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Uniqueness for First-Order Hamilton–Jacobi Equations and Hopf Formula

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We study uniqueness properties for a certain class of Cauchy problems for first-order Hamilton–Jacobi equations for which a solution is given by the Hopf formula. We prove various comparison and characterisation results concerning both convex generalized solutions and viscosity solutions. In particular, we show that the Hopf solution is the maximum convex generalized subsolution and the unique convex viscosity solution of the Cauchy problem. © 1987 Academic Press, Inc.

Nous étudions les propriétés d'unicité d'une certaine classe de problèmes de Cauchy pour les équations de Hamilton–Jacobi du premier ordre pour lesquels une solution est donnée par la formule de Hopf. Nous démontrons divers résultats de comparaison et de caractérisation concernant à la fois les solutions généralisées convexes et les solutions de viscosité. En particulier, nous montrons que la solution de Hopf est la sous-solution généralisée convexe maximale et l'unique solution de viscosité convexe du problème de Cauchy. © 1987 Academic Press, Inc.

INTRODUCTION

We consider the following Cauchy problem for first-order Hamilton–Jacobi equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + H(Du) = 0 & \text{in} \quad \mathbb{R}^N \times (0, T) \\ u(x, 0) = u_0(x) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{CP}$$

where u is the real-valued unknown function, $\partial u/\partial t$ and Du are, respectively, its time derivative and its gradient with respect to the space variables. H and u_0 are given continuous functions.

This paper discusses uniqueness, comparison and characterisation properties of solutions of (CP) in the special case *when u_0 is convex*. But, first let us explain what we mean by "solution."

The classical approach to treat problems like (CP) was to search

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generalized solutions, i.e., solutions in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^N \times (0, T))$ which satisfy the equation almost everywhere (cf., e.g., Douglis [13], Kruzkov [24–26], Fleming [14–16], Friedman [17], and Lions [29]). And for (CP), Hopf in [18] gave explicitly a generalized solution by

$$u(x, t) = \sup_{p \in \mathbb{R}^N} \{(p \mid x) - u_0^*(p) - tH(p)\} \tag{1}$$

provided, say, that

$$\lim_{|p| \to +\infty} \frac{u_0^*(p) + tH(p)}{|p|} = +\infty \qquad \text{uniformly for } t \in [0, T]. \tag{2}$$

In this formula, u_0^* is the Fenchel conjugate of u_0, defined by

$$u_0^*(p) = \sup_{q \in \mathbb{R}^N} ((p \mid q) - u_0(q)). \tag{3}$$

Recently, the lack of uniqueness of generalized solutions led Crandall and Lions to introduce in [7] (see also Crandall *et al.* [8]) the notion of *viscosity solutions*, a more restrictive notion which has very satisfying existence and uniqueness properties (see [2, 3], [2–10, 19–21, 29–33]). Moreover, Bardi and Evans [1] and Lions and Rochet [31] checked that u given by (1) is a viscosity solution of (CP). Let us finally emphasize that, except some recent works [11, 12, 19]), the uniqueness results for viscosity solutions of (CP) hold only for uniformly continuous solutions, which is not our general case. Our aim is to discuss uniqueness properties for (CP) of both generalized and viscosity solutions, and in the case of non uniqueness, to give a characterisation of the solution u given by (1) among all the other solutions. Let us immediately point out that the non uniqueness features come from the non restricted growth at infinity.

In part I, we are interested in convex generalized solutions of (CP). By "convex," we mean convex both in x and t. First, we prove that u given by (1)—denoted by u_{Hopf}—is the maximum convex generalized subsolution of (CP). We give also explicitly a minimum convex generalized supersolution of (CP)—denoted by $u_{\min}$. Then, we show that if u_0 is C^1, $u_{\text{Hopf}} = u_{\min}$ for all H satisfying (2), and so in this case we have uniqueness and comparison of convex generalized solutions. If u_0 is not C^1, uniqueness depends strongly on H and in the case of non uniqueness we build a continuum of solutions from $u_{\min}$ to u_{Hopf}.

In part II, we show that u_{Hopf} is the unique convex viscosity solution of (CP) and that we have comparison results with viscosity sub- and supersolutions. Let us point out that we do not need any assumptions on H and on u_0 (except (2)). We consider also viscosity solutions which are only convex in x. We prove that u_{Hopf} is the minimum viscosity supersolution of

(CP) convex in x and we show that, in general, comparison results with viscosity subsolution convex in x is false.

Finally, we consider uniqueness results for continuous—non necessarily convex—viscosity solutions. We prove that u_{Hopf} is the minimum viscosity supersolution of (CP) in a set of functions defined by some growth at infinity connected with the growth of H at infinity. These assumptions are proved to be optimal. This result is based on a result of Lions *et al.* [32] and is analogous to results obtained by Crandall and Lions [12].

I. Convex Generalized Solutions

The aim of this section is to describe the set of convex generalized solutions. To do so, we need

Definition I.1. We denote by $R(Du_0)$ the set

$$R(Du_0) = \{p \in \mathbb{R}^N \mid \exists y \in \mathbb{R}^N \text{ such that } u_0 \text{ is differentiable at } y \text{ and } Du_0(y) = p\}$$

Remark. Since we consider only convex continuous u_0, hence in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^N)$, by Rademacher theorem, u_0 is differentiable almost everywhere and then $R(Du_0)$ is nonempty.

Our results are the following.

Theorem I.2. *Let $u_{\min}$ defined by*

$$u_{\min}(x, t) = \sup_{p \in R(Du_0)} \{p \cdot x - u_0^*(p) - tH(p)\}$$

then $u_{\min}$ is a convex generalized solution of (CP).

Theorem I.2. *Let v (resp. w) be a convex generalized subsolution (resp. supersolution) of* (CP) *with $v(\cdot, 0) = v_0(\cdot) \leqslant u_0(\cdot)$ (resp. $w(\cdot, 0) = w_0(\cdot) \geqslant u_0$), then*

$$v \leqslant u_{\text{Hopf}} \qquad (\textit{resp. } u_{\min} \leqslant w).$$

This result means that u_{Hopf} is the *maximum convex generalized subsolution of* (CP) and $u_{\min}$ is the *minimum convex generalized supersolution of* (CP). From these theorems, we deduce the following results.

Corollary I.1. *If $u_0 \in C^1(\mathbb{R}^N)$ then $u_{\min} = u_{\text{Hopf}}$ and there exists a unique generalized solution of* (CP).

Remark I.2. It is worth noting that, in this case, we have more than a uniqueness result: we also have a comparison result in view of Theorem I.2.

COROLLARY I.2. *For all $\lambda > 0$, the function u_λ defined by*

$$u_\lambda(x, t) = \sup_{d(p, R(Du_0)) \leqslant \lambda} \{p \cdot x - u_0^*(p) - tH(p)\}$$

is a convex generalized solution of (CP). *So, when $u_{\min} \neq u_{\text{Hopf}}$, we have a continuum of convex generalized solutions.*

Let us give a very simple example of the situation above. We consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + |Du| = 0 & \text{in} \quad \mathbb{R} \times (0, \infty) \\ u(x, 0) = |x| & \text{in} \quad \mathbb{R}. \end{cases}$$

In this case, we have

$$R(Du_0) = \{+1; -1\}$$

and

$$u_0^*(p) = 0 \qquad \text{if} \quad |p| \leqslant 1; \qquad +\infty \quad \text{if} \quad |p| > 1.$$

It is easy to compute $u_{\min}$ and u_{Hopf}

$$u_{\min}(x, t) = |x| - t$$
$$u_{\text{Hopf}}(x, t) = (|x| - t)^+$$

and for $0 \leqslant \lambda \leqslant 1$, we have

$$u_\lambda(x, t) = (1 - \lambda)(|x| - t) + \lambda(|x| - t)^+.$$

Remark I.3. In the situation of non uniqueness, the Corollary I.2 does not give all the convex generalized solutions of (CP). For example, one can build other solutions by replacing the set

$$\{p \in \mathbb{R}^N \mid d(p, R(Du_0)) \leqslant \lambda\}$$

by any set B such that $R(Du_0) \subset B \subset \mathbb{R}^N$. Moreover B may depend on t, etc.

Remark I.4. Even if u_0 is not C^1, we can have a uniqueness property for (CP); but this depends strongly on the Hamiltonian H. Let us consider, for example, the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - |Du| = 0 & \text{in} \quad \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = |x| & \text{in} \quad \mathbb{R}^N. \end{cases}$$

One proves easily that $u_{\min} = u_{\text{Hopf}} = |x| + t$.

All these phenomena of non uniqueness have many connections with the characteristics method for first-order Hamilton–Jacobi equations (see [5, 22, 23, 27, 29]). To make precise this vague claim, we give the following result.

PROPOSITION I.1. *Let $H \in C^1(\mathbb{R}^N)$ and let u_0 be a convex continuous function. We assume that, for all $t \in (0, T)$, the map $p \to u_0^*(p) + tH(p)$ is strictly convex. We define for each $x \in \mathbb{R}^N$, $t \in (0, T)$ and $\lambda > 0$ the set*

$$X_\lambda(x, t) = \{x + tH'(p) \quad \textit{for} \quad p \in \partial u_0(x),\, d(p, R(Du_0)) \leqslant \lambda\},$$

where $\partial u_0(x)$ is the subdifferential of u_0 at x. And we denote by D_t^λ the set

$$D_t^\lambda = \overline{\bigcup_{x \in \mathbb{R}^N} X_\lambda(x, t)}.$$

Then

$$D_t^\lambda = \{ y \in \mathbb{R}^N \mid u_{\text{Hopf}}(y, t) = u_\lambda(y, t)\},$$

where u_λ is defined in Corollary I.2.

In fact for H and u_0 satisfying the assumptions of Proposition I.1, this result gives a necessary and sufficient condition for uniqueness, which is

$$D_t^0 = \mathbb{R}^N \qquad \text{for all } t \in (0, T)$$

Indeed, if $\lambda = 0$, $u_\lambda = u_{\min}$ and, moreover the set $X_0(x, t)$ is given by

$$X_0(x, t) = \{x + tH'(Du_0(x))\} \qquad \text{if} \quad u_0 \text{ is differentiable at } x$$

$$X_0(x, t) = \varnothing \qquad \text{if} \quad u_0 \text{ is not differentiable at } x.$$

There are the "classical characteristics;" in this case the lack of surjectivity of $x \to X_0(x, t)$ leads to non uniqueness. In fact, the solution is under determined on the set $(D_t^0)^c$. On the contrary, if $\lambda = +\infty$, the "good case" is obtained by constructing "generalized characteristics" as it was remarked in [29] and on a particular example in [4]; these "generalized characteristics" are given by the set

$$X_\infty(x, t) = \{x + tH'(p),\ p \in \partial u_0(x)\}.$$

Remark I.5. The situation described above holds in particular when H is C^1 and convex and when either H or u_0^* is strictly convex.

Let us give a simple example of the situation described in Proposition I.1. We consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{|Du|^2}{2} = 0 & \text{in} \quad \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = |x|. \end{cases}$$

Then one has

$$R(Du_0) = S^{N-1} = \{p \in \mathbb{R}^N \mid |p| = 1\}$$

and

$$u_\lambda(x, t) = \sup_{1-\lambda \leqslant |p| \leqslant 1} \left\{ p \cdot x - \frac{t|p|^2}{2} \right\}.$$

The Hopf solution is obtained for $\lambda = 1$. Let us compute it. It is easy to see that if $|x|/t \geqslant 1$, the supremum is achieved for $|p| = 1$ and in the other case for $p = x/t$. Now if we look at the supremum for u_λ. Obviously if $|x|/t \geqslant 1 - \lambda$, it is the same as u_{Hopf}. On the contrary, if $|x|/t < 1 - \lambda$, the strict concavity of $p \to p \cdot x - (|p|^2/2t)$ implies that $u_\lambda(x, t) < u_{\text{Hopf}}(x, t)$. In this example

$$X_\lambda(x, t) = X_0(x, t) = \left\{ x + t \frac{x}{|x|} \right\} \qquad \text{if} \quad x \neq 0$$

$$X_\lambda(0, t) = \{t \cdot p,\ 1 - \lambda \leqslant |p| \leqslant 1\}.$$

One proves easily that

$$D_t^\lambda = \{(x, t) \in \mathbb{R}^N \times \mathbb{R} \mid |x|/t \geqslant (1 - \lambda)\}.$$

Now, we turn to the *proof of Theorem* I.1. Let us write by D the set of differentiability points of u_0, then $u_{\min}$ may be written

$$u_{\min}(x, t) = \sup_{y \in D} \{\nabla u_0(y)\, x - u_0^*(\nabla u_0(y)) - tH(\nabla u_0(y))\}.$$

First we are going to prove that $u_{\min}(x, 0) = u_0(x)$. Recall that

$$u_0^*(\nabla u_0(y)) = \nabla u_0(y) \cdot y - u_0(y).$$

So

$$u_{\min}(x, 0) = \sup_{y \in D} \{\nabla u_0(y)(x-y) + u_0(y)\}. \tag{4}$$

Using the convexity of u_0, we obtain

$$u_{\min}(x, 0) \leqslant u_0(x). \tag{5}$$

Since D is dense, letting $y \to x$ we conclude that

$$u_{\min}(x, 0) = u_0(x).$$

Now let us prove that $u_{\min}$ satisfies the equation almost everywhere. Let $(x, t) \in \mathbb{R}^N \times]0, t[$, by (2) we know that the supremum is achieved at one point $p_0 \in \overline{R(Du_0)}$ then

$$u_{\min}(x, t) = p_0 \cdot x - u_0^*(p_0) - tH(p_0).$$

Let $(y, s) \in \mathbb{R}^N \times [0, T]$ then

$$u_{\min}(y, s) \geqslant p_0 \cdot y - u_0^*(p_0) - sH(p_0)$$

and so

$$u_{\min}(y, s) \geqslant u_{\min}(x, t) + p_0 \cdot (y - x_0) - (s-t)\, H(p_0).$$

We deduce from this inequality that, if $u_{\min}(x, t)$ is differentiable at (x, t) then

$$Du_{\min}(x, t) = p_0$$

$$\frac{\partial u_{\min}}{\partial t}(x, t) = -H(p_0).$$

Then at each differentiability point we have

$$\frac{\partial u_{\min}}{\partial t}(x, t) + H(Du_{\min}(x, t)) = 0.$$

The same method proves the Corollary I.2. Now, we turn to the *proof of Theorem* I.2.

Let us first prove that $v \leqslant u_{\text{Hopf}}$. Since v is convex continuous, v is in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^N \times (0, T))$, then by Rademacher theorem, v is differentiable almost everywhere and by assumption, the derivatives of v satisfy a.e. the inequality

$$\frac{\partial v}{\partial t} + H(Dv) \leqslant 0.$$

More precisely, using the convexity of v, it is easy to see that this inequality holds in each differentiability point of v. (See [29] or, for example, the proof of Lemma I.1 which gives all the ideas to prove that.)

Let (x, t) a differentiability point of v, then since v is convex for all $y \in \mathbb{R}^N$ we have

$$v(y, 0) \geqslant v(x, t) + Dv(x, t)(y-x) + (0-t)\frac{\partial v}{\partial t}(x, t).$$

Using the two inequalities above, we get

$$v(x, t) \leqslant Dv(x, t) \cdot x - [Dv(x, t) \cdot y - v(y, 0)] - tH(Dv(x, t)).$$

Taking the infimum in y and using the definition of the Fenchel conjugate of $v(y, 0)$, we obtain

$$v(x, t) \leqslant Dv(x, t) \cdot x - v^*(\cdot, 0)(Dv(x, t)) - tH(Dv(x, t)),$$

and finally

$$v(x, t) \leqslant \sup_{p \in \mathbb{R}^N} \{p \cdot x - v^*(\cdot, 0)(p) - tH(p)\}.$$

Since the Fenchel conjugate is order-reversing, we conclude

$$v(x, t) \leqslant u_{\text{Hopf}}(x, t).$$

This inequality is true at any differentiability point of v, but since v and u_{Hopf} are continuous and since the set of differentiability point of v is dense, we can conclude that

$$v \leqslant u_{\text{Hopf}} \quad \text{in} \quad \mathbb{R}^N \times (0, T).$$

Next, we turn to the proof of the second inequality. In order to do so, we need

LEMMA I.1. *Let w satisfy the assumptions of Theorem* I.1 *and let y be a differentiability point of $w(\cdot, 0) = w_0(\cdot)$, then for all $(x, t) \in \mathbb{R}^N \times (0, T)$, we have*

$$w(x, t) \geqslant w_0(y) + \nabla w_0(y)(x-y) - tH(\nabla w_0(y)).$$

Remark I.6. This lemma means that, in some sense, the inequality satisfied by w holds for $t=0$ and that we can apply a convex inequality by using it. In fact, the method used in the proof shows that the inequality satisfied by w holds for *any* differentiability point of w and not only almost everywhere.

Let us first prove Theorem I.2 *by using Lemma* I.1. For $(x, t) \in \mathbb{R}^N \times (0, T)$ we have

$$w(x, t) \geqslant \nabla w_0(y) \cdot x - [\nabla w_0(y) \cdot y - w_0(y)] - tH(\nabla w_0(y)).$$

Then

$$w(x, t) \geqslant \nabla w_0(y) \cdot x - w_0^*(\nabla w_0(y)) - tH(\nabla w_0(y)).$$

Taking the supremum in y yields

$$w(x, t) \geqslant \sup_{y \in \mathbb{R}^N} \{\nabla w_0(y) \cdot x - w_0^*(\nabla w_0(y)) - tH(\nabla w_0(y))\}.$$

And it is easy to see that the right-hand is exactly $u_{\min}(x, t)$.

Now, let us give the *proof of Lemma* I.1. Let y be a differentiability point of w_0 and let (y_n, t_n) be a sequence of differentiability points of w, where the inequality holds and which converges to $(y, 0)$. Such a sequence exists since the inequality holds almost everywhere.

By the convexity, we have for all $(x, t) \in \mathbb{R}^N \times (0, T)$,

$$w(x, t) \geqslant w(y_n, t_n) + \nabla w(y_n, t_n)(x - y_n) + \frac{\partial w}{\partial t}(y_n, t_n)(t - t_n).$$

Assuming $t > 0$, we can use the inequality for $t_n \leqslant t$ and then

$$w(x, t) \geqslant w(y_n, t_n) + \nabla w(y_n, t_n)(x - y_n) - (t - t_n)\, H(\nabla w(y_n, t_n)).$$

But since w is in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^N \times (0, T))$, $\nabla w(y_n, t_n)$ is bounded and so we can take a subsequence such that $\nabla w(y_n, t_n)$ converges to p; then for all $t > 0$,

$$w(x, t) \geqslant w_0(y) + p \cdot (x - y) - tH(p).$$

Since w_0 is differentiable at y this implies that

$$p = \nabla w_0(y)$$

(letting $t \to 0$, one has $p \in \partial w_0(y) \ldots$). This concludes the proof of Lemma I.1.

There only remains to *prove the Corollary* I.1. The proof is based on the following lemma.

LEMMA I.2. *Let u_0 being convex continuous then*

$$\mathrm{Dom}(u_0^*) = \overline{R(\partial u_0)},$$

where

$$\mathrm{Dom}(u_0^*) = \{p \in \mathbb{R}^N \mid u_0^*(p) < +\infty\}$$

and

$$R(\partial u_0) = \{p \in \mathbb{R}^N \mid \exists y \in \mathbb{R}^N,\ p \in \partial u_0(y)\}.$$

Now it is easy to prove Corollary I.1 by using Lemma I.2. Since $u_0 \in C^1$ then for all $x \in \mathbb{R}^N$

$$\partial u_0(x) = \{Du_0(x)\}$$

and then

$$R(\partial u_0) = R(Du_0).$$

Now

$$u_{\mathrm{Hopf}}(x, t) = \sup_{p \in \mathbb{R}^N} \{p \cdot x - u_0^*(p) - tH(p)\}.$$

Therefore

$$u_{\mathrm{Hopf}}(x, t) = \sup_{p \in \mathrm{Dom}(u_0^*)} \{p \cdot x - u_0^*(p) - tH(p)\}.$$

Using Lemma I.2

$$u_{\mathrm{Hopf}}(x, t) = \sup_{p \in \overline{R(Du_0)}} \{p \cdot x - u_0^*(p) - tH(p)\} = u_{\min}(x, t),$$

which ends the proof of Corollary I.1.

We do not give the proof of Lemma I.1 which is a classical result (see for instance H. Brezis [5]).

Finally, we prove proposition I.1.

Proof of Proposition I.1. The idea of the proof is very simple; we consider in the Hopf formula at (x, t) the unique point p where the supremum is achieved. Let us recall that the uniqueness is a consequence of the strict convexity of $u_0^* + tH$; this point p satisfies

$$x \in \partial(u_0^* + tH)(p).$$

Using the fact that H is C^1, we get

$$x - tH'(p) \in \partial u_0^*(p).$$

Hence

$$p \in \partial u_0(x - tH'(p)).$$

Let us define y by $y = x - tH'(p)$ then

$$\begin{cases} p \in \partial u_0(y) \\ x = y + tH'(p) \end{cases}$$

and this property is equivalent to

$$u_{\text{Hopf}}(x, t) = p \cdot x - u_0^*(p) - tH(p)$$

since $u_0^* + tH$ is strictly convex and H is C^1. Now, we have to consider two cases:

(i) $(x, t) \in D_t^\lambda$, then using the uniqueness of p, one proves easily that $p \in \{q \in \mathbb{R} \mid d(q, R(Du_0) \leqslant \lambda\}$ and then

$$u_{\text{Hopf}}(x, t) = u_\lambda(x, t).$$

(ii) $(x, t) \notin D_t^\lambda$, by the same argument, one sees that

$$d(p, R(Du_0)) > \lambda$$

and the strict concavity of $(q \rightarrow q \cdot x - u_0^*(q) - tH(q))$ implies that

$$u_{\text{Hopf}}(x, t) > u_\lambda(x, t).$$

And the proof is complete.

Remark I.7. Let us conclude this section by a remark on generalized solutions (not necessarily convex ones). We consider two examples

(i) $\max(0; t - |x|)$ and 0 are generalized solutions of $(\partial u/\partial t) - |Du| = 0$ in $\mathbb{R}^N \times (0, \infty)$ and $u_{\text{Hopf}} \equiv 0$.

(ii) $\min(0; |x| - t)$ and 0 are generalized solutions of $(\partial u/\partial t) + |Du| = 0$ in $\mathbb{R}^N \times (0, \infty)$ and $u_{\text{Hopf}} \equiv 0$.

These examples shows that, in the general case, u_{Hopf} is neither a maximum nor a minimum generalized solution of (CP) in $W^{1,\infty}_{\text{loc}}$.

Nevertheless some particular cases are known; *in particular if H is convex*, Lions proves in [29] that the Oleinik Lax formula, i.e.,

$$\tilde{u}(x, t) = \operatorname*{Inf}_{y \in \mathbb{R}^N} \left(u_0(y) + tH^* \left(\frac{y-x}{t} \right) \right)$$

gives the maximum generalized subsolution of (CP) in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^N \times (0, T)) \cap C(\mathbb{R}^N \times (0, T))$. And an easy minmax argument shows that $\tilde{u}$ is equal to u_{Hopf}.

II. Convex Viscosity Solutions

In this part, we prove that u_{Hopf} is the unique convex viscosity solution of (CP) and the minimum viscosity supersolution convex in x of (CP). If we compare to the first section, it is enough to have a comparison result with viscosity supersolution convex in x. In fact, we have even the more precise

THEOREM II.1. *Let v be a continuous viscosity supersolution of* (CP) *with $v(\cdot, 0) = v_0(\cdot) \geqslant u_0(\cdot)$. We assume*

$$v(x, t) \geqslant -C(1 + |x|) \qquad in \quad \mathbb{R}^N \times (0, T). \tag{$*$}$$

Then

$$v \geqslant u_{\text{Hopf}} \qquad in \quad \mathbb{R}^N \times (0, T).$$

COROLLARY II.1. *Let u and v be, respectively, convex viscosity sub- and supersolution of* (CP) *with $u(x, 0) = u_0(x) \leqslant v(x, 0) = v_0(x)$. Then*

$$u \leqslant v.$$

This last result implies in particular uniqueness of convex viscosity solution of (CP).

COROLLARY II.2. *Let $v \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^N \times (0, T))$, convex in x, be a viscosity supersolution of* (CP) *with $v(\cdot, 0) = v_0(\cdot) \geqslant u_0(\cdot)$. Then*

$$v \geqslant u_{\text{Hopf}}.$$

This result means that u_{Hopf} is the *minimum viscosity supersolution of* (CP) *convex in x.*

Remark II.1. It is worth noting that we have a comparison result for viscosity sub and supersolution although it is false in general for convex generalized solutions.

Remark II.2. We do not need any assumption on H, except (2) and its continuity.

Remark II.3. Comparison results with viscosity subsolutions convex only in x are false in general. Take in dimension 1

$$u(x, t) = \max(0; t|x|^2 - |x|^{3/2}).$$

It is easy to see that u is convex in x, but not in (x, t). Moreover $u(x, 0) = 0$. Now, we claim that u is viscosity subsolution of

$$\frac{\partial u}{\partial t} - |Du|^6 = 0 \qquad \text{in} \quad \mathbb{R}^N \times (0, T)$$

for T small enough. For the proof of this claim, we refer to the third section in which a more general counterexample is given. In this example, $u_{\text{Hopf}} \equiv 0$ and we have even $u \geqslant u_{\text{Hopf}}$.

First, we give the *proof* of *Corollary* II.1 which is easy using Theorem II.1. Since v is convex, v satisfies the assumptions of Theorem II.1, in particular $(*)$. Hence

$$v(x, t) \geqslant \sup_{p \in \mathbb{R}^N} (p \cdot x - u_0^*(p) - tH(p)).$$

By Theorem I.2, we have

$$u(x, t) \leqslant \sup_{p \in \mathbb{R}^N} (p \cdot x - u_0^*(p) - tH(p)).$$

The two inequalities yield the result.

We do not give the proof of Corollary II.2 which is very easy. Now, we turn to the *proof* of *Theorem* II.1. The proof consists in showing that if $u_0^*(p) < +\infty$ then

$$v(x, t) \geqslant p \cdot x - u_0^*(p) - tH(p).$$

If this claim is proved, it suffices to take the supremum in p in the right-hand side and to use the order-reversing property of the Fenchel conjugate to conclude.

Now, let us prove our claim. If we denote by w the function

$$w(x, t) = v(x, t) - p \cdot x + u_0^*(p) + tH(p).$$

The claim consists of proving that $w \geqslant 0$.

Now let us remark that w is viscosity supersolution of

$$\frac{\partial w}{\partial t}+H(Dw+p)-H(p)=0 \qquad \text{in} \quad \mathbb{R}^N\times(0,T)$$

and

$$w(x,0)=w_0(x)=v_0(x)-p\cdot x+u_0^*(p)\geqslant 0.$$

Let us also observe that, since v is convex, we have

$$w(x,t)\geqslant -C_1-C_2|x|, \qquad C_1, C_2\in\mathbb{R}.$$

The following lemma concludes.

LEMMA II.1. *Let $G\in C(\mathbb{R}^N)$ and let w be a continuous viscosity supersolution of*

$$\frac{\partial w}{\partial t}+G(Dw)=0 \qquad \textit{in} \quad \mathbb{R}^N\times(0,T)$$

We assume

(i) $G(0)=0$,

(ii) $w(x,0)=w_0(x)\geqslant 0$,

(iii) $w(x,t)\geqslant -C_1-C_2|x|$.

Then

$$w\geqslant 0.$$

Remark II.4. Let us first observe that we have no assumption concerning the convexity of w. This result is valid for any continuous viscosity supersolution satisfying in particular (iii).

Remark II.5. The assumption (i) ensures that 0 is viscosity solution of the problem; if we do not assume $G(0)=0$ we can only conclude

$$w(x,t)\geqslant -G(0)\,t.$$

Proof of Lemma II.1. We introduce $z(x,t)=\mathrm{Inf}(w(x,t);0)$; z satisfies the same assumptions as w, and z also satisfies

$$z(x,t)\leqslant 0.$$

We then can use a result of Crandall *et al.* [9] to conclude that $z\geqslant 0$ and therefore $w\geqslant 0$. For the sake of completeness, let us give a simple proof in our particular case. The idea of the proof consists in comparing $w(x, t)$ and $\varepsilon|x|^2$, for ε small enough. For that, we introduce the function z defined by

$$z(x, t) = w(x, t) + \varepsilon|x|^2.$$

Our claim is to prove that $z \geqslant 0$ for ε small enough.

First, we consider points x such that $|x| \to +\infty$,

$$z(x, t) \geqslant -C_1 - C_2|x| + \varepsilon|x|^2.$$

Using this inequality, one shows easily that, for ε small enough, if $|x| \geqslant R_\varepsilon$, where

$$R_\varepsilon = 2C_2\varepsilon^{-1}$$

we have

$$z(x, t) \geqslant 0 \qquad \text{for} \quad |x| \geqslant R_\varepsilon,\ t \in (0, T).$$

Now, it is enough to work in $B(0, R_\varepsilon)$. We see that in $B(0, R_\varepsilon) \times (0, T)$, z is viscosity supersolution of

$$\frac{\partial z}{\partial t} + G(Dz - 2\varepsilon x) = 0.$$

But since $|x| \leqslant R_\varepsilon$, z is viscosity supersolution of

$$\frac{\partial z}{\partial t} + \tilde{G}(Dz) = 0,$$

where $\tilde{G}(q) = \sup_{|p-q| \leqslant 4C_2} G(p)$.

Moreover,

$$z(x, 0) \geqslant 0 \qquad \text{in} \quad B(0, R_\varepsilon)$$
$$z(x, t) \geqslant 0 \qquad \text{in} \quad \partial B(0, R_\varepsilon) \times (0, T).$$

Noting that $\tilde{G}(0) \geqslant 0$ and that $-\tilde{G}(0) \cdot t$ is viscosity subsolution of the same problem, we deduce from the classical comparison result of viscosity solutions (cf. [7, 8, 29]) that

$$z(x, t) \geqslant -\tilde{G}(0) \cdot t \qquad \text{in} \quad B(0, R_\varepsilon) \times (0, T)$$

and obviously, the same inequality holds in $\mathbb{R}^N \times (0, T)$. Hence

$$w(x, t) \geqslant -\varepsilon|x|^2 - \tilde{G}(0) \cdot t.$$

Letting $\varepsilon \to 0$, we conclude

$$w(x, t) \geqslant -\tilde{G}(0)\, t \geqslant -\tilde{G}(0) \cdot T$$

with $\tilde{G}(0) = \sup_{|q| \leqslant 4C_2} G(q)$. To obtain the result, it suffices to remark that we have proved that w satisfies (iii) with $C_1 = -\tilde{G}(0)\, T$ and $C_2 = 0$. Therefore, we can take $C_2 = \eta$ for any $\eta > 0$ and the proof above gives

$$w(x, t) \geqslant \sup_{|q| \leqslant 4\eta} G(q) \cdot T.$$

Letting $\eta \to 0$, since $G(0) = 0$ and G is continuous, we conclude.

III. Remarks on Uniqueness of General Viscosity Solutions

In this section, we discuss uniqueness of general continuous viscosity solutions—not only convex viscosity solutions. Let us first mention that uniqueness results for general (i.e., not only uniformly continuous) viscosity solutions have been obtained by Crandall and Lions [11, 12] and Ishii [19]. Let us particularly point out the case of Lipschitz Hamiltonians for which one has comparizon results in $C(\mathbb{R}^N \times (0, T))$, i.e., in the largest class of functions we consider (cf. [7, 29]). The result we prove is that u_{Hopf} is the minimum viscosity supersolution of (CP) in a class of functions determined by a restriction on the growth at infinity connected with the growth of the Hamiltonian at infinity. This result is based on a result of Lions *et al.* [32]. And these comparison results are proved to be optimal. Let us finally mention that they are analogous to those obtained by Crandall and Lions [12], where the class of functions is determined by the growth of the norm of their generalized gradients, compared to the growth of the Lipschitz coefficient of the Hamiltonian at infinity.

The basic result—due to Lions *et al.* [32]—is the following

THEOREM III.1. *Let w be a continuous viscosity supersolution of*

$$\frac{\partial w}{\partial t} + G(Dw) = 0 \qquad \text{in} \quad \mathbb{R}^N \times (0, T).$$

We assume that

(i) $w(\cdot, 0) = w_0(\cdot) \geqslant C$; $C \in \mathbb{R}$,

(ii) $w(x, t) \geqslant C_1 + C_2|x|^n$; $C_1, C_2 \in \mathbb{R}$, $n > 1$,

(iii) *G is continuous and satisfies*

$$G(p) \leqslant E|p|^q + F \qquad \text{with} \quad q \leqslant \frac{n}{n-1}.$$

Then

$$w(x,t) \geqslant C - t \cdot F \qquad in \quad \mathbb{R}^N \times (0, T).$$

Moreover if $q = 1$, *the conclusion holds without any assumption on the behaviour of* w *then* $|x| \to +\infty$.

Remark III.1. Of course, a similar result holds for viscosity subsolutions with easy adaptations of assumptions.

Remark III.2. The result of Theorem III.1 extends Lemma II.1 which gives the case $n = 1$, or equivalently $n \leqslant 1$.

Remark III.3. This result is optimal: we give at the end of this section a counterexample to the comparison result when $q > n/(n-1)$.

From Theorem III.1, we can deduce the following results:

COROLLARY III.1. *Let* H *satisfying*

$$H(p) \leqslant E|p|^q + F \qquad for\ all\ p \in \mathbb{R}^N.$$

Let E_n *be defined by*

$$E_n = \{w \in C(\mathbb{R}^N \times (0, T)) \mid w(x,t) \geqslant C_1 + C_2|x|^n;\ C_1, C_2 \in \mathbb{R}\}.$$

Then if $q > 1$, u_{Hopf} *is the* minimum viscosity supersolution of (CP) in E_n *with* $n \leqslant q/(q-1)$. *If* $q \leqslant 1$, *the same result holds in* $C(\mathbb{R}^N \times (0, T))$.

Remark III.4. This result is the only general result we can prove. Some others comparison results can be obtained by using results of Crandall and Lions [12]. The question of the uniqueness of the viscosity solution u_{Hopf} in $C(\mathbb{R}^N \times (0, T))$ or even in E_n is open.

Proof of Theorem III.1. The idea of the proof is due to Lions *et al.* [32]. First, we reduce to the case when $C = 0$ and $F = 0$. Then the idea consists of examining the problem in B_R and to use the Oleinik–Lax formula.

In B_R, we have

$$\frac{\partial u}{\partial t} + E|Dw|^q \geqslant 0 \qquad \text{in} \quad B_R \times (0, T).$$

Moreover

$$w(x, 0) = w_0(x) \geqslant 0 \qquad \text{in} \quad B_R$$

and

$$w(x,t) \geqslant C_R(t)(1+R^n) \qquad \text{on} \quad \partial B_R \times (0,T),$$

where $C_R(\cdot)$ is bounded and satisfies $C_R(t) \to 0$ when $t \to 0^+$. Now in $B_R \times (0,T)$, we can apply uniqueness results for viscosity solutions (cf. [7, 8, 29]) and then

$$w(x,t) \geqslant w_R(x,t),$$

where w_R is the unique viscosity solution of

$$\begin{cases} \dfrac{\partial z}{\partial t} + E|Dz|^q = 0 & \text{in} \quad B_R \times (0,T) \\ z(x,0) = 0 \quad \text{in } B_R; & z(x,t) = C_R(t)(1+R^n) \quad \text{on } \partial B_R \times (0,T); \end{cases}$$

w_R is given by the Oleinik–Lax formula (cf. [29]).

If $q > 1$, we find

$$w_R(x,t) = \operatorname*{Inf}_{y \in B_R} \left(t^{-1/q-1} E_1(q)|x-y|^{q/q-1}\right)$$
$$\wedge \operatorname*{Inf}_{\substack{y \in \partial B_R \\ 0 \leqslant s \leqslant t}} \left((t-s)^{-1/q-1} E_1(q)|x-y|^{q/q-1} + C_R(t)(1+R^n)\right),$$

where

$$E_1(q) = (q-1)\, E^{-1/q-1}\, q^{-q/q-1} \qquad \text{and} \qquad a \wedge b = \operatorname{Inf}(a,b).$$

Since $E_1(q) \geqslant 0$, the first infimum is achieved for R large enough at $y = x$ and is equal to 0. For the second one, if you consider t such that

$$t < (E_1(q)(\max_t C_R(t))^{-1})^{q-1} = \delta.$$

Then

$$(t-s)^{-1/q-1} E_1(q) \geqslant t^{-1/q-1} E_1(q) \geqslant \max_t C_R(t) \geqslant C_R(t),$$

and then the second infimum tends to infinity when $R \to +\infty$. Passing to the limit, we get

$$w_R(x,t) \geqslant 0 \qquad \text{for} \quad t \leqslant \delta.$$

Now consider

$$T_0 = \sup\{t \leqslant T \mid w(\cdot,t) \geqslant 0\}.$$

If $T_0 = T$, we have the result. Assume on the contrary that $T_0 < T$, then by considering $v(x, t) = w_R(x, t + (T_0 - \delta/2))$. v satisfies the assumptions of Theorem III.1 and so we can apply the result proved above, i.e.,

$$v(x, t) \geq 0 \qquad \text{for} \quad t \leq \delta$$

and so

$$w_R(x, t) \geq 0 \qquad \text{for} \quad t \leq T_0 + \delta/2,$$

which contradicts the definition of T_0.

If $q = 1$, we obtain

$$w_R(x, t) = \operatorname*{Inf}_{\substack{y \in B_R \\ |y - x| \leq Et}} \{0\} \wedge \operatorname*{Inf}_{\substack{y \in \partial B_R \\ 0 \leq s \leq t \\ |y - x| \leq E|t - s|}} \{C_R(t)(1 + R^n)\}.$$

It suffices to remark that if R is large enough

$$\{y \in \partial B_R \mid |y - x| \leq E|t - s|\} = \varnothing \qquad \text{for all } 0 \leq s \leq t.$$

And so

$$w_R(x, t) \geq 0.$$

In the two cases, we have proved $w_R \geq 0$; since $w \geq w_R$ the proof is complete.

Using Theorem III.1, the proof of Corollary III.1 is a straightforward adaptation of the ideas of Theorem II.1, so we skip it. Now, we introduce a counterexample of Theorem III.1 and of Corollary III.1 when the assumption $q \leq n/(n-1)$ is not satisfied.

Let us define w by

$$w(x, t) = \text{Min}(0; -t|x|^n + |x|^k),$$

where $k < n$ will be choosen later on. We can already remark that

$$w(x, 0) = 0.$$

We need

LEMMA III.1. *Let A be defined by*

$$A = \{(x, t) \in \mathbb{R}^N \times]0, T[\mid w(x, t) = -t|x|^n + |x|^k\}.$$

There exists $k<n$ such that $-t|x|^n+|x|^k$ is a viscosity supersolution of

$$\frac{\partial u}{\partial t}+|Du|^q=0 \qquad \text{in a neighbourhood of } A, \text{ for } t \text{ small enough.}$$

Let us first conclude by using this result. In the neighbourhood of A, w is the minimum of two viscosity supersolution of

$$\frac{\partial u}{\partial t}+|Du|^q=0. \tag{P}$$

Therefore w is also a viscosity supersolution of (P) in this neighbourhood. Moreover, in the open complementary of A, $w\equiv 0$ and the same result holds. And then we can conclude that w is a viscosity supersolution of (P) in $\mathbb{R}^N\times\,]0, T[$, T small enough. Finally since $k<n$ for all $t>0$,

$$w(x,t)\to -\infty \qquad \text{when} \quad |x|\to +\infty$$

so w is certainly not non negative.

Finally, we *prove Lemma* III.1. Let A_ε defined by

$$A_\varepsilon=\{(x,t)\in\mathbb{R}^N\times\,]0,T[\;|\;|x|^{k-n}t^{-1}<1+\varepsilon\}.$$

It is easy to see that A_ε is a neighbourhood of A and we are going to prove that for convenient k and ε, A_ε satisfies the property announced in Lemma III.1. Let us denote by z the quantity

$$z(x,t)=-t|x|^n+|x|^k.$$

And let us compute

$$\frac{\partial z}{\partial t}+|Dz|^q=-|x|^n+|nt|x|^{n-1}-k|x|^{k-1}|^q,$$

which is equal to

$$|x|^n\,[-1+|x|^{(n-1)q-n}\cdot t^q|n-k|x|^{k-n}\cdot t^{-1}|^q].$$

Now, if we impose

$$n>k(1+\varepsilon)$$

(this is possible since we have to choose $k<n$). With this choice the bracket is greater than

$$-1+|x|^{(n-1)q-n}\,t^q|n-(1+\varepsilon)\,k|^q.$$

Now estimating $|x|$ by $[(1+\varepsilon)\,t]^{\gamma}$ with $\gamma=(k-n)^{-1}$ we conclude that the bracket is greater than

$$-1+(1+\varepsilon)^{\delta_1}\,t^{\delta_2}|n-(1+\varepsilon)\,k|^q,$$

with

$$\delta_1=[(n-1)\,q-n](k-n)^{-1}$$
$$\delta_2=q+[(n-1)\,q-n](k-n)^{-1}.$$

But we can choose $k<n$ such that $\delta_2<0$, then we choose ε and it is easy to see that for t small enough, the quantity above is positive and the proof is complete.

Remark III.6. A similar counterexample can be built in the same way for viscosity subsolutions.

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