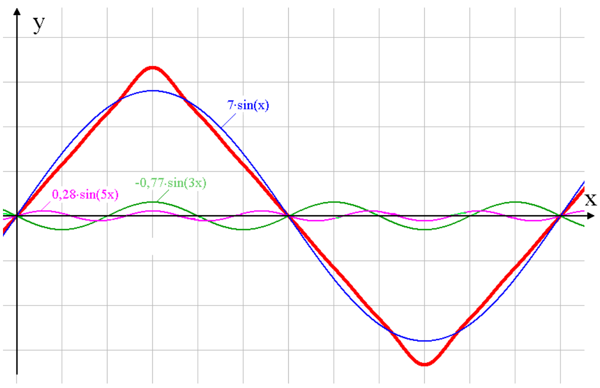

Figure 3.11: The Fourier Series of the Triangle Signal

Chapter 4

The method of separation of variables

In this chapter we will describe a method for solving initial-boundary value problems associated to linear and homogeneous PDEs and homogeneous boundary conditions.

4.1 Description of the method

Such a method consists in the following three main steps.

Step 1. One searches for solutions of the homogeneous problem, which are called product solutions or separated solutions. These solutions have the following special form

$$u(x, t) = X(x)T(t). \tag{4.1}$$

We notice that X is a function of x only and T is a function of t.

In general such solutions should satisfy certain additional conditions. It turns out that X and T satisfy suitable linear ordinary differential equations (ODEs) which are easily derived from the given PDE.

Step 2. We use a generalization of the superposition principle to generate out of the separated solutions a more general solution of the homogeneous PDE, in the form of an infinite series of product solutions.

Step 3. We compute the coefficients of the series to satisfy the initial conditions.

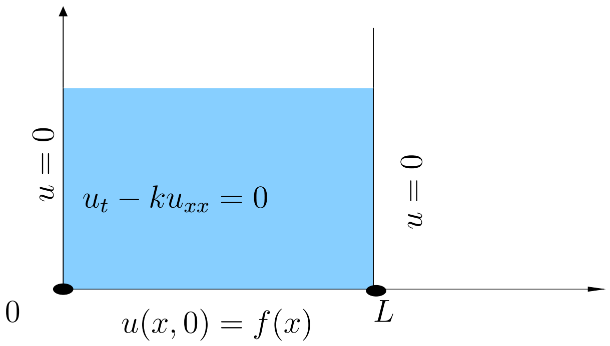

Figure 4.1: The initial boundary conditions for the heat equation and the domain

4.2 Heat equation with homogeneous boundary conditions

We consider the following initial-boundary value problem associated to the heat equation:

$$\begin{cases} u_t - ku_{xx} = 0, & 0 < x < L, \ t > 0 \\ u(0,t) = u(L,t) = 0, & t \geq 0 \\ u(x,0) = f(x), & 0 \leq x \leq L \end{cases} \tag{4.2}$$

where f is a given initial condition and $k > 0$. We assume the compatibility conditions

$$f(0) = f(L) = 0.$$

The equation and the domain are drawn in Fig. 4.1.

The problem defined above corresponds to the evolution of the temperature $u(x,t)$ in a homogeneous one-dimensional heat conducting rod of length L, whose initial temperature (at time $t = 0$) is known and is such that its two ends are immersed in a zero temperature bath.

The problem (4.2) is an initial boundary value problem that is linear and homogeneous. Recall that the condition on the boundary is called Dirichlet condition. We apply the method of separation of variables described above.

We start by looking for solutions of the heat equation that satisfy the boundary conditions that have the special form

$$u(x,t) = X(x)T(t). \tag{4.3}$$

At this step we do not take into account the initial condition $u(x,0) = f(x)$. Obviously we are not interested in the zero solution $u(x,t) = 0$. Therefore we seek functions X and T that do not vanish identically. By plugging the solution into the PDE we get

$$XT_t = kX_{xx}T.$$

Now we carry out a simple but decisive step: the separation of variables step. We move to one side of the PDE all the functions that depend only on x and to the other side the functions that depend only on t. We thus write

$$\frac{T_t}{kT} = \frac{X_{xx}}{X}.$$

Since x, t are independent variables, there exists a constant denoted by λ (which is called the separation constant) such that

$$\frac{T_t}{kT} = \frac{X_{xx}}{X} = -\lambda. \tag{4.4}$$

Equation (4.4) leads to the following ODEs:

$$\begin{cases} \dfrac{d^2X}{d^2x} = -\lambda X, & 0 < x < L \\ \dfrac{dT}{dt} = -\lambda kT, & t > 0. \end{cases} \tag{4.5}$$

The above ODEs are coupled only by the separation constant λ. The function u satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ if and only if

$$u(0,t) = X(0)T(t) = 0 \quad \forall t > 0$$
$$u(L,t) = X(L)T(t) = 0 \quad \forall t > 0.$$

The above two conditions are satisfied if and only if either $T(t) = 0$ for all $t > 0$ (which gives the trivial solution) or $X(0) = X(L) = 0$ (which represents the interesting case).

Problem for X:
The function X should be a solution of the boundary value problem

$$\begin{cases} \dfrac{d^2X}{d^2x} = -\lambda X, & 0 < x < L \\ X(0) = X(L) = 0. \end{cases} \tag{4.6}$$

The problem (4.6) is called eigenvalue problem. A nontrivial solution of the problem (4.6) is called an eigenfunction of (4.6) with an eigenvalue λ. It is known that the general solution of the second order linear ODE in (4.6) is of the form

1. $X(x) = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x}$, with $\alpha, \beta \in I\!R$, if $\lambda < 0$.
2. $X(x) = \alpha + \beta x$, with $\alpha, \beta \in I\!R$, if $\lambda = 0$.
3. $X(x) = \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x)$, with $\alpha, \beta \in I\!R$, if $\lambda > 0$.

Next we examine separately the three cases.

Negative eigenvalue: $\lambda < 0$.

The solution is $X(x) = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x}$. By imposing $X(0) = X(L) = 0$ we get $\alpha = \beta = 0$. The system (4.6) does not admit negative eigenvalues.

Zero eigenvalue: $\lambda = 0$.

The solution is $X(x) = \alpha + \beta x$. By imposing $X(0) = X(L) = 0$ we get $\alpha = \beta = 0$. The system (4.6) does not admit zero eigenvalues.

Positive eigenvalue: $\lambda > 0$.

The solution is $X(x) = \alpha \cos(\sqrt{\lambda}x) + \beta \sin(\sqrt{\lambda}x)$. The condition $X(0) = 0$ gives $\alpha = 0$. The boundary condition $X(L) = 0$ implies either $\beta = 0$ or $\sin(\sqrt{\lambda}L) = 0$, which is the interesting case. Therefore $\sqrt{\lambda}L = n\pi$, $n > 0$. Hence λ is an eigenvalue if and only if

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots \tag{4.7}$$

The corresponding eigenfunctions are

$$X_n(x) = \sin(\frac{n\pi x}{L}) \tag{4.8}$$

and they are defined up to a multiplicative constant.

Thus the set of all solutions of (4.6) is an infinite sequence of eigenfunctions, each associated with a positive eigenvalue. We use the notation $X_n(x) = \sin(\lambda_n x)$ where for all $n > 0$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$.

Problem for T:

The general solution of $\frac{dT}{dt} = -\lambda kT$ is $T(t) = Be^{-k\lambda t}$. Substituting λ_n we obtain the sequence $T_n(t) = B_n e^{-k\lambda_n t}$.

The sequence of separated solutions is given by

$$u_n(x,t) = X_n(x)T_n(t) = B_n \sin(\frac{n\pi x}{L})e^{-k\lambda_n t}.$$

The superposition principle implies that any finite linear combination

$$u(x,t) = \sum_{n=1}^{N} B_n \sin(\frac{n\pi x}{L})e^{-k\lambda_n t}$$

of the separated solutions is still a solution of the heat equation and the Dirichlet boundary conditions.

Consider now the initial condition. If

$$f(x) = \sum_{n=1}^{M} C_n \sin(\frac{n\pi x}{L}),$$

then the solution of the original problem (4.2) is given by

$$u(x,t) = \sum_{n=1}^{M} C_n \sin(\frac{n\pi x}{L})e^{-k\lambda_n t}.$$

Hence we are able to solve the problem for a certain family of initial conditions.

Suppose now that $f(x)$ admits the following Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}).$$

In this case a natural candidate for a solution of (4.2) is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}) e^{-k\lambda_n t}. \tag{4.9}$$

Formally it satisfies all the required conditions of the problem (4.2). Actually one should verify that u given by (4.9) is differentiable once with respect to t and twice with respect to x and we can differentiate the series term by term.

Exercise 4.2.1 Solve the initial-boundary value problem

$$\begin{cases} u_t - u_{xx} = 0, & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t \geq 0 \\ u(x,0) = f(x), & 0 \leq x \leq \pi \end{cases} \tag{4.10}$$

where

a) $f(x) = 5\cos(2x)\sin(3x)$,

b)

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \pi/2 \\ \pi - x & \text{if } \pi/2 \leq x \leq \pi. \end{cases} \tag{4.11}$$

Sketch of solution: In this case $k = 1$ and $L = \pi$. The solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 t}. \tag{4.12}$$

The coefficients are determined by the initial condition. Setting $t = 0$ in (4.12) we obtain

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx.$$

a) Consider the case $f(x) = 5\cos(2x)\sin(3x)$.

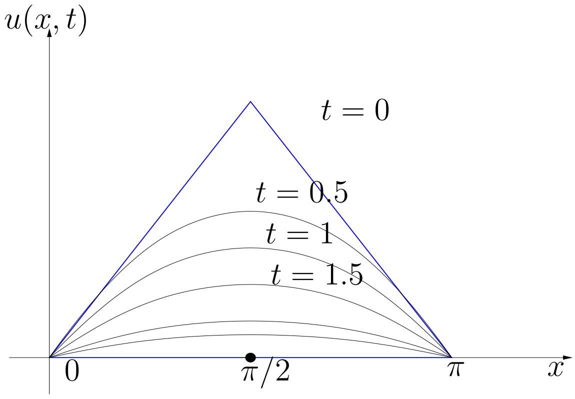

Figure 4.2: The temperature in Exercise 4.2.1

We recall the following trigonometric formula (Werner formula):

$$\cos(\alpha)\sin(\beta) = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}.$$

By applying this formula to $\alpha = 3x$ and $\beta = 2x$ we get

$$5\cos(2x)\sin(3x) = 5\frac{\sin(5x) - \sin(x)}{2}.$$

In this case we can chose directly $b_2 = -5/2$, $b_5 = 5/2$ and $b_n = 0$ for all $n \neq 2, 5$.

b) Consider the function (4.11). In this case we can show that $b_n = 0$ when n is even. When $n = 2j+1$ is odd, we have

$$b_{2j+1} = (-1)^j \frac{4}{\pi(2j+1)^2}.$$

(The computations are left to the reader). The final solution is

$$u(x,t) = \sum_{n=0}^{\infty} (-1)^n \frac{4}{\pi(2n+1)^2} e^{-(2n+1)^2 t} \sin(2n+1)x.$$

Figure 4.2 shows the initial heat distribution plotted in blue. The black curves show the temperature distrubution after $t = 0.25$. The temperature on the rod decreases with time, approaching the constant temperature of

0^o as $t \to \infty$. Heat flows from the hot part in the middle of the rod toward the endpoints. However, the end-points are kept at a fixed temperature, so heat is constantly being removed from the rod at the endpoints. As the heat is removed, the rod approaches the temperature of 0 degree.

4.3 Wave equation with homogeneous boundary conditions

We now apply the method of separation of variables to solve the problem of a vibrating string without external forces. We consider the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < L, \ t > 0 \\ u(0,t) = u(L,t) = 0, & t \geq 0 \\ u(x,0) = f(x), & 0 \leq x \leq L \\ u_t(x,0) = g(x), & 0 \leq x \leq L \end{cases} \tag{4.13}$$

where f, g are given functions and $c > 0$. We assume the compatibility conditions

$$f(0) = f(L) = 0 = g(0) = g(L).$$

At the first stage we compute nontrivial separated solutions of the wave equations, i.e. solutions of the form

$$u(x,t) = X(x)T(t).$$

Substituting into the wave equation yields:

$$X(x)T''(t) = c^2 X''(x)T(t),$$

or, upon dividing by $c^2 T(t)X(x)$

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''}{X(x)}.$$

Setting each term equal to a constant $-\lambda$, we obtain the two ODEs

$$\begin{cases} \dfrac{d^2X}{d^2x} = -\lambda X, & 0 < x < L \\ \dfrac{d^2T}{d^2t} = -\lambda c^2 T, & t > 0. \end{cases} \tag{4.14}$$

Next we substitute $u(x,t) = X(x)T(t)$ into the boundary conditions in (4.20) to obtain

$$u(0,t) = X(0)T(t) = 0$$
$$u(L,t) = X(L)T(t) = 0\,,$$

which gives $X(0) = X(L) = 0$. Therefore we are led to the boundary value problem

$$\begin{cases} \dfrac{d^2X}{d^2x} = -\lambda X, \qquad 0 < x < L \\ X(0) = X(L) = 0. \end{cases} \tag{4.15}$$

It is exactly the same as the heat flow example above. The eigenvalues and the eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \ n = 1, 2, \ldots$$

Solving the ODE for $T(t)$,we get

$$T(t) = C\sin\left(\frac{n\pi ct}{L}\right) + D\cos\left(\frac{n\pi ct}{L}\right).$$

Therefore we have constructed infinitely many product solutions of the wave equation of the form

$$u_n(x,t) = \left[C_n\sin\left(\frac{n\pi ct}{L}\right) + D_n\cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right), \ n = 1, 2, \ldots$$

These product solutions $u_n(x,t)$ represent modes of vibrations. The temporal part is periodic in time with period $\frac{2L}{nc}$; the spatial part is a standing wave with frequency $\frac{nc}{2L}$. These product solutions also satisfy the boundary conditions in (4.20) but do not satisfy the given initial conditions. So we form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty}\left[C_n\sin\left(\frac{n\pi ct}{L}\right) + D_n\cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \tag{4.16}$$

and select the constants C_n and D_n such that the initial conditions hold.

Thus we require

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

The right side is the Fourier sine series of the function $f(x)$ on the interval $(0, L)$. Therefore, the coefficients D_n are the Fourier coefficients given by

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx. \tag{4.17}$$

To apply the other condition $u_t(x,0) = g(x)$ we need to calculate the time derivative of $u(x,t)$. We obtain

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left[C_n \cos\left(\frac{n\pi ct}{L}\right) - D_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right).$$

Thus

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} C_n \sin\left(\frac{n\pi x}{L}\right) = g(x).$$

Again the right side is the Fourier sine series of the function $g(x)$ on the interval $(0, L)$. Therefore, the coefficients $\frac{n\pi c}{L}C_n$ are the Fourier coefficients given by

$$\frac{n\pi c}{L} C_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx\,,$$

namely

$$C_n = \frac{2}{nc\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx. \tag{4.18}$$

Therefore the solution is given by the infinite series (4.16) where the coefficients are given by (4.17) and (4.18).

Let us interpret these results in the context of musical stringed instruments (with fixed ends). The vertical displacement is composed of a linear combination of simple product solutions

$$u_n(x,t) = \left[C_n \sin\left(\frac{n\pi ct}{L}\right) + D_n \cos\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right),\ n = 1, 2, \ldots$$

These are called the normal modes of vibration. The intensity of the sound produced depends on the amplitude $\sqrt{C_n^2 + D_n^2}$.

The time dependence is simple harmonic with frequency (number of oscillations in a second) equaling $\omega_n = \frac{cn}{2L}$. The sound produced consists of the superpositions of these infinite number of frequencies. The normal mode $n = 1$ is called the first harmonic or fundamental mode. In the case of a vibrating string the fundamental mode has frequency equal to $\omega = \frac{c}{2L}$. The other frequencies are multiples of the fundamental one.

Exercise 4.3.1 Let a vibrating string satisfy:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < L, \ t > 0 \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \\ u_t(x,0) = 0 & 0 \leq x \leq L. \end{cases} \tag{4.19}$$

Show that

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right],$$

where F is the odd periodic extension of $f(x)$.

Hints:

a) For all x, $F(x) = \sum_{n=1}^{+\infty} D_n \sin \frac{n\pi x}{L}$.

b) $\sin(a)\cos(b) = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$.

Exercise 4.3.2 Let a vibrating string satisfy:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < L, \ t > 0 \\ u(0,t) = u(L,t) = 0, & t \geq 0 \\ u(x,0) = 0, & 0 \leq x \leq L \\ u_t(x,0) = g(x), & 0 \leq x \leq L. \end{cases} \tag{4.20}$$

Show that

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(y)dy,$$

where G is the odd periodic extension of $g(x)$.

Hints:

a) For all x, $G(x) = \sum_{n=1}^{+\infty} C_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$.

b) $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$.

4.4 Nonhomogeneous problems

The problems we have considered so far are homogeneous: the PDE is linear homogeneous and the boundary conditions are of the form $u(a, t) = 0$. Such boundary conditions are called homogeneous. The essential feature of homogeneous problems is the Superposition Principle. If either the equation or the boundary conditions are not homogeneous the *method of separation of variables* does not work. But it holds the following principle (that we have already applied): The general solution of a nonhomogeneous boundary value problem is the sum of the general solution of the corresponding homogeneous problem (i.e. the problem where one replaces zero on the right hand side of both the equation and the boundary conditions) and a particular solution of the homogeneous one (that one has to guess in each particular case).

Example 4.4.1 We consider the problem

$$\begin{cases} u_t - ku_{xx} = 0, & 0 < x < L, \ t > 0 \\ u(0, t) = T_0 & t \geq 0 \\ u(L, t) = T_L, & t \geq 0 \\ u(x, 0) = f(x), & 0 \leq x \leq L \end{cases} \tag{4.21}$$

where f is a given initial condition and $k > 0$.

The strategy is to split the problem into parts. We first find the so called **steady-state** temperature that satisfies the boundary condition in (4.21). A **steady-state** temperature is one that does not depend on time. Then $u_t = 0$, so the heat equation simplifies to $u_{xx} = 0$. Hence we are looking for a function $u_s(x)$ defined in $0 \leq x \leq L$ such that

$$\begin{cases} \dfrac{\partial^2 u_s}{\partial x^2}(x) = 0, & 0 < x < L \\ u_s(0) = T_0 & \\ u_s(L) = T_L. & \end{cases} \tag{4.22}$$

The solution of (4.22) (the steady-state temperature) is given by

$$u_s(x) = (T_L - T_0)\frac{x}{L} + T_0.$$

It remains now to find $v = u - u_s$. It will be a solution of the heat equation, since both u and u_s are, and the heat equation is linear. The boundary and initial conditions that v satisfies can be calculated from those of u and u_s. Thus v must satisfy

$$\begin{cases} v_t - ku_{xx} = 0, & 0 < x < L,\ t > 0 \\ v(0,t) = 0 & t \geq 0 \\ v(L,t) = 0, & t \geq 0 \\ v(x,0) = g(x) = f(x) - u_s(x), & 0 \leq x \leq L. \end{cases} \tag{4.23}$$

The most important fact is that the boundary conditions for v are homogeneous and we can apply the method of separation of variables to determine it.

Having found the steady-temperature u_s and the temperature v, the solution to the original problem is $u(x,t) = u_s(x) + v(x,t)$.

Exercise 4.4.1 Find the solution to the initial-boundary value problem

$$\begin{cases} u_t - 2u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u(0,t) = 0 & t \geq 0 \\ u(\pi,t) = 2 & t \geq 0 \\ u(x,0) = \sin^2(x) & 0 \leq x \leq \pi. \end{cases} \tag{4.24}$$

Chapter 5

The Fourier Transform and Applications

5.1 Motivation from Fourier Series Identity

In solving boundary value problems on a finite interval $[-L, L]$ we have seen that we can use the complex form of a Fourier series:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}.$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}} dx.$$

The entire region of interest $-L < x < L$ is the domain of integration. We will extend these ideas to functions defined for $-\infty < x < \infty$.

The **Fourier series identity** follows by eliminating c_n (and using a dummy integration variable ξ to distinguish it from the space variable x):

$$f(x) = \sum_{-\infty}^{\infty} \left[\frac{1}{2L} \int_{-L}^{L} f(\xi) e^{-\frac{in\pi\xi}{L}} d\xi \right] e^{\frac{in\pi x}{L}}. \tag{5.1}$$

For periodic functions, $-L < x < L$, the allowable wave numbers ω (number of waves in 2π distance) are the infinite set of discrete values, $\omega = \frac{n\pi}{L} = 2\pi \frac{n}{2L}$.

The distance between successive values of the wave number is

$$\Delta\omega = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}.$$

From (5.1)

$$f(x) = \sum_{-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(\xi)e^{i\omega\xi}d\xi e^{-i\omega x}. \tag{5.2}$$

We observe that functions defined in $(-\infty, \infty)$ may be thought of in some sense as periodic functions with an infinite period.

We have $\Delta\omega \to 0$ as $L \to \infty$. Thus all possible wave numbers are allowable. The function $f(x)$ should be represented as "sum" of waves of all possible wave lengths. Equation (5.2) represents a sum of rectangles (starting from $\omega = -\infty$ and going to $\omega = +\infty$) of base $\Delta\omega$ and height

$$\frac{1}{2\pi} \int_{-L}^{L} f(\xi)e^{-i\omega\xi}d\xi\, e^{i\omega x}.$$

We expect as $L \to +\infty$ that the areas of the rectangles approach the Riemann sum. Since $\Delta\omega \to 0$ as $L \to \infty$ the equation (5.2) becomes the **Fourier integral identity**

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{+\infty} f(\xi)e^{-i\omega\xi}d\xi \right] e^{i\omega x} d\omega. \tag{5.3}$$

5.2 Definition and main properties

Definition 5.2.1 *Let f be an absolutely integrable function on the real line, namely $\int_{-\infty}^{+\infty} |f(x)|dx < +\infty$, (in short we write $f \in L^1(\mathbb{R})$). The Fourier transform of f is the function $\mathcal{F}[f] = \hat{f} \colon \mathbb{R} \to \mathbb{C}$ defined by the formula*

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{-\infty}^{+\infty} f(x)e^{-ix\xi}dx, \quad \forall \xi \in \mathbb{R}. \tag{5.4}$$

We observe the analogy with the definition of the Fourier coefficients in the complex Fourier series associated to a periodic function f:

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x)e^{-i\frac{n\pi}{L}x}dx.$$

There is no standard convention on how to define the Fourier transform. Some put a factor $\frac{1}{2\pi}$ or $\frac{1}{\sqrt{2\pi}}$ in front of the integral and some have even a factor of 2π in the exponential.

Definition 5.2.2 *Let $f \in L^1(I\!R))$. The inverse Fourier transform of F is the function $\mathcal{F}^{-1}[f] \colon I\!R \to \mathbb{C}$ defined by the formula*

$$\mathcal{F}^{-1}[f](x) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) e^{ix\xi} d\xi, \quad \forall x \in I\!R. \tag{5.5}$$

The following *Inversion Formula* holds.

Theorem 5.2.1 *Let $f \in L^1(I\!R)$ be such that $\hat{f} \in L^1(I\!R)$ ($\int_{-\infty}^{+\infty} |\hat{f}(w)| dw < +\infty$). Then f is continuous and the following holds*

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(w)\, e^{iwt} dw.$$

5.2.1 Main properties

In this section we list some properties of the Fourier Transform.

1. Linearity. The Fourier transform is linear:

$$\mathcal{F}[\alpha f + \beta g] = \alpha \mathcal{F}[f] + \beta \mathcal{F}[f]$$

for every f, g absolutely integrable, and $\alpha, \beta \in I\!R$.

2. Symmetrization. For every function $f \colon I\!R \to R$ we define, $\tilde{f}(x) := f(-x)$. which is called the *symmetrized* of f. Then

$$\begin{aligned} \mathcal{F}[\tilde{f}](\xi) &= \int_{-\infty}^{+\infty} \tilde{f}(x) e^{-ix\xi} dx = \int_{-\infty}^{+\infty} f(-x) e^{-ix\xi} dx \\ &= \int_{+\infty}^{-\infty} f(x) e^{-i(-x)\xi} d(-x) = \int_{-\infty}^{+\infty} f(x) e^{-ix(-\xi)} dx = \mathcal{F}[f](-\xi). \end{aligned}$$

3. Scaling property.

Let $\lambda \neq 0$ and $f \in L^1(I\!R)$, then the Fourier transform of $g(x) = f(\lambda x)$ is

$$\mathcal{F}[g](\xi) = \frac{1}{|\lambda|}\mathcal{F}[f](\frac{\xi}{\lambda}).$$

Proof. We suppose first $\lambda > 0$:

$$\mathcal{F}[g](\xi) = \int_{-\infty}^{+\infty} f(\lambda x)e^{-ix\xi}dx = \int_{-\infty}^{+\infty} f(\theta)e^{-i\theta\frac{\xi}{\lambda}}\frac{d\theta}{\lambda};$$

if $\lambda < 0$ then we exchange the extremes of integration. □

4. Translation.

For every $f \in L^1(I\!R)$ and $a \in I\!R$ we define $\tau_a f(x) = f(x+a)$. We have

$$\mathcal{F}[\tau_a f](\xi) = e^{ia\xi}\mathcal{F}[f](\xi).$$

Proof.

$$\mathcal{F}[\tau_a f](\xi) = \int_{-\infty}^{+\infty} f(x+a)e^{-ix\xi}dx = \int_{-\infty}^{+\infty} f(\theta)e^{-i(\theta-a)\xi}d\theta = e^{ia\xi}\mathcal{F}[f](\xi). \ \square.$$

5. Derivative.

(i) If f, $tf \in L^1(I\!R)$ then the Fourier transform of f is differentiable and

$$D(\mathcal{F}[f])(\xi) = \int_{-\infty}^{+\infty} (-ix)f(x)e^{-ix\xi}dx.$$

More generally if the map $x \mapsto x^m f(x)$ is in $L^1(I\!R)$ then the Fourier transform of f is differentiable m times and for all $k = 1, \ldots, m$ we have

$$D^k(\mathcal{F}[f])(\xi) = \int_{-\infty}^{+\infty} (-ix)^k f(x)e^{-ix\xi}dx.$$

(ii) If the k-derivative $D^k f \in L^1(I\!R)$ then

$$\mathcal{F}[D^k f](\xi) = (i\xi)^k\mathcal{F}[f].$$

5. Convolution. The convolution of f and g is defined by

$$(f * g)(x) := \int_{I\!R} f(x-y)g(y)dy.$$

Observe that $(f * g)(x) = (g * f)(x)$. One has

$$\mathcal{F}[f * g](\xi) = \hat{f}(\xi)\hat{g}(\xi) \text{ and } \mathcal{F}[fg](\xi) = \frac{1}{2\pi}(\hat{f} * \hat{g})(\xi).$$

We resume the main properties of the Fourier transform in the following table:

Function	**Fourier Transform**	**Remarks**
$f(x)$	$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-ix\xi}dx$	Definition
$af(x) + bg(x)$	$a\hat{f}(\xi) + b\hat{g}(\xi)$	Linearity
$f(x-a)$	$e^{-ia\xi}\hat{f}(\xi)$	Shift in the domain.
$f(ax)$	$\frac{1}{\lvert a\rvert}\hat{f}(\frac{\xi}{a})$	Scaling in the domain
$\hat{f}(x)$	$2\pi f(-\xi)$	Duality
$\frac{d^n}{dx^n}f(x)$	$(i\xi)^n\hat{f}(\xi)$	
$x^n f(x)$	$i^n \frac{d^n}{d\xi^n}\hat{f}(\xi)$	
$f * g(x)$	$\hat{f}(\xi)\hat{g}(\xi)$	$f * g$ is the convolution of f and g
$f(x)g(x)$	$\frac{1}{2\pi}(\hat{f} * \hat{g})(\xi)$	

5.3 Computation of some common Fourier transforms

5.3.1 Characteristic function of an interval

Let $T > 0$, the Fourier transform of

$$f(x) = \mathbb{1}_{[-T/2,T/2]}(x) = \begin{cases} 1 & \text{if } x \in [-T/2, T/2] \\ 0 & \text{otherwise} \end{cases}$$

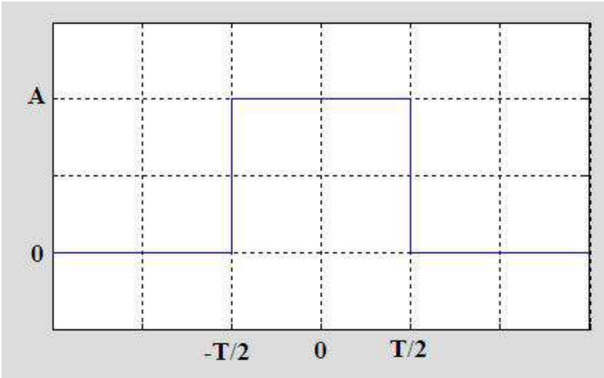

Figure 5.1: Characteristic function of an interval or box function

is given by

$$\begin{aligned}\mathcal{F}[\mathbb{1}_{[-T/2,T/2]}](\xi) &= \int_{-T/2}^{+T/2} e^{-ix\xi}dx = \frac{e^{-i\xi x}}{-i\xi}\,|_{x=-T/2}^{x=T/2} \\ &= \frac{e^{\frac{i\xi T}{2}} - e^{-\frac{i\xi T}{2}}}{i\xi} = \frac{\sin T\xi/2}{\xi/2} := T sinc(\frac{T\xi}{2})\,,\end{aligned}$$

where $sinc(x) := \frac{\sin(x)}{x}$.

For an arbitrary interval $[a,b]$, set $c = (a+b)/2$, $T = b-a$, we have $\mathbb{1}_{[a,b]}(x) = \tau_c \mathbb{1}_{[-T/2,T/2]}(x)$ and thus (by the translation property)

$$\mathcal{F}[\mathbb{1}_{[a,b]}](\xi) = e^{-ic\xi}2\xi^{-1}\sin(\frac{T\xi}{2}) = e^{-i\xi(a+b)/2}2\xi^{-1}\sin(\frac{(b-a)\xi}{2}).$$

(see Fig. 5.1 and Fig. 5.2).

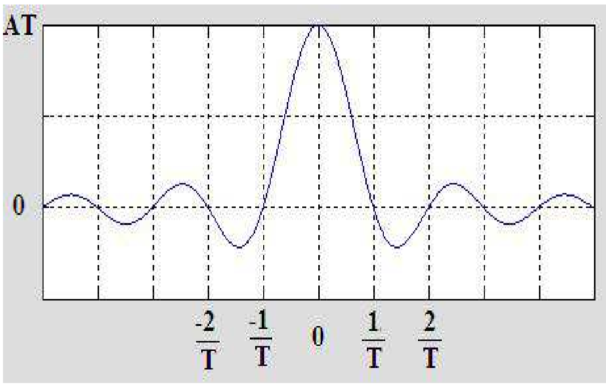

Figure 5.2: The *sinc* function is the Fourier Transform of the box function

5.3.2 Triangular functions

Consider the function $f(x) = (1-|x|)^+ := \sup(1-|x|,0)$

$$\begin{aligned}
\mathcal{F}[f](\xi) &= \int_{-1}^{1}(1-|x|)e^{-ix\xi}dx = \int_{-1}^{0}(1+x)e^{-ix\xi}dx + \int_{0}^{1}(1-x)e^{-ix\xi}dx \\
&= \int_{0}^{1}(1-x)(e^{-ix\xi}+e^{ix\xi})dx = 2\int_{0}^{1}(1-x)\cos(x\xi)dx \\
&= 2\frac{\sin(\xi x)}{\xi}(1-x)|_{x=0}^{x=1} + \frac{2}{\xi}\int_{0}^{1}\sin(\xi x)dx \\
&= \frac{2-2\cos\xi}{\xi^2} = 4\frac{\sin^2\xi/2}{\xi^2} = sinc^2(\frac{\xi}{2}),
\end{aligned}$$

(see Fig. 5.4 and Fig. 5.5).

5.3.3 Gaussian

For every $a > 0$ the function e^{-ax^2} is in $L^1(I\!R)$. We compute its Fourier transform by two methods.

$$\mathcal{F}[f](\xi) = \int_{-\infty}^{+\infty} e^{-ax^2}e^{-ix\xi}dx = \int_{-\infty}^{+\infty} e^{-(ax^2+ix\xi)}dx.$$

Method 1.

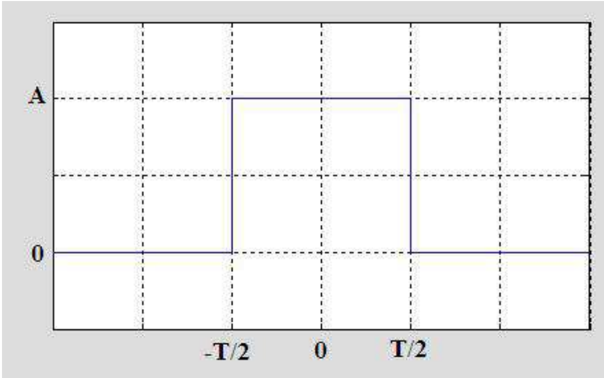

Figure 5.3: Characteristic function of an interval or box function

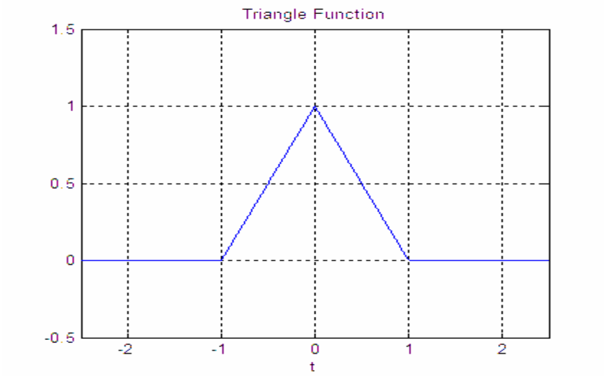

Figure 5.4: The triangle function.

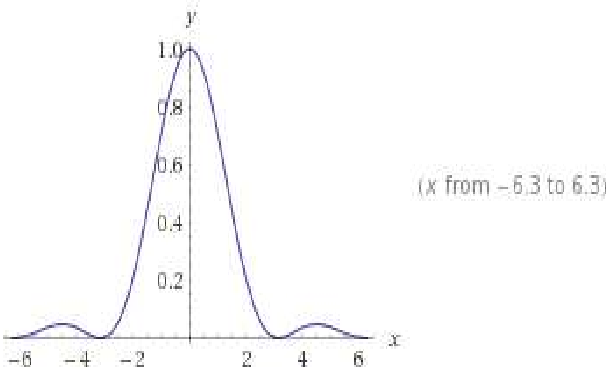

Figure 5.5: The Fourier Transform of the triangle function.

We write the quantity $ax^2 + ix\xi$ as sum of two squares:

$$\begin{aligned} ax^2 + ix\xi &= ax^2 + 2i\sqrt{a}x\frac{\xi}{2\sqrt{a}} \pm (\frac{i\xi}{2\sqrt{a}})^2 \\ &= (\sqrt{a}x + \frac{i\xi}{2\sqrt{a}})^2 + \frac{\xi^2}{4a}. \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{F}[f](\xi) &= \int_{-\infty}^{+\infty} e^{-ax^2} e^{-ix\xi} dx \\ &= e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{+\infty} e^{-(\sqrt{a}x + \frac{i\xi}{2\sqrt{a}})^2} dx \\ & \quad \text{by setting } y = x + \tfrac{i\xi}{2a} \\ &= e^{-\frac{\xi^2}{4a}} \int_{-\infty}^{+\infty} e^{-ay^2} dy = \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}}. \end{aligned}$$

The above change of variable can be justified in the framework of complex analysis. Precisely $\forall \alpha \in \mathbb{C}$ it holds: $\int_{-\infty}^{+\infty} e^{-(x+\alpha)^2} dx = \int_{-\infty}^{+\infty} e^{-x^2} dx$.

Method 2.

We set

$$g(\xi) := \int_{-\infty}^{+\infty} e^{-ax^2} e^{-ix\xi} dx = \int_{-\infty}^{+\infty} e^{-(ax^2 + ix\xi)} dx.$$

Observe that

$$\begin{aligned} g'(\xi) &= \int_{-\infty}^{+\infty} (-ix) e^{-ax^2} e^{-ix\xi} dx \\ & \quad \text{integration by parts} \\ &= \frac{i}{2a} \int_{-\infty}^{+\infty} (e^{-ax^2})' e^{-ix\xi} dx \\ &= \frac{i}{2a} e^{-ax^2} e^{-ix\xi} |_{-\infty}^{+\infty} - \frac{i}{2a} \int_{-\infty}^{+\infty} (-i\xi) e^{-ax^2} e^{-ix\xi} dx \\ &= -\frac{\xi}{2a} g(\xi). \end{aligned}$$

Hence the Fourier transform of f satisfies the following ODE:

$$g'(\xi) + \frac{\xi}{2a} g(\xi) = 0.$$

The solution is

$$g(\xi) = g(0)e^{-\frac{\xi^2}{4a}} = \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}}.$$

A similar computation shows that

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\xi) e^{ix\xi} d\xi = e^{-ax^2}.$$

Thus in this case $f = \mathcal{F}^{-1}[\hat{f}]$.

Exercise 5.3.1 **1.** Show that the inverse Fourier transform of $e^{-a|\xi|}$, $a > 0$ is

$$\frac{a}{\pi} \frac{1}{x^2 + a^2}.$$

2. (a) $\mathcal{F}[f](\xi) = 2\pi(\mathcal{F}^{-1})(-\xi)$.

Formula (a) states that if a transform is known, so is its inverse, and conversely.

(b) $\mathcal{F}[e^{iax} f](\xi) = \mathcal{F}(\xi - a)$.

5.4 Application 1: Solution of the heat equation in $I\!R$

In this section we illustrate how to use Fourier transform to solve the heat equation on an infinite interval. We consider the following initial-value problem:

$$\begin{cases} u_t - k u_{xx} = 0, \quad x \in I\!R, \ t > 0 \\ u(x, 0) = f(x). \end{cases} \tag{5.6}$$

Physically this problem is a model of heat flow in a infinitely long bar where the initial temperature $f(x)$ is prescribed. In a chemical or biological context, the equation governs variations under a diffusion process. Notice that there are no boundary conditions, so we do not prescribe boundary conditions explicitly. However, for problems on infinite domains, conditions at infinity are sometimes either stated explicitly or understood. We are going to write the equation for u as an ODE for the Fourier transform of u with respect to x.

We write

$$\begin{aligned}\hat{u}(\xi,t) &= \int_{-\infty}^{+\infty} u(x,t)e^{-i\xi x}dx \\ u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}(\xi,t)e^{i\xi x}d\xi.\end{aligned}$$

By differentiating u we get

$$\begin{aligned}u_t(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}_t(\xi,t)e^{i\xi x}d\xi \\ u_{xx}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (i\xi)^2\hat{u}(\xi,t)e^{i\xi x}d\xi.\end{aligned}$$

Since $u_t - u_{xx} = 0$, $\hat{u}$ satisfies the following initial-value problem:

$$\begin{cases} \hat{u}_t + k\xi^2\hat{u} = 0, \quad \xi \in I\!R,\ t > 0 \\ \hat{u}(\xi,0) = \hat{f}(\xi). \end{cases} \tag{5.7}$$

The solution of (5.7) is

$$\hat{u}(\xi,t) = \hat{f}(\xi)e^{-k\xi^2 t}. \tag{5.8}$$

In order to obtain u, we take the inverse Fourier transform of both sides of (5.8) and get

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)e^{-k\xi^2 t}e^{i\xi x}d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(z)e^{-i\xi z}dz\right)e^{-k\xi^2 t}e^{i\xi x}d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(z)\underbrace{\left(\int_{-\infty}^{+\infty} e^{-k\xi^2 t}e^{-i\xi(z-x)}d\xi\right)}_{=\mathcal{F}[e^{-ktx^2}](z-x)}dz \\ &= \int_{-\infty}^{+\infty} f(z)K(t,z-x)dz\,, \end{aligned} \tag{5.9}$$

where $K(t,x)$ is called the heat kernel of the heat equation and it is defined by

$$K(t,x) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}.$$

We observe that

1. $K(t,x) > 0$;
2. $\int_{-\infty}^{+\infty} K(t,z-x)dz = 1$;
3. If $f \equiv C$ then $u(x,t) \equiv C$;
4. Even if f is zero outside a small interval, the solution $u(x,t)$ is strictly positive for every $(x,t) \in I\!R \times [0,+\infty)$. Thus, a signal propagated by the heat/diffusion equation travels infinitely fast. According to this model, if odors diffuse, a bear would instantly smell a newly opened can of tuna ten miles away. Moreover initial signals propagated by the heat equation are immediately smoothed out.

The heat kernel is the temperature that results from an initial unit heat source, i.e. injecting a unit amount of heat at $x = 0$ at time $t = 0$.

Example 5.4.1 We solve the following problem

$$\begin{cases} u_t - ku_{xx} = 0, & x \in I\!R,\ t > 0 \\ u(x,0) = f(x)\,, \end{cases} \tag{5.10}$$

where

$$f(x) = \begin{cases} T_1 & x > 0 \\ T_2 & x \le 0 \end{cases}$$

Case 1. We first suppose that $T_1 = -T_2 = T$. We use the formula (5.9)

$$\begin{aligned} u(x,t) &= \underbrace{-\frac{T}{\sqrt{4\pi kt}}\int_{-\infty}^{0} e^{-\frac{(x-z)^2}{4kt}}dz}_{(1)} \\ &+ \underbrace{\frac{T}{\sqrt{4\pi kt}}\int_{0}^{+\infty} e^{-\frac{(x-z)^2}{4kt}}dz}_{(2)}. \end{aligned}$$

To compute (1) we set $y = \frac{x-z}{\sqrt{4kt}}$ and we get

$$(1) = -\frac{T}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-y^2}dy.$$

To compute (2) we set $y = \frac{z-x}{\sqrt{4kt}}$ and we get

$$(2) = \frac{T}{\sqrt{\pi}}\int_{-\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-y^2}dy.$$

Thus

$$(1) + (2) = \frac{T}{\sqrt{\pi}}\int_{-\frac{x}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} e^{-y^2}dy.$$

Case 2. T_1 and T_2 arbitrary.

We observe that if u is a solution of the problem (5.6) with initial data f then $\tilde{u} = u - C$ is still a solution of the heat equation with initial data $\tilde{f} = f - C$. In the case of the problem (5.10) if we choose $C = \frac{T_1+T_2}{2}$ then

$$\tilde{f}(x) = \begin{cases} T & x > 0 \\ -T & x \le 0 \end{cases}$$

with $T = \frac{T_1-T_2}{2}$.

We apply the case 1 and get

$$\tilde{u}(x,t) = \frac{1}{\sqrt{\pi}} \frac{T_1 - T_2}{2} \int_{-\frac{x}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} e^{-y^2} dy.$$

Thus

$$u(x,t) = \frac{1}{\sqrt{\pi}} \frac{T_1 - T_2}{2} \int_{-\frac{x}{\sqrt{4kt}}}^{\frac{x}{\sqrt{4kt}}} e^{-y^2} dy + \frac{T_1 + T_2}{2}.$$

Suppose for instance that $T_1 = 100$ and $T_2 = 0$ then the solution is given (by using that e^{-y^2} is even and that $\int_0^\infty e^{-y^2} dy = \sqrt{\pi}/2)$

$$u(x,t) = 50 + \frac{100}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-y^2} dy.$$

We note again that the temperature in non zero at all x for any $t > 0$ even though $u = 0$ for $x < 0$ and $t = 0$. The thermal energy spreads at an infinite propagation speed. It contrasts with the finite propagation speed of the wave equation.

5.5 Application 2: Solution of the Wave equation in $I\!R$

We consider the *Cauchy problem* for the one-dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad x \in I\!R, \ t > 0 \\ u(x,0) = f(x), \\ u_t(x,0) = g(x) \qquad\qquad . \end{cases} \tag{5.11}$$

We have already solved this problem in Section 2.2 by performing a suitable change of variable. Now we are going to solve it by using the Fourier transform of the solution with respect to the space variable x:

$$\begin{aligned} \hat{u}(\xi) &= \int_{-\infty}^{+\infty} u(x,t) e^{-i\xi x} dx \\ u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{u}(\xi,t) e^{i\xi x} d\xi. \end{aligned}$$

By arguing as in the case of heat equation we find that $\hat{u}$ satisfies the following ODE: $\hat{u}$ satisfies the following initial-value problem:

$$\begin{cases} \hat{u}_{tt} + c^2\xi^2\hat{u} = 0, \quad \xi \in I\!R,\ t > 0 \\ \hat{u}(\xi, 0) = \hat{f}(\xi) \\ \hat{u}_t(\xi, 0) = \hat{g}(\xi). \end{cases} \tag{5.12}$$

The solution of (5.12) is

$$\hat{u}(\xi, t) = \alpha_1(\xi)\sin(|\xi|ct) + \alpha_2(\xi)\cos(|\xi|ct) \tag{5.13}$$

where α_1, α_2 are determined by the initial conditions:

$$\alpha_1(\xi) = \frac{\hat{g}(\xi)}{c|\xi|}, \quad \alpha_2(\xi) = \hat{f}(\xi).$$

In order to obtain u we take the inverse Fourier transform of both sides of (5.13) and get

$$\begin{aligned} u(x,t) &= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\xi)\cos(\xi ct)e^{i\xi x}d\xi}_{(3)} \\ &+ \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{g}(\xi)\frac{\sin(\xi ct)}{c\xi}e^{i\xi x}d\xi}_{(4)}\,. \end{aligned} \tag{5.14}$$

We recall that

$$\cos(\xi ct) = \frac{e^{i\xi ct} + e^{-i\xi ct}}{2}$$

and

$$\frac{\sin(\xi ct)}{c\xi} = \int_0^t \cos(c\xi s)ds.$$

Therefore

$$\begin{aligned} (3) &= \frac{1}{4\pi}\left(\int_{-\infty}^{+\infty} \hat{f}(\xi)e^{i\xi(x+ct)}d\xi + \int_{-\infty}^{+\infty} \hat{f}(\xi)e^{i\xi(x-ct)}d\xi\right) \\ &= \frac{f(x+ct) + f(x-ct)}{2}. \end{aligned}$$

$$
\begin{aligned}
(4) &= \frac{1}{4\pi}\int_0^t \left(\int_{-\infty}^{+\infty} \hat{g}(\xi)\frac{e^{i\xi(x+cs)}+e^{i\xi(x-cs)}}{2}d\xi\right) \\
&= \frac{1}{2}\left[\int_0^t g(x+cs)ds + \int_0^t g(x-cs)ds\right] \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.
\end{aligned}
$$

(3) + (4) give again the d'Alembert formula.

5.6 Application 3: Solve an ODE

We consider the following second order linear ODE:

$$-u''(x) + u(x) = e^{-2|x|}.$$

We take the Fourier transform of both sides and get

$$(\xi^2+1)\mathcal{F}[u](\xi) = \mathcal{F}[e^{-2|x|}](\xi).$$

The Fourier transform of $e^{-2|x|}$ is $\frac{4}{4+\xi^2}$ (see exercise 5.3.1). Hence

$$\mathcal{F}[u](\xi) = \frac{1}{(\xi^2+1)}\frac{4}{(\xi^2+4)}$$

and

$$
\begin{aligned}
u(x) &= \mathcal{F}^{-1}[\frac{1}{(\xi^2+1)}] * \mathcal{F}^{-1}[\frac{4}{(\xi^2+4)}] \\
&= \frac{e^{-|x|}}{2} * e^{-2|x|} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}e^{-2|y|}dy.
\end{aligned}
$$

We consider first $x \geq 0$.

We have

$$|x-y| + 2|y| = \begin{cases} y+x & 0 \leq y \leq x \\ x-3y & y < 0 \\ 3y-x & y > x. \end{cases}$$

Therefore

$$\frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|}e^{-2|y|}dy = \frac{1}{2}\int_{-\infty}^{0} e^{3y-x}dy$$
$$+\frac{1}{2}\int_{0}^{x} e^{-y-x}dy + \frac{1}{2}\int_{-\infty}^{0} e^{-3y+x}dy$$
$$\frac{1}{2}\left[\frac{e^{3y-x}}{3}|_{-\infty}^{0} - e^{-x-y}|_{0}^{x} - \frac{e^{-3y+x}}{3}|_{x}^{+\infty}\right]$$
$$= \frac{2}{3}e^{-x} - \frac{1}{3}e^{-2x}.$$

The case $x < 0$ is left to the reader.

Chapter 6

The Laplace equation

The Laplace equation $\Delta u = 0$ occurs frequently in applied sciences, in particular in the study of the *steady state phenomena*. Its solutions are called *harmonic* functions. For instance, the equilibrium position of a perfectly elastic membrane is a harmonic function as it is the velocity potential of a homogeneous fluid. Also, the steady temperature of a homogeneous and isotropic body is a harmonic function and in this case Laplace equation constitutes the stationary counterpart (time independent) of the diffusion equation.

More generally Poisson's equation $-\Delta u = f$ plays an important role in the theory of *conservative fields* (electrical, magnetic, gravitational,..) where the vector field derived from the gradient of a potential.

For example let $\mathbf{E}$ be a force field due to a distribution of electric charges in a domain $\Omega \subset I\!R^3$. Then $div(\mathbf{E}) = 4\pi\rho$, where ρ represents the density of the charge distribution.When a potential u exists such that $\nabla u = -\mathbf{E}$ then $\Delta u = div(\nabla u) = -4\pi\rho$, which is Poisson's equation. If the electric field is created by charges located outside Ω, then $\rho = 0$ in Ω and u is harmonic therein.

In this chapter we limit the discussion to functions $u(x, y)$ in two independent variables, although most of the analysis can be generalized to higher dimensions.

6.1 The Laplace's Equation in a half-plane

Now we work a problem involving Laplace's equation in the upper half plane. Consider

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in I\!R, y > 0 \\ u(x,0) = f(x) & x \in I\!R. \end{cases} \tag{6.1}$$

We also append the condition that the solution stays bounded as $y \to +\infty$. We take the Fourier transform of the PDE on x with y as a parameter and we obtain

$$\hat{u}_{yy} - \xi^2 \hat{u} = 0\,,$$

which has general solution

$$\hat{u}(\xi, y) = a(\xi) e^{-|\xi| y} + b(\xi) e^{|\xi| y}.$$

The boundedness condition on u forces $b(\xi) = 0$ for all $\xi \in I\!R$

$$\hat{u}(\xi, y) = \alpha(\xi) e^{-|\xi| y}.$$

Upon applying Fourier transform to the boundary condition, we get $\alpha(\xi) = \hat{f}(\xi)$. Therefore, the solution in the transform is

$$\hat{u}(\xi, y) = \hat{f}(\xi) e^{-|\xi| y}.$$

The solution is given by

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{-|\xi| y} e^{i\xi x} d\xi. \tag{6.2}$$

Now observe that (see Exercise 5.3.1)

$$\mathcal{F}^{-1}[e^{-y|\xi|}] = \frac{y}{\pi} \frac{1}{x^2 + y^2}.$$

Therefore

$$\begin{aligned} u(x,y) &= \mathcal{F}^{-1}[e^{-y|\xi|}] * f \\ &= (\frac{y}{\pi} \frac{1}{x^2 + y^2}) * f = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(\tau)}{(x-\tau)^2 + y^2} d\tau. \end{aligned}$$

Exercise 6.1.1 Find a bounded solution to the Neumann problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in I\!R, y > 0 \\ u_y(x,0) = g(x) & x \in I\!R. \end{cases} \tag{6.3}$$

Hint: Let $v = u_y$ and reduce the problem to a Dirichlet problem. The solution is

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(x-\xi) \ln(y^2 + \xi^2) d\xi + C.$$

6.2 Laplace equation in a disc

We consider the disc $B_a((0,0)) := \{(x,y) \in I\!R^2 : \; x^2 + y^2 \leq a\}$,with $a > 0$. We solve the following Dirichlet problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2 + y^2 \leq a \\ u(x,y) = f(x,y) & x^2 + y^2 = a. \end{cases} \tag{6.4}$$

Step 1. Polar coordinates.
$x = r\cos(\theta), \; y = r\sin(\theta)$, $r = \sqrt{x^2+y^2}$, $\theta = \arctan(\frac{y}{x})$. We have

$$M := \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$$

and

$$\begin{aligned} M^{-1} &= \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = (det(M)^{-1}) \begin{pmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r^{-1}\sin(\theta) & r^{-1}\cos(\theta) \end{pmatrix} = \begin{pmatrix} r^{-1}x & r^{-1}y \\ -r^{-2}y & r{-2}x \end{pmatrix}. \end{aligned}$$

Step 2. Laplacian in polar coordinates.
Let

$$u(x,y) = u(r\cos(\theta), r\sin(\theta)) = v(r,\theta).$$

By **chain rule** (derivative of composition) we have

$$\begin{aligned}
u_x &= v_r r_x + v_\theta \theta_x \\
u_{xx} &= v_{rr} r_x^2 + 2v_{r\theta} r_x \theta_x + v_r r_{xx} \\
&\quad + v_{\theta\theta}\theta_x^2 + v_\theta \theta_{xx} \\
&= v_{rr}\frac{x^2}{r^2} - 2v_{\theta r}\frac{xy}{r^3} + v_r\frac{y^2}{r^3} \\
&\quad + v_{\theta\theta}\frac{y^2}{r^4} + v_\theta\frac{2xy}{r^4}
\end{aligned}$$

$$\begin{aligned}
u_y &= v_r r_y + v_\theta \theta_y \\
u_{yy} &= v_{rr} r_y^2 + 2v_{r\theta} r_y \theta_y + v_r r_{yy} \\
&\quad + v_{\theta\theta}\theta_y^2 + v_\theta \theta_{yy} \\
&= v_{rr}\frac{y^2}{r^2} + 2v_{\theta r}\frac{xy}{r^3} + v_r\frac{x^2}{r^3} \\
&\quad + v_{\theta\theta}\frac{x^2}{r^4} - v_\theta\frac{2xy}{r^4}.
\end{aligned}$$

Therefore

$$\boxed{u_{xx} + u_{yy} = v_{rr} + \tfrac{1}{r}v_r + \tfrac{1}{r^2}v_{\theta\theta}}.$$

Step 3. Solve the problem in polar coordinates.
We set

$$\tilde{f}(\theta) = f(a\cos\theta, a\sin(\theta)).$$

We observe that $\tilde{f}$ is 2π-periodic.

We are lead to solve the following problem

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0, & 0 < r < a,\ 0 < \theta < 2\pi \\ \\ v(a,\theta) = \tilde{f}(\theta) & 0 \le \theta \le 2\pi. \end{cases} \tag{6.5}$$

We apply the method of separation of variables: we write $v(r,\theta) = R(r)\Theta(\theta)$, with $R \in C^2((0,a)) \cap C([0,a])$ and $\Theta \in C^2(\mathbb{R})$, 2π periodic. By differentiating and plugging into the equation we get

$$r^2 R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0.$$

Now we *separate* the variables and we get

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Problem for Θ.

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(0) = \Theta(2\pi). \end{cases} \tag{6.6}$$

Since we want periodic solutions, the admissible values of λ are $\lambda_n = n^2$, $n \geq 0$ and the solutions are

$$\Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta).$$

Problem for R.

We solve

$$r^2R'' + rR' - n^2R = 0.$$

Case 1. $n = 0$.

$r^2R'' + rR' - n^2R = 0 \leftrightarrow (rR')' = 0 \ \leftrightarrow R(r) = c_1 \log r + c_2$.

Since we look for solutions having finite limit as $r \to 0^+$, we choose $c_1 = 0$.

Case 2. $n > 0$.

We set $r = e^s$ and $S(x) = R(e^x)$. $S(x)$ satisfied the following ODE:

$$S''(x) - n^2S(x) = 0$$

whose general solution is given by

$$S(x) = c_n e^{nx} + d_n e^{-nx}.$$

Thus $R(r) = c_n r^n + d_n r^{-n}$. We choose again $d_n = 0$, in order that the solutions have finite limits as $r \to 0^+$.

Hence for every $n \geq 0$ we find

$$v_n(r, \theta) = r^n(a_n \cos(n\theta) + b_n \sin(n\theta)).$$

In order that the boundary condition is satisfied, we look for our solution as infinite combination of the v_n:

$$v(r,\theta) = a_0 + \sum_{n=1}^{+\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta))$$

with

$$\tilde{f}(\theta) = v(a,\theta) = a_0 + \sum_{n=1}^{+\infty} a^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

This is the complete Fourier series for $\tilde{f}$:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\theta')d\theta' \\ a^n a_n &= \frac{1}{\pi}\int_0^{2\pi} \tilde{f}(\theta')\cos(n\theta')d\theta' \\ a^n b_n &= \frac{1}{\pi}\int_0^{2\pi} \tilde{f}(\theta')\sin(n\theta')d\theta'. \end{aligned}$$

Hence

$$\begin{aligned} v(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\theta')d\theta' \\ &+ \frac{1}{\pi}\sum_{n=1}^{+\infty}\frac{r^n}{a^n}\int_0^{2\pi} \tilde{f}(\theta')[\cos(n\theta')\cos(n\theta) + \sin(n\theta')\cos(n\theta)]d\theta' \\ &= \frac{1}{\pi}\int_0^{2\pi} \tilde{f}(\theta')\left[\frac{1}{2} + \sum_{n=1}^{+\infty}\frac{r^n}{a^n}\cos(n(\theta-\theta'))\right]d\theta' \\ &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\theta')\left(\frac{1-(\frac{r}{a})^2}{1+(\frac{r}{a})^2 - 2\frac{r}{a}\cos(\theta-\theta')}\right)d\theta'. \end{aligned}$$

We thus get the following representation formula in polar coordinates:

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\theta')P(\frac{r}{a},\theta-\theta')d\theta',$$

where for every q, t:

$$P(q,t) = \frac{1-q^2}{1-2q\cos t + q^2}.$$

P is called Poisson Kernel. For $r = 0$ we get

$$v(r,0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(\theta')d\theta',$$

namely the value of the solution at the origin is equal to the average of $\tilde{f}$ on the circumference $x^2 + y^2 = a^2$ (Mean Value Property).

Example 6.2.1 Solve the following problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } x^2 + y^2 < 1 \\ u(x,y) = y^2 & \text{in } x^2 + y^2 = 1 \end{cases} \tag{6.7}$$

Solution.

We pass in polar coordinates by looking for a solution of the type $v(r,\theta) = u(r\cos(\theta), r\sin(\theta))$.

The problem becomes

$$\begin{cases} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} = 0, & 0 < r < 1,\ 0 < \theta < 2\pi \\ v(a,\theta) = \sin^2(\theta) & 0 \le \theta \le 2\pi. \end{cases} \tag{6.8}$$

We find

$$v(r,\theta) = a_0 + \sum_{n=1}^{+\infty} r^n(a_n\cos(n\theta) + b_n\sin(n\theta))$$

with

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} \sin^2(\theta')d\theta' \\ a^n a_n &= \frac{1}{\pi}\int_0^{2\pi} \sin^2(\theta')\cos(n\theta')d\theta' \\ a^n b_n &= \frac{1}{\pi}\int_0^{2\pi} \sin^2(\theta')\sin(n\theta')d\theta'. \end{aligned}$$

In this case

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

thus $b_n = 0$, for every $n \geq 0$, $a_0 = \frac{1}{2}$, $a_2 = -\frac{1}{2}$ and $a_n = 0$ for $n \neq 0, 2$.

Hence $v(r, \theta) = \frac{1}{2} - \frac{1}{2}r^2 \cos(2\theta)$ or upon returning to Cartesian coordinates $u(x, y) = \frac{1}{2}(1 - x^2 + y^2)$.

6.2.1 Two main properties of harmonic functions

Harmonic functions satisfies the following two principles:

Mean value principle. If u is harmonic ($\Delta u = 0$) in a open set $\Omega \subset I\!R^2$, then the value $u(x, y)$, $(x, y) \in \Omega$ is the average of u on the boundary of the disc centered at (x, y) and contained in Ω :

$$u(x, y) = \frac{1}{2\pi a} \int_{\partial B((x,y),a)} u(x', y') d\sigma. \tag{6.9}$$

As we have already remarked, the formula (6.9) is a consequence of Poisson formula, but it holds in every dimensions $n \geq 3$ as well.

Maximum and minimum principles. If $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is harmonic ($\Delta u = 0$) in a open bounded set $\Omega \subset I\!R^2$, then

$$\max\{u(x, y), \quad (x, y) \in \Omega\} = \max\{u(x, y), \quad (x, y) \in \partial\Omega\}$$

and

$$\min\{u(x, y), \quad (x, y) \in \Omega\} = \min\{u(x, y), \quad (x, y) \in \partial\Omega\}.$$

Exercise 6.2.1 1. Solve the following problem

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } x^2 + y^2 < 1 \\ u(x, y) = 1 + x^2 & \text{in } x^2 + y^2 = 1 \end{cases} \tag{6.10}$$

2. Compute $u(0, 0)$ by using Poisson Formula.

3. Show that $u(x, y)$ is positive.

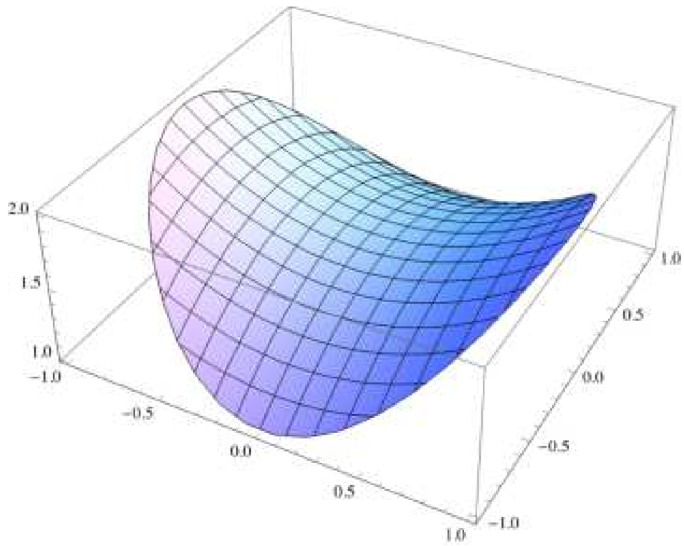

Figure 6.1: Solution of $u_{xx} + u_{yy} = 0$, $u(x, y) = 1 + x^2$ on the unit disk

Exercise 6.2.2 Determine a solution of the following problem

$$\begin{cases} u_{xx} + u_{yy} = -|x|, & \text{in } x^2 + y^2 < 1 \\ u(x, y) = 0 & \text{in } x^2 + y^2 = 1 \end{cases} \tag{6.11}$$

Hint: Look for solutions of the type $u(x, y) = f(r)$, with $f \in C^2(I\!R)$ even function, (see Fig. (6.2)).

6.3 The Laplace equation on a rectangle

In this section we consider a rectangle of the type $R = [0, a] \times [0, b]$ and we solve the Dirichlet problem for the Laplace equation on R.

$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{in } (0, a) \times (0, b) \\ u(x, y) = f(x) & \text{in } \partial R. \end{cases} \tag{6.12}$$

For simplicity we suppose that f is zero on the sides of R except on $[0, a] \times \{0\}$, moreover f is continuous and $f(0, 0) = f(a, 0) = 0$ (see Figure 6.3).

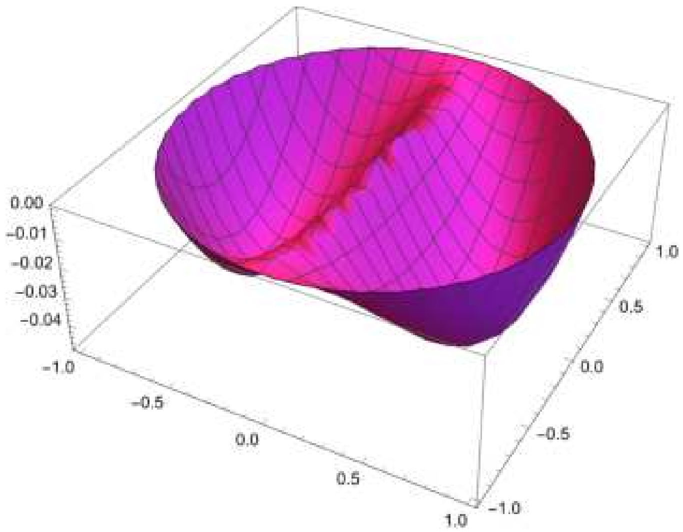

Figure 6.2: Solution of $u_{xx} + u_{yy} = -|x|$, $u(x, y) = 0$ on the unit disk

We apply the method of separation of the variables: we look for solutions of the type $u(x, y) = U(x)V(y)$. By plugging these functions into the Laplace equation, and separating the variables we get

$$\frac{U''(x)}{U(x)} = -\frac{V''(y)}{V(y)} = \lambda.$$

We obtain the differential equations

$$U''(x) - \lambda U(x) = 0 \;\text{ and }\; V''(y) + \lambda V(y) = 0.$$

For $0 < y < b$ it holds $U(0)V(y) = U(a)V(y) = 0$. If V is not identically zero, this implies that $U(0) = U(a) = 0$. As we have already seen in the case of heat equation, it must be $\lambda = \lambda_n = -\frac{n^2\pi^2}{a^2}$ and $U_n(x) = \sin(\frac{n\pi}{a}x)$. The equation for V, $V''(y) - \frac{n^2\pi^2}{a^2}V(y) = 0$ has the fundamental set of solutions $e^{\omega y}$, $e^{-\omega y}$, where $\omega = \frac{n\pi}{a}$. While these are standard solutions, it will be convenient to use

$$\sinh \omega y = \frac{e^{\omega y} - e^{-\omega y}}{2}, \text{and} \cosh \omega y = \frac{e^{\omega y} + e^{-\omega y}}{2}.$$

These functions are linear combinations of $e^{\omega y}$, $e^{-\omega y}$, so they are solutions to the equations as well. Thus $V_n(y) = c_1 \sinh \omega y + c_2 \cosh \omega y$. The condition

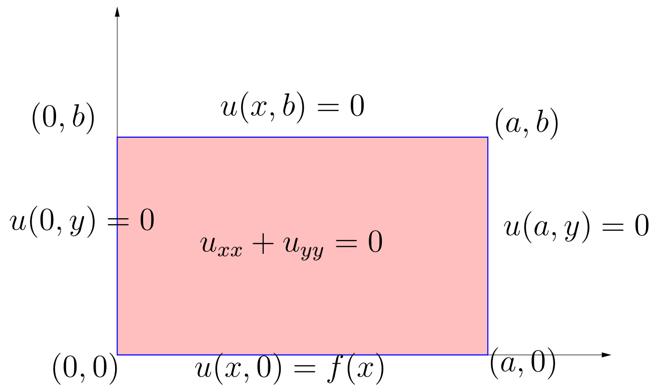

Figure 6.3: The Dirichlet problem for the rectangle R

$V(b) = 0$ implies that $V_n(y) = c \sinh \omega(y-b)$. We look for a solution of the type

$$u(x,y) = \sum_{n=1}^{+\infty} c_n \sin(\frac{n\pi}{a}x) \sinh \frac{n\pi}{a}(y-b).$$

Since $u(x,0) = f(x)$, we get

$$c_n \sinh \frac{n\pi}{a}(-b) = \frac{2}{a} \int_0^a f(x) \sin(\frac{n\pi}{a}x) dx$$

By using that sinh is an odd function, we get

$$c_n = \frac{-2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \sin(\frac{n\pi}{a}x) dx.$$

In a similar way one can solve the Dirichlet problem in a rectangle for boundary conditions which do not vanish in only one side, Then by using the *superposition principle* one can get the solution for an arbitrary data.

Chapter 7

The Laplace Transform and Applications

The Laplace transform (in short LT) offers another technique for solving linear differential equations with constant coefficients. It is a particular useful technique when the right hand side of a differential equation is a discontinuous function. Such functions often arise in applications in mechanics and electric circuits where the forcing term may be an impulse function that takes effect for some short time duration, or it may be a voltage that is turned off and on.

The process of solving a differential equation using the Laplace transform method consists of three steps:

Step 1. The given differential equation is transformed into an algebraic equation, called the **subsidiary equation.**

Step 2. The subsidiary equation is solved by purely algebraic manipulations.

Step 3. The solution in Step 2. is transformed back, resulting in the solution of the given problem.

7.1 Definition and examples

In this section we learn about Laplace transform and some of their properties. Roughly speaking the LT, when applied to a function, changes that function into a new function by using a process that involves integration.

Definition 7.1.1 *Let $f\colon [0,+\infty) \to I\!R$, its Laplace transform is defined as follows:*

$$F(s) = \mathcal{L}[f](s) = \int_0^{+\infty} f(t)e^{-st}dt. \tag{7.1}$$

Here we must assume that $f(t)$ is such that the integral exits (that is, has some finite value). This assumption is usually satisfied in applications.

The given function $f(t)$ is called the **inverse Laplace Transform** of $F(s)$ and it is denoted by $\mathcal{L}^{-1}(F)$ that is, we shall write

$$f(t) = \mathcal{L}^{-1}[F](t).$$

Notation. Original functions are denoted by **lower cases** and their LT by the same **letters in capital**.

Example 7.1.1 Let $f(s) = 1$ when $t \geq 0$. Find $F(s)$.

Solution. From (7.1) we obtain by integration

$$F(s) = \int_0^{+\infty} e^{-st}dt = -\frac{1}{s}e^{-st}|_0^{+\infty} = \frac{1}{s}, \quad (s > 0).$$

Such an integral is called **improper integral** and, by definition, is evaluated according the rule

$$\int_0^{+\infty} e^{-st}dt = \lim_{T\to+\infty} \int_0^{T} e^{-st}dt.$$

By arguing by induction one can show that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0.$$

Example 7.1.2 Let $f(t) = e^{at}$, when $t \geq 0$, and $a \in I\!R$. Find $F(s)$.

Solution. Again by (7.1)

$$F(s) = \int_0^{+\infty} e^{at}e^{-st}dt = \frac{1}{a-s}e^{-(s-a)t}|_0^{+\infty} = \frac{1}{s-a}, \quad (s > a).$$

Example 7.1.3 Let us consider the Heaviside function

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

The Heaviside function represents a force that is turned on at time $t = 0$, and left on thereafter. One has $\mathcal{L}[H](s) = \frac{1}{s}$ as in the Example 7.1.1. This should not be surprising, since $f(t) = H(t) = 1$ for $t > 0$, and the Laplace transform only looks at the values of a function for $t > 0$.

Exercise 7.1.1 1. Derive the formulas

$$\mathcal{L}[\cos(at)] = \frac{s}{a^2 + s^2}, \quad \mathcal{L}[\sin(at)] = \frac{a}{a^2 + s^2}.$$

2. Compute the LT of

$$f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 1 & 1 < t < +\infty \end{cases}$$

(**Solution.** $F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$.)

Definition 7.1.2 *A function $f(t)$ is of exponential order if there are positive constants C and a such that*

$$|f(t)| \leq Ce^{at}, \quad \text{for all } t > 0.$$

Theorem 7.1.1 *Suppose f is a piecewise continuous function defined on $[0, +\infty)$, which is of exponential order. Then the Laplace transform $\mathcal{L}[f](s)$ exists for large values of s. Specifically, if $|f(t)| \leq Ce^{at}$, then $\mathcal{L}[f](s)$ exists at least for $s > a$.*

Theorem 7.1.1 says that the LT exists for a wide variety of functions. However there are functions for which the LT does not exist. For example, the function $f(t) = e^{t^2}$ is not of exponential growth and its LT does not exist.

Exercise 7.1.2 Let f be a piecewise continuous function such that $|f(t)| \leq Ce^{at}$ for all $t > 0$.

a) Show that

$$|f(t)e^{-st}| \leq C^{-(s-a)t}, \quad \text{for all } t > 0.$$

b) $\mathcal{L}[f](s)$ exists at least for $s > a$.

c) $|\mathcal{L}[f](s)| \leq \frac{C}{s-a}$, provided $s > a$

d) Why can't $F(s) = \frac{s+4}{s-2}$ be the LT of any function of exponential order?

7.2 Basic properties of the Laplace transform

This section will discuss the most important properties of the Laplace transform.

1. Linearity.

Theorem 7.2.1 *[Linearity of LT] The LT is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a and b the Laplace transform of $af(t)+bg(t)$ exists and*

$$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)].$$

Example 7.2.1 Find the LT of $\cosh(at)$ and $\sinh(at)$.

Solution. Since $\cosh(at) = \frac{1}{2}(e^{at} + e^{-at})$ and $\sinh(at) = \frac{1}{2}(e^{at} - e^{-at})$, we obtain

$$\begin{aligned}
\mathcal{L}[\cosh(at)] &= \frac{1}{2}(\mathcal{L}[e^{at}] + \mathcal{L}[e^{-at}]) \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2} \\
\mathcal{L}[\sinh(at)] &= \frac{1}{2}(\mathcal{L}[e^{at}] - \mathcal{L}[e^{-at}]) \\
&= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2 - a^2}.
\end{aligned}$$

2. The Laplace transform of derivatives.

This property is the key tool when using the Laplace transform to solve differential equations. Under suitable regularity properties of f and its derivatives up to the order k we have:

$$\mathcal{L}[f'](s) = s\mathcal{L}[f](s) - f(0); \tag{7.2}$$

$$\mathcal{L}[f''](s) = s^2\mathcal{L}[f](s) - sf(0) - f'(0); \tag{7.3}$$

$$\mathcal{L}[f^{(k)}](s) = s^k\mathcal{L}[f](s) - \sum_{j=0}^{k-1} s^{k-1-j} f^j(0). \tag{7.4}$$

Proof of (7.2).

$$\begin{aligned} \mathcal{L}[f](s) &= \int_0^{+\infty} e^{-st} f(t)dt \\ &= \left[\frac{f(t)e^{-st}}{-s}\right] |_0^{+\infty} - \int_0^{+\infty} \frac{e^{-st}}{-s} f'(t)dt \\ &= \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}[f'](s). \end{aligned}$$

This yields the formula (7.2). $\Box$

3. The Laplace transform of the product of an exponential with a function.

$$\mathcal{L}[e^{ct}f](s) = \mathcal{L}[f](s-c). \tag{7.5}$$

Example 7.2.2 Compute the LT of the function $g(t) = e^{2t}\sin(4t)$.

Solution. The LT of $f(t) = \sin(4t)$ is $F(s) = \frac{4}{s^2+16}$. By using (7.5) with $c = 2$, the LT of g is $G(s) = F(s-2) = \frac{4}{(s-2)^2+16}$.

4. The derivative of a Laplace transform.

Theorem 7.2.2 *Suppose f is a piecewise continuous function of exponential order and let $F(s)$ be its Laplace transform. Then*

$$\mathcal{L}[tf](s) = -F'(s).$$

More generally, if n is any positive integer, then

$$\mathcal{L}[t^n f](s) = (-1)^n F^{(n)}(s).$$

Example 7.2.3 Compute the LT of $t^2 e^{3t}$.

Solution. Let $f(t) = e^{3t}$. Its LT is $F(s) = \frac{1}{s-3}$. By using Theorem 7.2.2 with $n = 2$, we obtain

$$\mathcal{L}[t^2 e^{3t}](s) = (-1)^2 F''(s) = \frac{2}{(s-3)^3}.$$

5. The Laplace transform of a translate of a function.

$$\mathcal{L}[H(t-c)f(t-c)](s) = e^{-cs}F(s).$$

6. The convolution product.

$$\mathcal{L}[f * g](s) = \mathcal{L}[f](s)\mathcal{L}[g](s).$$

7.3 The inverse Laplace transform

The Laplace transform turns a differential equation into an algebraic equation, which allows us to find the Laplace transform of the solution.

We could then find the solution to the original differential equation if we knew how to find the function with the given Laplace transform. This is the problem of finding the **inverse Laplace transform**.

Although there is a general integral formula for the inverse Laplace transform, it requires a contour integral in the complex plane. We will be able to get by using the linearity of the inverse Laplace transform and the knowledge we continue to accumulate about the transforms of specific functions.

Example 7.3.1 Compute the inverse Laplace transform of

$$F(s) = \frac{1}{s-2} - \frac{16}{s^2+4}.$$

We have

$$\mathcal{L}^{-1}[\frac{1}{s-2}] = e^{2t} \quad \text{and} \quad \mathcal{L}^{-1}[\frac{16}{s^2+4}] = \sin(2t).$$

Hence by the linearity of the inverse Laplace transform,

$$\mathcal{L}^{-1}[F](t) = e^{2t} - 8\sin(2t).$$

7.3.1 The inverse Laplace transform of rational functions

Most of the Laplace transforms that arises in the study of differential equations are rational functions. This means that the method of partial fractions will frequently be useful in computing inverse Laplace transform.

A property that we use in the sequel is

Second translation property.

Let $a \in \mathbb{R}$,

$$\mathcal{L}^{-1}[e^{-sa}F(s)] = H(t-a)f(t-a). \tag{7.6}$$

Example 7.3.2 Compute the inverse Laplace transform of the function $F(s) = \frac{1}{s^2-2s-3}$.

Solution. The denominator factors as $s^2 - 2s - 3 = (s-3)(s+1)$. Hence the partial fraction decomposition has the form

$$F(s) = \frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}.$$

One finds $A = 1/4$ and $B = -1/4$. Thus

$$\frac{1}{(s-3)(s+1)} = \frac{1}{4}(\frac{1}{s-3} - \frac{1}{s+1}).$$

Using the linearity of the inverse LT, we obtain the answer

$$\mathcal{L}^{-1}[F](t) = \frac{e^{3t} - e^{-t}}{4}.$$

Example 7.3.3 Compute the inverse Laplace transform of the function $F(s) = \frac{1}{s^2+4s+13}$.

Solution. We complete the square in the dominator to obtain

$$F(s) = \frac{1}{(s+2)^2+3^2}.$$

Thus

$$\mathcal{L}^{-1}[F](t) = \mathcal{L}^{-1}\left[\frac{1}{3}\frac{3}{(s+2)^2+3^2}\right] = \frac{1}{3}e^{-2t}\sin(3t).$$

Exercise 7.3.1 Compute the inverse Laplace transform of the following functions: $F(s) = \frac{2s+5}{s^2+4}$, $F(s) = \frac{2s-3}{(s-1)^2+5}$, $F(s) = \frac{2s^2+s+13}{(s-1)((s+1)^2+4))}$.

7.3.2 Using the Laplace transform to solve differential equations

The following example illustrates the general method by which the LT is used to solve initial value problems.

Example 7.3.4 Use the LT to find the solution to the initial value problem

$$\begin{cases} y'' + y = \cos(2t) \\ y(0) = 0 \\ y'(0) = 1\,, \end{cases}$$

Let's take the LT of the equation. We get

$$Y(s) + s^2Y(s) - sy(0) - y'(0) = \frac{s}{s^2+4}.$$

Substituting the initial conditions and solving for Y, we get

$$Y(s) = \frac{s^2+s+4}{(s^2+1)(s^2+4)}.$$

The denominator has two irreducible quadratic factors. We look for a decomposition of the following type

$$\frac{s^2+s+4}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

We find: $A = 1/3$, $B = 1$, $C = -1/3$ and $D = 0$ (see Appendix B). Hence

$$Y(s) = \frac{s}{3(s^2+1)} + \frac{1}{s^2+1} - \frac{s}{3(s^2+4)}.$$

Then from the Table 2 and the linearity, we get

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[y](t) = \frac{1}{3}\mathcal{L}^{-1}[\frac{s}{s^2+1}] \\ &+ \mathcal{L}^{-1}[\frac{1}{s+1}] - \frac{1}{3}[\frac{s}{s^2+4}] \\ &= \frac{1}{3}\cos t + \sin t - \frac{1}{3}\cos 2t. \end{aligned}$$

Exercise 7.3.2 Determine for $t > 0$ the solution of

$$\begin{cases} y''(t) + y(t) = r(t) \\ y(0) = 1 \\ y'(0) = 0\,, \end{cases}$$

where

$$r(t) = \begin{cases} t, & 0 < t \leq \pi/2 \\ \pi - t, & \pi/2 < t \leq \pi \\ 0, & t > \pi. \end{cases}$$

Solution. $y(t) = \int_0^t \sin(t-t')r(t')dt' + \cos(t) = \dots.$

7.3.3 The Damped harmonic oscillator

Consider the following initial value problem

$$\begin{cases} y''(t) = -\omega^2 y(t) - 2ky'(t) + g(t) \\ y(0) = 0 \\ y'(0) = 0\,, \end{cases}$$

This equation describes the motion of a vibrating spring. The quantity $\omega^2 y(t)$ represents an elastic force which is proportional to the distance from the origin, $2ky'(t)$ represents the attrition of the air proportional to the velocity and $g(t)$ is an external force.

We denote by $Y(s)$ and $G(s)$ the Laplace transforms of $y(t)$ and $g(t)$. By taking the LT of the equation we get

$$s^2Y(s) + \omega^2 Y(s) + 2ksY(s) = G(s)$$

and

$$Y(s) = \frac{G(s)}{s^2 + w^2 + 2ks}.$$

We consider the case $k < \omega$ (weak damping) :

$$\begin{aligned} s^2 + w^2 + 2ks &= (s+k)^2 + \omega^2 - k^2 \\ &= (s+k)^2 + \overline{\omega}^2 \,, \end{aligned}$$

where $\overline{\omega} = \sqrt{\omega^2 - k^2}$. We set $F(s) = \frac{1}{(s+k)^2+\overline{\omega}^2}$. We have $Y(s) = F(s)G(s)$. Thus

$$\mathcal{L}^{-1}[Y(s)] = \int_0^t f(t-t')g(t')dt'.$$

where $f(t) = \mathcal{L}^{-1}[F](t)$.

Recall that $\mathcal{L}[\sin(at)] = \frac{a}{a^2+s^2}$, $\mathcal{L}[e^{ct}f](s) = F(s-c)$.

Thus

$$\mathcal{L}\left[e^{-kt}\frac{\sin(\overline{\omega}t)}{\overline{w}}\right] = \frac{1}{(s+k)^2+\overline{\omega}^2}.$$

Therefore

$$y(t) = \frac{1}{\overline{\omega}}\int_0^t e^{-k(t-t')}\sin(\overline{\omega}(t-t'))g(t')dt'.$$

The function $h(t) = e^{-kt}\frac{\sin(\overline{\omega}t)}{\overline{w}}$ is called **Green Function**: $h(t-t')$ represents the effect that the external force $g(t')$ at the time t' has on the solution at the time t. We observe that the external force at the time t' influences the solution at $t \geq t'$ (this is the so-called **causality principle**).

Let us take for instance

$$g(t) = \begin{cases} a, & t_0 < t < t_1 \\ 0, & otherwise \,, \end{cases}$$

with $0 < t_0 < t_1$. We apply to an oscillator an external constant force during the interval of time $[t_0, t_1]$. We write the solution $y(t) = \int_0^t h(t-t')g(t')dt'$ as $\int_0^\infty H(t-t')h(t-t')g(t')dt'$, where $H(t)$ is the Heaviside function.

By our choice of g we get

$$y(t) = a \int_{t_0}^{t_1} H(t-t')h(t-t')dt'.$$

7.3.4 Using the Laplace Transform to solve PDEs

In this Section we show how to apply the Laplace transform to solve PDEs. We give two examples.

Example 7.3.5 We consider the following initial-boundary value problem:

$$\begin{cases} w_t - cw_x = 0, & x > 0, t > 0 \\ w(0,t) = f(t) & t > 0 \\ w(x,0) = 0 & x > 0. \end{cases} \tag{7.7}$$

We set

$$W(x,s) = \int_0^\infty w(x,t)e^{-st}dt \quad \text{and} \quad F(s) = \int_0^\infty f(t)e^{-st}dt.$$

We have

$$\mathcal{L}[w_t](s) = sW(x,s) - w(x,0) = sW(x,s)$$
$$\mathcal{L}[w_x](s) = \int_0^\infty w_x(x,t)e^{-st}dt = W_x(x,s)$$
$$W(0,s) = \int_0^\infty w(0,t)e^{-st}dt = F(s).$$

The problem for $W(x,s)$ is

$$\begin{cases} sW - cW_x = 0, & x > 0 \\ \\ W(0,s) = F(s), & s > 0 \end{cases} \tag{7.8}$$

The solution is $W(x,s) = F(s)e^{\frac{s}{c}x}$. Hence

$$w(x,t) = \mathcal{L}^{-1}[F(s)e^{\frac{s}{c}x}]$$
by (7.6)
$$= f(t+\frac{x}{c})H(t+\frac{x}{c}) = f(t+\frac{x}{c}).$$

Observe that $t + \frac{x}{c} > 0$.

Example 7.3.6 We consider the $1D$-wave equation in the domain $x > 0, t > 0$.

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & x > 0, t > 0 \\ w(0,t) = f(t) & t > 0 \\ \lim_{x\to+\infty} u(x,t) = 0 & s > 0 \\ w(x,0) = w_t(x,0) = 0 & x > 0. \end{cases} \tag{7.9}$$

We set

$$W(x,s) = \int_0^\infty w(x,t)e^{-st}dt, \quad \text{and} \quad F(s) = \int_0^\infty f(t)e^{-st}dt.$$

We have

$$\mathcal{L}[w_{tt}](s) = s^2 W(x,s)$$
$$\mathcal{L}[w_{xx]}(s) = \int_0^\infty w_{xx}(x,t)e^{-st}dt = W_{xx}(x,s)$$
$$W(0,s) = \int_0^\infty w(0,t)e^{-st}dt = F(s).$$

The problem for $W(x,s)$ is

$$\begin{cases} s^2 W - c^2 W_{xx} = 0, & x > 0 \\ W(0,s) = F(s), & s > 0 \\ \lim_{x\to+\infty} W(x,s) = 0 & s > 0 \end{cases} \tag{7.10}$$

The general solution of the equation in (7.10) is $W(x,s) = A(s)e^{\frac{s}{c}x} + B(s)e^{-\frac{s}{c}x}$. By imposing the two boundary conditions $W(0,s) = F(s)$ and $\lim_{x\to+\infty} W(x,s) = 0$ we get

$$W(x,s) = F(s)e^{-\frac{s}{c}x}.$$

Hence by applying (7.6) we obtain

$$w(x,t) = H(t - \frac{x}{c})f(t - \frac{x}{c}) = \begin{cases} f(t - \frac{x}{c}) & x < ct \\ 0 & x > ct. \end{cases}$$

Table 2: A small table of Laplace transform

$f(t)$	$\mathcal{L}(f)(s) = F(s)$
1	$\frac{1}{s}, \quad s > 0$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2} \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2} \quad s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$
$\underbrace{\int_0^t f(\tau)g(t-\tau)d\tau}_{\text{convolution}}$	$F(s)G(s)$

Appendix A

A short review of linear first and second order ODEs

A.1 First Order Linear Ordinary Differential Equations

First order linear ordinary differential equations (written in normal form) are equations of the form

$$y' + a(t)y = f(t) \tag{A.1}$$

where a, f are continuous functions defined on an interval $I \subset I\!R$.

If $f \equiv 0$ then the equation is called *homogeneous* :

$$z' + a(t)z = 0. \tag{A.2}$$

The equation (A.1) is called *complete*.

The following property holds:

The **general integral** *of (A.1) is obtained by adding to the general integral of (A.2) a particular solution of (A.1).*

A.1.1 Solution of the homogeneous equation

i) Let $A(t)$ be a primitive of $a(t)$ (namely $A'(t) = a(t)$).

ii) We multiply the equation (A.2) by $e^{A(t)}$:

$$z'(t)e^{A(t)} + a(t)z(t)e^{A(t)} = \frac{d}{dt}\left(z(t)e^{A(t)}\right) = 0.$$

Thus we get

$$z(t)e^{A(t)} = constant =: C.$$

We remark that the set of solutions of (A.2) is a vector space of dimension 1.

A.1.2 A particular solution of the complete equation

We apply the *method of constants variation.* We consider

$$\bar{y}(t) = c(t)e^{-A(t)}$$

where $c(t)$ is a function that we have to determine in order that $\bar{y}$ is a solution of (A.1). We have $\bar{y}'(t) = c'(t)e^{-A(t)} - c(t)e^{-A(t)}$. By plugging $\bar{y}$ and $\bar{y}'$ into (A.1) we get $c(t) = \int f(t)e^{A(t)}dt$. Thus

$$\bar{y}(t) = e^{-A(t)} \int f(t)e^{A(t)}dt.$$

The general integral of (A.1) is given by

$$y(t) = Ce^{-A(t)} + e^{-A(t)} \int f(s)e^{A(s)}ds. \tag{A.3}$$

The constant C will be determined by an initial condition

$$y(t_0) = y_0. \tag{A.4}$$

If we choose $A(t)$ in such a way that $A(t_0) = 0$ (namely $A(t) = \int_{t_0}^{t} a(s)ds$), the integral of (A.1) satisfying (A.4) will be

$$y(t) = y_0e^{-A(t)} + e^{-A(t)} \int_{t_0}^{t} f(s)e^{A(s)}ds. \tag{A.5}$$

We remark that in (A.3) the symbol $\int f(s)e^{A(s)}ds$ denotes a general primitive of $f(s)e^{A(s)}$. In (A.5) a particular primitive of $f(s)e^{A(s)}$ appears.

A.2 Second Order Linear Ordinary Differential Equations

A second order differential equation (written in normal form) is called linear if it is of the type

$$y'' + a(t)y' + b(t)y = g(t) \tag{A.6}$$

where a, b, g are continuous functions defined in an interval $I \subset I\!R$.

If $g \equiv 0$ then the equation is called *homogeneous* otherwise it is called *complete.*

The following two properties hold:

1. The set of solutions of the homogeneous equation

$$z'' + a(t)z' + b(t)z = 0 \tag{A.7}$$

is a vector space of dimension 2.

2. The general integral of (A.6) is obtained by summing up the general integral of (A.7) with a particular solution of (A.6).

A.2.1 Homogeneous equation with constant coefficients

We consider

$$z'' + az' + bz = 0 \tag{A.8}$$

with $a, b \in I\!R$. We consider the polynomial $P(\lambda) = \lambda^2 + a\lambda + b$ which is called the *characteristic polynomial* associated to the equation (A.8). We consider the following three cases.

i) If $\Delta := a^2 - 4b > 0$ then $P(\lambda)$ has two roots: $\lambda_1, \lambda_2 \in I\!R$. Then $z_1(t) = e^{\lambda_1 t}$, $z_2(t) = e^{\lambda_2 t}$ are two distinct solutions of (A.8) and the general integral of (A.8) is

$$z(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

ii) If $\Delta := a^2 - 4b < 0$ then $P(\lambda)$ has two complex roots $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$. In this case the general integral of (A.8) is

$$z(t) = e^{\alpha t}(c_1 \sin(\beta t) + c_2 \cos(\beta t)).$$

Another useful way to write the general integral is

$$z(t) = e^{\alpha t} A \cos(\beta t + \varphi),$$

with $A, \varphi \in I\!R$.

iii) If $\Delta = 0$ then (A.8) has one root $\lambda = -a/2$. In this case the general integral of (A.8) is

$$z(t) = e^{-a/2}(c_1 + c_2 t).$$

.

Thus in each case we have two independent solutions:

$$e^{\lambda_1 t}, \quad e^{\lambda_2 t} \qquad (\Delta > 0);$$

$$e^{\alpha t}\cos(\beta t), \quad e^{\alpha t}\sin(\beta t) \qquad (\Delta < 0);$$

$$e^{\lambda t}, \quad te^{\lambda t} \qquad (\Delta = 0).$$

A.2.2 Nonhomogeneous equation with constant coefficients

$$y'' + ay' + by = g(t), \quad a, b \in I\!R. \tag{A.9}$$

1. $g(t) = p_n(t)$, polynomial of oder n. One looks for a *particular* solution of the type:

$$\begin{cases} \bar{y}(t) = q_n(t), & \text{if } b \neq 0 \\ \bar{y}(t) = tq_n(t) & \text{if } b = 0 \text{ and } a \neq 0 \\ \bar{y}(t) = t^2 q_n(t) & \text{if } b = 0 \text{ and } a = 0 \end{cases}$$

2. $g(t) = Ae^{\lambda t}$, $\lambda \in \mathbb{C}$

One looks for a solution of the type $y(t) = e^{\lambda t}\gamma(t)$. One finds

$$\gamma'' + \gamma'(2\lambda + a) + \gamma(\lambda^2 + a\lambda + b) = A. \tag{A.10}$$

It is enough to find any γ satisfying (A.10).

- If $\lambda^2 + a\lambda + b \neq 0$, namely if λ is not a root of the characteristic polynomial then we take

$$\gamma = \frac{A}{\lambda^2 + a\lambda + b} \quad \text{and} \quad y(t) = \frac{Ae^{\lambda t}}{\lambda^2 + a\lambda + b}.$$

- If $\lambda^2 + a\lambda + b = 0$ and $2\lambda + a \neq 0$ then

$$\gamma' = \frac{A}{2\lambda + a} \quad \text{and} \quad y(t) = \frac{Ate^{\lambda t}}{2\lambda + a}.$$

- If $\lambda^2 + a\lambda + b = 0$ and $2\lambda + a = 0$ then

$$\gamma = \frac{A}{2}t^2 \quad \text{and} \quad y(t) = \frac{A}{2}t^2 e^{\lambda t}.$$

We remark that terms of the type $e^{\lambda t}$ with $\lambda \in \mathbb{C}$ include the following cases:

$$\cos(\omega t), \ \sin(\omega t), \ e^{\alpha t}\cos(\omega t), \ e^{\alpha t}\sin(\omega t), \ \alpha, \omega \in I\!R.$$

Appendix B

Decomposition of rational functions

We consider a function $f(t) = \frac{M(t)}{N(t)}$ where M, N are polynomial of degree respectively m, n, with $m < n$.

1. **Case of real and distinct roots**

   We suppose that the equation $N(x) = 0$ has n different real roots
   $$\alpha_1, \alpha_2, \ldots, \alpha_n\,,$$
   namely $N(t) = (t-\alpha_1)(t-\alpha_2)\cdots(t-\alpha_n)$. Then one looks for a decomposition of the type
   $$\frac{M(t)}{N(t)} = \frac{A_1}{t-\alpha_1} + \frac{A_2}{t-\alpha_2} + \cdots + \frac{A_n}{t-\alpha_n}.$$

2. **Case of real and multiple roots**

   Suppose that the equation $N(x) = 0$ has again real roots but they are not distinct. To fix ideas, we assume that it has three real distinct roots $\alpha_1, \alpha_2, \alpha_3$ of multiplicity respectively r, s, p with $r + s + p = n$, namely $N(t) = (t-\alpha_1)^r(t-\alpha_2)^s(t-\alpha_3)^p$. In this case one looks for a decompositions of $f(t)$ of the form
   $$\begin{aligned}\frac{M(t)}{N(t)} &= \frac{A_1}{t-\alpha_1} + \frac{A_2}{(t-\alpha_1)^2} + \cdots + \frac{A_t}{(t-\alpha_1)^r}\\ &+ \frac{B_1}{t-\alpha_2} + \frac{B_2}{(t-\alpha_2)^2} + \cdots + \frac{B_s}{(t-\alpha_2)^s}\\ &+ \frac{C_1}{t-\alpha_2} + \frac{C_2}{(t-\alpha_2)^2} + \cdots + \frac{C_p}{(t-\alpha_3)^p}.\end{aligned}$$

3. **Case of complex roots**

We consider the following example.

$$N(t) = (t^2 + at + b)(t - \alpha_1)^r$$

where α_1, a, b are real numbers and the polynomial $t^2 + ax + b$ is irreducible(namely it has complex roots). Then we look for a decomposition of the following type:

$$\begin{aligned}\frac{M(t)}{N(t)} &= \frac{A_1}{t-\alpha_1} + \frac{A_2}{(t-\alpha_1)^2} + \cdots + \frac{A_t}{(t-\alpha_1)^r} \\ &+ \frac{Bt + C}{t^2 + at + b}.\end{aligned}$$

EXAMPLE

Let $\frac{M(t)}{N(t)} = \frac{t^3+t-2}{(t+1)^2(t^2-t+1)}$. Since the polynomial is irreducible we look for a decomposition of the type

$$\frac{M(t)}{N(t)} = \frac{A}{t+1} + \frac{B}{(t+1)^2} + \frac{Ct + D}{t^2 - t + 1}.$$

The coefficients A, B, C, D have to satisfy

$$\begin{cases} A + C = 1 \\ B + 2C + D = 0 \\ -B + C + 2D = 1 \\ A + B + D = -2 \end{cases}$$

One gets $A = 0, B = -4/3, C = 1, D = -2/3$.

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