Program of Measure and Integration FS21 and Some Possible Questions for the Oral Exam ETH, D-MATH

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For the preparation of the oral exam you should take as guideline the class content that you find in

https://people.math.ethz.ch/∼ fdalio/MASSundINTEGRALFS21

Almost all the results in the lecture notes have been proved during the course except the following ones:

i) CHAPTER 1: Theorem 1.4.2, Lemma 1.9.4, Theorem 1.9.5

ii) CHAPTER 4: Theorem 4.1.5 (only statement), Lemma 4.3.2, Proposition 4.3.9, Theorem 4.4.1

Remark 0.1 Theorem 3.4.1 corresponds to Serie 11.5, Exercise 3.6.7 part 2) corresponds to Serie 11.2, Corollary 3.7.7 corresponds to Serie 12.1 and Exercise 3.7.11 corresponds to Serie 12.2, Exercise 3.7.12 corresponds to Serie 12.3.

The oral exam lasts 20 minutes (the time for the discussion of the grade is included) and you will be asked in general the statement and the proof of two results. In the case we are not completely satisfied by the answers or if we are not completely sure for the maximal grade we will ask (if the time permits it) a third question. It is very important to be very precise in the statement of a result and to explain clearly and rigorously the key steps of its proof. A question can consist in stating a definition and then proving a related property or in describing an example/counter-example related to a specific topic. We will ask you also the examples that have been treated during the lectures. We advise you to look at also the class notes in order to have an idea of what have been exactly done during the course .

We list below some (but not all) possible questions concerning Chapter 1 (just to give you an idea):

1. Can you give the definition of an algebra and σ-algebra and mention some concrete examples? Could you also describe examples of algebras which are not σ-algebras?

2. In an uncountable set X, consider the class

$$\mathcal{E} = \{A \in \mathcal{P}(X) \mid A \text{ is countable or } A^c \text{ is countable}\}. \tag{1}$$

   Could you show that $\mathcal{E}$ is a σ-algebra which is strictly smaller than $\mathcal{P}(X)$ (here "countable" stands for "at most countable")?

3. Could you give the definition of μ is additive and μ σ-additive functions and show some examples? Could you show that an additive function is σ-additive if and only if it is σ-subadditive?

4. Could you explain the Carathéodory criterion of measurability and show that the set

$$\Sigma = \{A \subseteq X\colon A \text{ is } \mu\text{-measurable}\}$$

   is a σ-algebra?

5. Could you prove the following Theorem?

   Theorem:

   Let (X, Σ, μ) be a Measure Space and let $A_k \in \Sigma$, $k \in I\!N$. Then the following conditions hold:

i) $A_j \cap A_\ell = \emptyset \; (j \neq \ell) \Longrightarrow \mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k)$, (**$\sigma$-additivity**).

ii) $A_1 \subset A_2 \subset \cdots \subset A_k \subset A_{k+1} \subset \ldots \Longrightarrow \mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \lim_{k\to+\infty} \mu(A_k))$

iii) $A_1 \supset A_2 \supset \cdots \supset A_k \supset A_k \supset \ldots, \quad \mu(A_1) < +\infty$. Then

$$\mu\Big(\bigcap_{k=1}^{\infty} A_k\Big) = \lim_{k\to+\infty} \mu(A_k).$$

6. Could you prove the following Theorem?

   Teorem:

   Let $\mathcal{K}$ be a covering for X, $\lambda\colon \mathcal{K} \to [0,+\infty]$ with $\lambda(\emptyset) = 0$. Then

   $$\mu(A) = \inf\Big\{\sum_{j=1}^{\infty} \lambda(K_j) : \; K_j \in \mathcal{K},\, A \subseteq \bigcup_{j=1}^{\infty} K_j\Big\} \tag{2}$$

   is a measure on X.

7. What is a pre-measure? Could you explain and prove the Carathéodory-Hahn extension theorem?

8. Could you state and prove the Theorem about the **Uniqueness of Charathéodory-Hahn extension** ?

9. Definition of Lebesgue measure and show

   Theorem:

   For every $A \subseteq \mathbb{R}^n$ it holds

   $$\mathcal{L}^n(A) = \inf_{A\subseteq G} \mathcal{L}^n(G), \quad G \text{ open}. \tag{3}$$

10. Could you state and prove a regularity property of Lebesgue measure (see results of Section 1.3)

11. Can you describe in details Vitali's set? Why isn't it Lebesgue measurable?

12. Could you give an example of a sequence of disjoint sets which are not Lebesgue measurable for which the σ-additivity of the Lebesgue measure does not hold?

13. Comparison between Lebesgue and Jordan measures. In particular show that if $A \subseteq \mathbb{R}^n$ is bounded, then

    i) $\underline{\mu}(A) \leq \mathcal{L}^n(A) \leq \overline{\mu}(A)$.

    ii) If A is Jordan-measurable, then A is $\mathcal{L}^n$-measurable and $\mathcal{L}^n(A) = \mu(A)$.

14. What is a Borel regular measure? Show that $\mathcal{L}^n$ is Borel regular.

15. An uncountable $\mathcal{L}^n$-null set: the Cantor Triadic Set. Describe it in details.

16. Definition of ***s*-dimensional Hausdorff measure** on $\mathbb{R}^n$. Show:

    Theorem: For $s \geq 0$, $\mathcal{H}^s$ is a Borel regular measure on $\mathbb{R}^n$.

17. Show the following

    Lemma:

    Let $A \subseteq \mathbb{R}^n$ and $0 \leq s < t < +\infty$. It holds

    i) $\mathcal{H}^s(A) < +\infty \Longrightarrow \mathcal{H}^t(A) = 0$.
    ii) $\mathcal{H}^t(A) > 0 \Longrightarrow \mathcal{H}^s(A) = +\infty$.

    and give the definition of the **Hausdorff dimension** of a set $A \subseteq \mathbb{R}^n$ i

18. Could you estimate the n-Hausdorff measure of $Q = [-1, 1]^n$ (see Example 1.8.6)?

19. Describe the Cantor Dust and show that its Hausdorff dimension is 1.

20. What is a metric measure? Show that the Hausdorff measure is metric? Do you know other examples of metric measures?

21. Definition of Radon measures. Examples and counter-examples. Show a particular property of a Radon measure.