A Model of Set-Theory in Which Every Set of Reals is Lebesgue Measurable

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A model of set-theory in which every set of reals is Lebesgue measurable[*]

By Robert M. Solovay[**]

We show that the existence of a non-Lebesgue measurable set cannot be proved in Zermelo-Frankel set theory (**ZF**) if use of the axiom of choice is disallowed. In fact, even adjoining an axiom **DC** to **ZF**, which allows countably many consecutive choices, does not create a theory strong enough to construct a non-measurable set.

Let **ZFC** be Zermelo-Frankel set theory together with the axiom of choice. Let **I** be the statement: There is an inaccessible cardinal[1].

THEOREM 1. *Suppose that there is a transitive ε-model of* **ZFC** + **I**. *Then there is a transitive ε-model of* **ZF** *in which the following propositions are valid.*

(1) *The principle of dependent choice* (= **DC**, cf. III. 2.7.)

(2) *Every set of reals is Lebesgue measurable* (**LM**).

(3) *Every set of reals has the property of Baire.*[2]

(4) *Every uncountable set of reals contains a perfect subset* (**P**).

(5) *Let* $\{A_x : x \in \mathbf{R}\}$ *be an indexed family of non-empty set of reals with index set the reals. Then there are Borel functions,* h_1, h_2 *mapping* $\mathbf{R}$ *into* $\mathbf{R}$ *such that*

(a) $\{x \mid h_1(x) \notin A_x\}$ *has Lebesgue measure zero.*

(b) $\{x \mid h_2(x) \notin A_x\}$ *is of first category.*

Remarks. 1. It is known that the theory **ZFC** + **I** has a transitive ε-model if **ZF** + **DC** + **P** does; cf. [10, pp. 213–214]. Thus the hypothesis of Theorem 1 (that **ZFC** + **I** has a transitive ε-model) cannot be weakened. However it does seem likely that the existence of a transitive model of **ZF** + **DC** + **LM** is a consequence, in **ZFC**, of the existence of a transitive

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[1] In the presence of the axiom of choice, we identify cardinals with initial ordinals. A cardinal ℵ is *regular*, if each order unbounded subset of ℵ has power ℵ. A cardinal ℵ is inaccessible, if it is regular, uncountable, and for ℵ′ < ℵ, $2^{ℵ'} < ℵ$.

[2] A set of reals A has the Baire property if there is an open set U such that $(A - U) \cup (U - A)$ is of the first category.

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model of **ZFC**.

2. Our proofs use Cohen's forcing method. In the usual way (cf. [8, pp. 132–133]), they can be recast as finitistic relative consistency proofs.

3. We take this opportunity to describe some recent work on the model of Theorem 1. Mathias has shown that the following Ramsey-like theorem holds in the model. Divide the set of all infinite subsets of ω into two disjoint pieces. Then there is an infinite subset A of ω such that every infinite subset of A lies in the same piece as A.

Levy and the author have shown that in this model every set of reals is the union of $\aleph_1$ Borel sets. This should be contrasted with the following consequence of **DC** and the axiom of determinateness (**AD**), due to Moschovakis: Every union of $\aleph_1$ Borel sets is Σ^1_2. It follows that **AD** fails in the model.

This result might seem to throw cold water on a conjecture of the author that a suitable large cardinal axiom will imply that **AD** holds in $L[\mathbf{R}]$. Closer inspection shows that there is no conflict between the conjecture and this result.

4. It is fairly easy to deduce parts (2) and (3) of Theorem 1 from part (5). Nevertheless, we have included our original proofs of (2) and (3) since they are much simpler and more natural than the proof of (5). (The ideas in our proof of (2) will be used in a forthcoming paper of the author to show that the existence of measurable cardinals implies that every Σ^1_2 set of reals is Lebesgue measurable.)

5. Proposition (5) of Theorem 1 was suggested to the author by Mycielski. According to Mycielski, (5) implies that every subset of $\mathbf{R}^3$ has a newtonian capacity. The author is totally ignorant of the theory of capacities, so will simply pass on (slightly re-phrased) the relevant portion of Mycielski's letter in the hope that some knowledgeable reader may understand it.

Let C be Choquet's paper, *Theory of capacities*, Annales de l'Institute de Fourier, 5 (1955), 131–292. Mycielski's remark is that using (5), we can establish 37.1 of C for arbitrary subsets A of $\mathbf{R}^2$. This statement generalized to the case $A \subseteq X \times Y$ where X is an arbitrary separable measure space implies capacitability of all sets with respect to the classical capacities interpreted as in 49.3 and 49.4 of C. (In the application, X is the set of brownian trajectories with the Wiener measure, and $Y = [0, \infty)$.)

6. The reader will find in [10] a detailed discussion of various forms of the axiom of choice whose failure follows from (1)–(4) of Theorem 1, e.g., the axiom of choice for families of two-element sets.

We add a brief discussion on the Hahn-Banach theorem. Of course the

Hahn-Banach theorem for *separable* Banach spaces follows readily from **DC**. On the other hand, one can deduce from (3) of Theorem 1 that there is no finitely additive probability measure on the power set of ω which vanishes on singletons. It follows that the Hahn-Banach theorem fails in the model of Theorem 1.

Of course, the axiom of choice is true, and so there are non-measurable sets. It is natural to ask if one can explicitly describe a non-Lebesgue measurable set.[3] Our next theorem bears on this question.

We say that a set of reals A is *definable* from a set x_0 if there is a set-theoretical formula $\Psi(x, y)$ (having free only the variables x and y) such that

$$A = \{y \in \mathbf{R} \colon \Psi(x_0, y)\} \ .$$

Because of the familiar difficulties about the "undefinability of truth" it is not clear how to express the notion "definable" by a set-theoretical formula. However, Myhill and Scott [11] have shown that the notion "A is definable from some countable sequence of ordinals" is expressible by a set-theoretical formula. Thus we can formulate in set-theory the propositions referred to in the following theorem.

THEOREM 2. *Suppose that* **ZFC** $+$ **I** *has a transitive ε-model. Then so does the theory* **ZFC** $+$ **GCH** *together with analogs of* (2) *through* (5) *of Theorem* 1. (*We state the analog of* (2):

(2$'$) *Every set of reals definable from a countable sequence of ordinals is Lebesgue measurable.*)

Remark. Since a real can be coded into a countable sequence of zeros and ones, every set definable from a real is, *ipso facto*, definable from a countable sequence of ordinals. In particular, every projective set of reals is definable from a countable sequence of ordinals.

McAloon has simplified the author's original proof of Theorems 1 and 2. (We present McAloon's version of the proof below.) As McAloon and the author independently noticed, McAloon's version of the proof allows one to prove the following

THEOREM 3. *Assume the hypotheses of Theorem* 1. *Then there is a transitive ε-model of* **ZFC** *in which* $2^{\aleph_0} = \aleph_2$ *and the analogs of* (2)–(5) *of Theorem* 1 *referred to in Theorem* 2 *are valid.*

Remarks. 1. It is clear from the proof (cf. III 3.9) that Theorem 3 remains true if "$2^{\aleph_0} = \aleph_2$" is replaced by "$2^{\aleph_0} = \aleph_\Lambda$" for a wide variety of reasonable Λ; e.g., $\Lambda = 3$, $\Lambda = \aleph_2$, etc.

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[3] This question was suggested to the author by Milnor.

2. A natural question suggested by Theorem 3 is whether there are models of the sort described in Theorem 3 in which Martin's axiom holds. We can construct such models using a remark of Kunen but we need to take the large cardinal in the ground model weakly compact as well as strongly inaccessible. It seems likely that this large cardinal assumption can be appreciably weakened.

Our paper is divided into three main sections. Section I begins with general remarks on the notion of forcing. I.1 gives a precise mathematical interpretation of the concept of "generic". (Cohen did not need such a definition; he simply constructed and studied one generic filter.) The advantage of having such a precise notion is shown in I.2, where we relate generic filters on various partially ordered sets; cf. especially Lemma 2.3 relating a generic filter on the product of two partially ordered sets with the filters on the factors. I.3 describes a model due to A. Levy [9], which is the model of our Theorem 2 and gives proofs of basic facts about this model (also due to Levy). In I.4, we prove an important lemma which allows us to enlarge the ground model of Levy's construction so as to absorb a specified real of the extension.

Section II.1 contains foundational material about the relations between Borel sets of a transitive ε-model $\mathfrak{M}$ and an extension $\mathfrak{N}$ of $\mathfrak{M}$. We show that there is a natural way to prolong the Borel sets of $\mathfrak{M}$ to sets of $\mathfrak{N}$ which preserves most properties of the Borel set. II.2 defines and studies one of the main technical devices of the paper, the notion of a random real. (Roughly speaking a real is random if it avoids all the sets of measure zero that one can explicitly define. An alternative heuristic definition is that random reals are those reals whose binary expansions are obtained by tossing an honest coin infinitely many times.)

Finally III puts the material of I and II together and proves Theorems 1 through 3.

We close this introduction by thanking various people who in one way or another materially helped us in this work. The original problem of showing **ZF** + **LM** consistent was suggested to the author by Paul Cohen. (And of course Cohen's idea of forcing [2] is the *sine qua non* of our proof.) We are grateful to Levy for sending us a preprint of his work on the model $\mathfrak{N}$ (of Theorem 2) and for permission to incorporate proofs of his results into our paper. Ken McAloon made a vital simplification in our proof which reduced our original cumbersome verification of **DC** to a triviality. Finally, we are grateful to Hao Wang and the Rockefeller University for hospitality during the year when this paper was finally written.

I. THE MODEL

1. Generic filters

1.1. We are going to review briefly the formalism of forcing in a form suitable for applications in this paper. Proofs will not be given for these results. The reader familiar with [8] and [1] should be able to reconstruct the proofs, cf. § 1.10 for some discussion of the proofs.

1.2. **ZF** is Zermelo-Frankel set theory. **ZFC** is **ZF** plus the axiom of choice. Let $\mathfrak{M}$ be a transitive model of **ZFC**. We do not assume that $\mathfrak{M}$ is countable, but we shall assume, for convenience, that $\mathfrak{M}$ is a set.

Let $\mathscr{P}$ be a non-empty partially ordered set lying in $\mathfrak{M}$. We suppose in addition that the partial order $\leq$ on $\mathscr{P}$ is reflexive, i.e., if $x \in \mathscr{P}$, $x \leq x$. (We assume, of course, that the ordering $\leq$ lies in $\mathfrak{M}$.)

Two elements of $\mathscr{P}$, x and y, are *compatible* if $(\exists z \in \mathscr{P})$ $(x \leq z$ and $y \leq z)$. Otherwise, they are *incompatible*.

A subset X of $\mathscr{P}$ is *dense* if

(1) if $x \in X$, $y \in \mathscr{P}$, and $x \leq y$, then $y \in X$;

(2) if $x \in \mathscr{P}$, there is a $y \in X$ with $x \leq y$.[4]

1.3. Let G be a subset of $\mathscr{P}$. We say that G is an $\mathfrak{M}$-*generic filter*[5] on $\mathscr{P}$ if:

(1) If $x, y \in G$, then there is a $z \in G$, with $x \leq z$, and $y \leq z$.

(2) If $x \in G$, $y \in \mathscr{P}$, and $y \leq x$, then $y \in G$.

(3) Let $X \subseteq \mathscr{P}$, $X \in \mathfrak{M}$, and suppose that X is dense. Then $X \cap G$ is non-void.

1.4. Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}$. Then there is a transitive model $\mathfrak{M}[G]$ of **ZF** with the following properties:

(1) $\mathfrak{M} \subseteq \mathfrak{M}[G]$;

(2) $G \in \mathfrak{M}[G]$;

(3) if $\mathfrak{N}$ is a transitive model of **ZF** such that $\mathfrak{M} \subseteq \mathfrak{N}$ and $G \in \mathfrak{N}$ then $\mathfrak{M}[G] \subseteq \mathfrak{N}$.

It is clear that (1)–(3) characterize $\mathfrak{M}[G]$. $\mathfrak{M}[G]$ has the following additional properties:

(4) The axiom of choice holds in $\mathfrak{M}[G]$.

(5) Let α be an ordinal. Then $\alpha \in \mathfrak{M}[G]$ if and only if $\alpha \in \mathfrak{M}$.

1.5 We introduce a first order language $\mathscr{L}$ as follows: the predicates of

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[4] Our conventions are such that if $x \leq y$, y "gives *more* information" than x.

[5] Our original definition of generic was based on "complete sequences". The present approach is due to Levy [8].

$\mathcal{L}$ are ε and a one-place predicate S. We interpret $\mathcal{L}$ in $\mathfrak{M}[G]$ as follows: ε is interpreted in the obvious way; Sx holds in $\mathfrak{M}[G]$ if and only if $x \in \mathfrak{M}$.

We can formulate new instances of the replacement axiom involving the predicate S. All these instances are valid in $\mathfrak{M}[G]$.

1.6. Let $A \in \mathfrak{M}[G]$, with $A \subseteq \mathfrak{M}$. Then there is a model $\mathfrak{M}[A]$ of **ZFC** with the following properties:

(1) $\mathfrak{M} \subseteq \mathfrak{M}[A]$.

(2) $A \in \mathfrak{M}[A]$.

(3) If $\mathfrak{N}$ is a transitive model of **ZF**, $\mathfrak{M} \subseteq \mathfrak{N}$, and $A \in \mathfrak{N}$, then $\mathfrak{M}[A] \subseteq \mathfrak{N}$.

It is clear that (1)–(3) uniquely characterize $\mathfrak{M}[A]$, and that $\mathfrak{M}[A] \subseteq \mathfrak{M}[G]$. If we interpret $\mathcal{L}$ in $\mathfrak{M}[A]$, by interpreting ε in the obvious way and interpreting S as before, then all the instances of the replacement axiom expressible in $\mathcal{L}$ hold in $\mathfrak{M}[A]$.

1.7. If $\mathfrak{A}$ is a relational system, and Φ a sentence, we use the notation

$$\mathfrak{A} \models \Phi$$

to mean: Φ is true in $\mathfrak{A}$.

There is a formula $\Phi(v_1, v_2, v_3)$ of $\mathcal{L}$ with the following properties:

(1) $\mathfrak{M}[G] \models \Phi(x, y, z) \rightarrow x \in \mathfrak{M}, y \subseteq \mathfrak{M}$.

(2) If $\mathfrak{M}[G] \models \Phi(x, y, z)$ and $\mathfrak{M}[G] \models \Phi(x, y, z')$, then $z = z'$.

(3) Let $A \subseteq \mathfrak{M}$, $A \in \mathfrak{M}[G]$. Then

$$\mathfrak{M}[G] \models \Phi(x, A, z) \rightarrow z \in \mathfrak{M}[A] \ .$$

(4) Let A be as in (3), and let $z \in \mathfrak{M}[A]$. Then

$$\mathfrak{M}[G] \models \Phi(x, A, z) \longleftrightarrow \mathfrak{M}[A] \models \Phi(x, A, z) \ .$$

(5) $\mathfrak{M}[A] = \{z \mid (\exists x \in \mathfrak{M})(\mathfrak{M}[G] \models \Phi(x, A, z))\}$.

Roughly speaking Φ is constructed as follows. We can describe $\mathfrak{M}[A]$ as the set of denotations of terms of a certain ramified language $\mathcal{L}_*$; if t is a term of $\mathcal{L}_*$, and u is the collection of sets of $\mathfrak{M}$ of rank $\leq$ rank (t), then we can "compute" the denotation of t from t, u, and A. Then $\Phi(\langle t, u \rangle, A, z)$ holds just in case z is the denotation of t.

The existence of Φ with the properties just stated has several important consequences:

(a) Let A be as above. Let $x \in \mathfrak{M}[A]$. Then x is definable in

$$\langle \mathfrak{M}[A]; \varepsilon, S, A \rangle$$

from some element y of $\mathfrak{M}$.

(b) The predicate "$y \in \mathfrak{M}[A]$" is expressible in $\langle \mathfrak{M}[G]; \varepsilon, S \rangle$ (by $(\exists x)\Phi(x, A, y)$). Thus we can lay our hands on $\mathfrak{M}[A]$ inside $\mathfrak{M}[G]$.

1.8. We now make the following countability assumption on $\mathfrak{M}, \mathscr{P}$: *there are only countably many subsets of $\mathscr{P}$ lying in $\mathfrak{M}$.* This has the following important consequence. Let $p \in \mathscr{P}$. Then there is an $\mathfrak{M}$-generic filter G on $\mathscr{P}$ with $p \in G$.

1.9. *Forcing.* Let $\mathfrak{M}, \mathscr{P}$ be as in § 1.8. We enlarge $\mathscr{L}$ to a language $\mathscr{L}'$ as follows. If $x \in \mathfrak{M}$, we introduce a term $\underline{x}$; we also have a term $\underline{G}$. If G is a generic filter on $\mathscr{P}$, we interpret $\mathscr{L}'$ in $\mathfrak{M}[G]$ by letting $\underline{x}$ denote x, and letting $\underline{G}$ denote G.

We can arrange matters so that each formula of $\mathscr{L}'$ is (coded by) a set of $\mathfrak{M}$, and all the usual syntactical properties relevant to $\mathscr{L}'$ are expressible in $\mathfrak{M}$.

Let Φ be a sentence of $\mathscr{L}'$, and $p \in \mathscr{P}$. We say that *p forces Φ* if

$$\mathfrak{M}[G] \models \Phi$$

whenever G is an $\mathfrak{M}$-generic filter containing p. (Notation: $p \Vdash \Phi$.)

A fundamental fact about forcing is the connection between forcing and truth: If G is an $\mathfrak{M}$-generic filter on $\mathscr{P}$ and

$$\mathfrak{M}[G] \models \Phi \ ,$$

then Φ is forced by some $p \in G$.

It follows that if p does not force Φ, there is an extension p' of p such that p' forces $\neg \Phi$.

We say that *p decides Φ* if $p \Vdash \Phi$ or $p \Vdash \neg \Phi$. (Notation: $p \| \Phi$.) We write $\Vdash \Phi$, if for every $p \in \mathscr{P}$, $p \Vdash \Phi$.

Suppose now that $\Phi(w_0, \cdots, w_n)$ is a formula of $\mathscr{L}$. Then the relation

(1) $$p \Vdash \Phi(\underline{G}, \underline{x}_1, \cdots, \underline{x}_n)$$

is expressible in $\mathfrak{M}$; i.e., there is a formula $\Psi(x_0, \cdots, x_n)$ such that (1) holds if and only if

$$\mathfrak{M} \models \Psi(p, x_1, \cdots, x_n) \ .$$

1.10. We know of no proof of the results stated above which does not require preliminary indirect definitions of $\mathfrak{M}[G]$ and $\Vdash$. For example, one can extend $\mathscr{L}'$ to a ramified language $\mathscr{L}''$, and define $\mathfrak{M}[G]$ for any $G \subseteq \mathscr{P}$ as the set of denotations of terms of $\mathscr{L}''$; cf. [8] for a representative special case. One defines an auxiliary forcing relation, say $\Vdash'$, by induction on some ordinal measure of the complexity of a sentence of $\mathscr{L}''$. (For example, if $p, q \in \mathscr{P}$, we would have

$$p \Vdash' \underline{q} \in \underline{G}$$

if and only if $q \leqq p$.) The correct forcing relation, $p \Vdash \Phi$, is defined in terms

of $\Vdash'$ by $p \Vdash \Phi$ if and only if $p \Vdash' \neg \neg \Phi$ (" $\neg$ " is the negation symbol).

An alternative proof of these results can be given in terms of boolean valued models (cf. [12]).

1.11. Recall that a cardinal Ω is *regular* if each subset $A \subseteq \Omega$ of cardinality less than Ω has a sup less than Ω. A cardinal Ω is strongly inaccessible, if Ω is regular, greater than $\aleph_0$, and satisfies

$$\aleph < \Omega \to 2^\aleph < \Omega .$$

(Here $\aleph$ ranges over infinite cardinals.)

We shall need the following known result.

LEMMA. *Let $\mathfrak{M}$ and $\mathcal{P}$ be as in §1.8. Let $\Omega \in \mathfrak{M}$ be such that $\mathfrak{M} \models$ "Ω is strongly inaccessible, and the cardinality of $\mathcal{P}$ is less than Ω". Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}$. Then Ω is strongly inaccessible in $\mathfrak{M}[G]$.*

1.12. *Collapsing a cardinal.* Now let $\mathfrak{M}$ be a countable transitive model of ZFC. Let λ be a non-zero ordinal of $\mathfrak{M}$. Let $\mathcal{P}_\lambda$ be the set of functions f whose domain is a finite subset of ω and whose range is a subset of λ. We partially order $\mathcal{P}_\lambda$ by inclusion: $f \leq g$ if and only if $f \subseteq g$.

Let G be a generic filter on $\mathcal{P}_\lambda$. Then $\bigcup G$ is a function $F \colon \omega \to \lambda$ and F is surjective. It follows that λ is countable in $\mathfrak{M}[G]$; cf. the discussion of §3.2 for proof of a related result.

One can recover G from F as follows:

$$G = \{f \in \mathcal{P}_\lambda \colon f \subseteq F\} .$$

It follows that $\mathfrak{M}[G] = \mathfrak{M}[F]$.

We say that $F \colon \omega \to \lambda$ is a generic collapsing function (more precisely, an $\mathfrak{M}$-generic collapsing function), if F arises from a generic filter on $\mathcal{P}_\lambda$ in the manner just described.

LEMMA. *Let $F \colon \omega \to \lambda$ be an $\mathfrak{M}$-generic collapsing map. Then there is an $s \subseteq \omega$, $s \in \mathfrak{M}[F]$, with*

$$\mathfrak{M}[F] = \mathfrak{M}[s] .$$

PROOF. We put

$$s = \{2^n 3^m \mid F(n) \leq F(m)\} .$$

Clearly $s \in \mathfrak{M}[F]$. To complete the proof, we show that $F \in \mathfrak{M}[s]$.

Put $m \sim n$ if and only if $F(m) = F(n)$. Clearly the relation $\sim$ lies in $\mathfrak{M}[s]$. Therefore so does the set $A = \omega/\sim$ of equivalence classes. We order A by $[m] < [n]$ if and only if $2^n 3^m \notin s$. Clearly F induces a map F' of $\langle A, < \rangle$

onto $\langle \alpha, < \rangle$ in an order preserving way. Thus $\langle A, < \rangle$ is well-ordered. Hence it is well-ordered in $\mathfrak{M}[s]$. Let $F'': A \to \alpha'$ be an isomorphism of $\langle A, < \rangle$ with $\langle \alpha', < \rangle$ in $\mathfrak{M}[s]$. (F'' exists since

(1) $\mathfrak{M}[s]$ is a model of ZFC, and

(2) a theorem of ZFC asserts every well-ordered set is order isomorphic to an ordinal.)

The map

$$F'' \circ (F')^{-1}$$

is clearly the identity map so $F' \in \mathfrak{M}[s]$. It follows that $F \in \mathfrak{M}[s]$.

Remark. This lemma is the motivation for my paper [14].

2. Some lemmas on genericity

2.1. Throughout this section $\mathfrak{M}$ is a countable transitive model of ZFC. The following lemma is trivial but useful.

LEMMA. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be non-empty reflexive partially ordered sets lying in $\mathfrak{M}$, and let $\Psi \colon \mathcal{P}_1 \to \mathcal{P}_2$ be an order isomorphism lying in $\mathfrak{M}$. If $A \subseteq \mathcal{P}_1$, let*

$$\Psi_*(A) = \{\Psi(x) \colon x \in A\} \, .$$

Then for $G \subseteq \mathcal{P}_1$, G is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$ if and only if $\Psi_*(G)$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_2$. Moreover, $\mathfrak{M}[G] = \mathfrak{M}[\Psi_*(G)]$.

2.2. Now let $\mathcal{P}_1$, $\mathcal{P}_2$ be reflexive partially ordered sets lying in $\mathfrak{M}$. We suppose that $\mathcal{P}_1 \subseteq \mathcal{P}_2$, and that the order on $\mathcal{P}_1$ is the restriction of the order on $\mathcal{P}_2$.

Definition. $\mathcal{P}_1$ is *cofinal* in $\mathcal{P}_2$ if for every $x \in \mathcal{P}_2$ there is a $y \in \mathcal{P}_1$ with $x \leq y$.

LEMMA. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be non-empty reflexive partially ordered sets lying in $\mathfrak{M}$. Suppose that $\mathcal{P}_1$ is cofinal in $\mathcal{P}_2$. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}_2$. Then $G \cap \mathcal{P}_1$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$. The map Ψ, given by*

$$\Psi(G) = G \cap \mathcal{P}_1 \, ,$$

gives a bijection of the set of $\mathfrak{M}$-generic filters on $\mathcal{P}_2$ with the set of $\mathfrak{M}$-generic filters on $\mathcal{P}_1$. Moreover, $\mathfrak{M}[G] = \mathfrak{M}[\Psi(G)]$.

PROOF. Let G be a $\mathfrak{M}$-generic filter on $\mathcal{P}_2$. Then it is straightforward to verify that $G \cap \mathcal{P}_1$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$. To verify clause (1) of § 1.3, let $x, y \in G \cap \mathcal{P}_1$. Then there is a $z \in G$ with $x \leq z$, $y \leq z$. Since $\mathcal{P}_1$ is cofinal in $\mathcal{P}_2$ and G satisfies § 1.3, (2), we can assume $z \in \mathcal{P}_1$. For clause (3), note that if X is a dense subset of $\mathcal{P}_1$,

$$\{y \in \mathscr{P}_2 \mid (\exists x \in X)(x \leqq y)\}$$

is dense in $\mathscr{P}_1$. Clause (2) is trivial to verify.

Now suppose that G_1 and G_2 are distinct $\mathfrak{M}$-generic filters on $\mathscr{P}$. We show that

(1) $$\Psi(G_1) \neq \Psi(G_2) .$$

Let $p \in G_1$, $p \notin G_2$. (If necessary, we interchange G_1 and G_2 to get such a p.) Let

$$X = \{q \in \mathscr{P}_2 \mid q \geqq p \quad \text{or } p \text{ and } q \text{ are incompatible}\} .$$

Then X lies in $\mathfrak{M}$ and X is dense in $\mathscr{P}$. Pick $q \in G_2 \cap X$. If $q \geqq p$, we would have $p \in G_2$, contradicting our choice of p. Thus q is incompatible with p. Replacing q by an extension if necessary, we may also suppose $q \in \mathscr{P}_1$. Thus $q \in G_2 \cap \mathscr{P}_1$, but $q \notin G_1 \cap \mathscr{P}_1$, since any two members of G_1 have a common extension in G_1, and $p \in G_1$. This proves (1) and shows that G is one-to-one.

Let H be an $\mathfrak{M}$-generic filter on $\mathscr{P}_1$. Put $G = \{x \in \mathscr{P}_2 \mid (\exists y \in H)(x \leqq y)\}$. We leave to the reader to verify that G is an $\mathfrak{M}$-generic filter on $\mathscr{P}_2$ and that $\Psi(G) = H$. This shows Ψ is onto, and the lemma is proved. (The last sentence of the lemma is clear from our explicit description of Ψ^{-1}.)

2.3. We now consider the following situation: $\mathscr{P}_1$ and $\mathscr{P}_2$ are reflexive non-empty partially ordered sets lying in $\mathfrak{M}$. We define a partially ordered set $\mathscr{P}$ as follows: as a set,

$$\mathscr{P} = \mathscr{P}_1 \times \mathscr{P}_2 ;$$

let $\langle p_1, p_2 \rangle$, $\langle p'_1, p'_2 \rangle$ be elements of $\mathscr{P}$. Then

$$\langle p_1, p_2 \rangle \leqq \langle p'_1, p'_2 \rangle$$

if and only if $p_1 \leqq p'_1$ and $p_2 \leqq p'_2$.

The following lemma characterizes the $\mathfrak{M}$-generic filters on $\mathscr{P}$.

LEMMA. *Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}$. Then $G = G_1 \times G_2$ where G_1 is an $\mathfrak{M}$-generic filter on $\mathscr{P}_1$, and G_2 is an $\mathfrak{M}[G_1]$-generic filter on $\mathscr{P}_2$.*

Conversely, let G_1 be an $\mathfrak{M}$-generic filter on $\mathscr{P}_1$ and G_2 an $\mathfrak{M}[G_1]$-generic filter on $\mathscr{P}_2$. Then $G_1 \times G_2$ is an $\mathfrak{M}$-generic filter on $\mathscr{P}$.

PROOF. Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}$. Put

$$G_1 = \{x \in \mathscr{P}_1 \colon (\exists y \in \mathscr{P}_2)(\langle x, y \rangle \in G)\} .$$
$$G_2 = \{x \in \mathscr{P}_2 \colon (\exists y \in \mathscr{P}_1)(\langle y, x \rangle \in G)\} .$$

Clearly, $G \subseteq G_1 \times G_2$. Conversely, let $\langle x, y \rangle \in G_1 \times G_2$. Pick $x' \in \mathscr{P}_2$ and $y' \in \mathscr{P}_1$ with $\langle x, x' \rangle$ and $\langle y', y \rangle \in G$. Let $\langle z, z' \rangle \in G$ be a common extension of $\langle x, x' \rangle$ and $\langle y', y \rangle$. Then

$$\langle x, y \rangle \leqq \langle z, z' \rangle$$

so $\langle x, y \rangle \in G$. This proves $G = G_1 \times G_2$.

We next verify that G_1 is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$. Clauses (1) and (2) of § 1.3 are trivial to verify. We turn to clause (3). Let $X \subseteq \mathcal{P}_1$ be dense in $\mathcal{P}_1$, with $X \in \mathfrak{M}$. Then $X \times \mathcal{P}_2$ is a dense subset of $\mathcal{P}$ lying in $\mathfrak{M}$. Since G is $\mathfrak{M}$-generic, $G \cap X \times \mathcal{P}_2$ is non-empty; i.e., $G_1 \cap X$ is non-empty.

We now show that G_2 is an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_2$. Again, § 1.3 (1)–(2) is trivial to verify. Let $X_1 \in \mathfrak{M}[G_1]$ be a dense subset of $\mathcal{P}_2$. By § 1.9, there is a formula $\Phi(x)$ of $\mathcal{L}'$ such that

$$\mathfrak{M}[G_1] \models \Phi(x)$$

if and only if $x = X_1$. We use $\Phi(x)$ to translate any assertion about X_1 into a statement of $\mathcal{L}'$; i.e., replace $\Psi(X)$ by

$$(u)\big(\Phi(u) \rightarrow \Psi(u)\big) \, .$$

Let p_1 be an element of G_1 which forces "$(\exists!\, x)\Phi(x)$ and $\forall x\big(\Phi(x) \rightarrow x$ is a dense subset of $\mathcal{P}_2\big)$".

Let

$$X_2 = \{\langle p, q \rangle \in \mathcal{P}\colon \text{either } p \text{ is incompatible with } p_1 \text{ or } p_1 \leqq p \text{ and } p \Vdash q \in X_1\} \, .$$

We claim X_2 is a dense subset of $\mathcal{P}$.

Suppose first that $\langle p, q \rangle \in X_2$ and $\langle p, q \rangle \leqq \langle p', q' \rangle$. We must show $\langle p', q' \rangle \in X_2$. This is trivial unless $p_1 \leqq p \leqq p'$. If this is so, p' forces the following:

(1) X_1 is a dense subset of $\mathcal{P}_2$.

(2) $q \in X_1$.

(3) $q \leqq q'$.

Hence, p' forces $q' \in X_1$ and so $\langle p', q' \rangle \in X_2$.

Next let $\langle p, q \rangle \in \mathcal{P}$. We show that $\langle p, q \rangle$ has an extension $\langle p^*, q^* \rangle$ lying in X_2. If p is incompatible with p_1, $\langle p, q \rangle$ itself lies in X_2. So suppose that p and p_1 have the common extension p'. Then p' forces the following statements (since $p' \geqq p_1$):

(1) X_1 is a dense subset of $\mathcal{P}_2$.

(2) $q \in \mathcal{P}_2$.

(3) $(\exists x \in \mathcal{P}_2)(q \leqq x \text{ and } x \in X_1)$.

Hence there is a $q^* \in \mathcal{P}_2$, and a $p^* \geqq p'$ such that

$$p^* \Vdash \text{``}q \leqq q^* \text{ and } q^* \in X_1\text{.''} \, .$$

It follows that $\langle p^*, q^* \rangle \in X_2$. Clearly $\langle p, q \rangle \leqq \langle p^*, q^* \rangle$.

Thus X_2 is dense. Since forcing is definable in $\mathfrak{M}$ (§ 1.9), $X_2 \in \mathfrak{M}$. Since

G is $\mathfrak{M}$-generic, there is $\langle p^*, q^* \rangle \in G \cap X_2$. Now since p^* and p_1 both lie in G_1, they are compatible. Thus

$$p_1 \leqq p^* \;,$$

and $p^* \Vdash q^* \in X_1$. Since $p^* \in G_1$, we have

$$\mathfrak{M}[G_1] \models q^* \in X_1 \;.$$

This shows $G_2 \cap X_1$ contains q^*, and so is non-empty. Our proof that G_2 is $\mathfrak{M}[G_1]$-generic is complete. This proves the first half of the lemma.

Now suppose that G_1 is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$ and G_2 is an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_2$. Put $G = G_1 \times G_2$. We show G is an $\mathfrak{M}$-generic filter on $\mathcal{P}$. As usual clauses (1) and (2) of §1.3 are trivial to verify. Now let $X \in \mathfrak{M}$ be a dense subset of $\mathcal{P}$. We show that $X \cap G$ is non-empty.

Let $X' = \{q \in \mathcal{P}_2 \mid (\exists p \in G_1)(\langle p, q \rangle \in X)\}$. Clearly $X' \subseteq \mathcal{P}_2$, $X' \in \mathfrak{M}[G_1]$. We claim X' is dense in $\mathcal{P}_2$. Clearly, if $q \in X'$, and $q \leqq q'$, then $q' \in X'$. Next, let $q \in \mathcal{P}_2$. We consider

$$X'' = \{p \in \mathcal{P}_1 \mid (\exists q' \in \mathcal{P}_2)(q \leqq q' \text{ and } \langle p, q' \rangle \in X)\} \;.$$

One checks easily that X'' is a dense subset of $\mathcal{P}_1$, lying in $\mathfrak{M}$. Hence $\exists p \in G_1 \cap X''$. But then there is a $q' \in \mathcal{P}_2$ with $q \leqq q'$ and $\langle p, q' \rangle \in X$; i.e., $(\exists q' \geqq q)(q' \in X')$. We have now verified that X' is dense in $\mathcal{P}_2$.

Since G_2 is $\mathfrak{M}[G_1]$-generic, there is a $q \in G_2 \cap X'$; i.e., there is a $\langle p, q \rangle$ lying in $G_1 \times G_2 \cap X$. This completes our verification that $G_1 \times G_2$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}$.

COROLLARY. *Let G_1 be an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$ and G_2 an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_2$. Then G_1 is an $\mathfrak{M}[G_2]$-generic filter on $\mathcal{P}_1$.*

PROOF. By the lemma, $G_1 \times G_2$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1 \times \mathcal{P}_2$. By Lemma 2.1, $G_2 \times G_1$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_2 \times \mathcal{P}_1$. By the lemma, G_1 is an $\mathfrak{M}[G_2]$-generic filter on $\mathcal{P}_1$.

2.4. Let $\mathfrak{M}, \mathcal{P}_1, \mathcal{P}_2$ be as in §2.3. We make the following additional assumption on $\mathcal{P}_1$ and $\mathcal{P}_2$: $\mathcal{P}_1$ and $\mathcal{P}_2$ have a minimal element (which we name 0) such that $p \in \mathcal{P}_i \rightarrow 0 \leqq p$).

This assumption is quite harmless since if a reflexive partially ordered set $\mathcal{P}$ fails to satisfy it, we can simply add a new element 0 to $\mathcal{P}$ and decree that $0 \leqq p$, for all $p \in \mathcal{P} \cup \{0\}$. Since $\mathcal{P}$ is cofinal in $\mathcal{P} \cup \{0\}$, Lemma 2.2 says that $\mathcal{P}$ and $\mathcal{P} \cup \{0\}$ are equivalent for all our purposes.

LEMMA. *Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$. Let Φ be a sentence of $\mathscr{L}'$. Let $p = \langle p_1, p_2 \rangle$ be an element of $\mathcal{P}$. We suppose that*

$$p \Vdash \text{``}\mathfrak{M}[G_1] \models \Phi\text{''} \;.$$

(The sentence in quotes can be constructed by the techniques of § 1.7.) *Then*

$$\langle p_1, 0 \rangle \Vdash \text{``} \mathfrak{M}[G_1] \vDash \Phi \text{''} \,.$$

PROOF. Suppose not. Then there is an element $p' = \langle p_1', p_2' \rangle$ of $\mathscr{P}$ such that $p_1 \leqq p_1'$ and

$$p' \Vdash \text{``} \mathfrak{M}[G_1] \vDash \neg\, \Phi \text{''} \,.$$

Select G' a generic filter on $\mathscr{P}$ with $p' \in G'$. By Lemma 2.3, $G' = G_1' \times G_2'$, where G_1' is an $\mathfrak{M}$-generic filter on $\mathscr{P}_1$. Since $p' \in G'$, we have

$$\mathfrak{M}[G_1'] \vDash \neg\, \Phi \,.$$

Pick an $\mathfrak{M}[G_1']$-generic filter G_2'' on $\mathscr{P}_2$ with $p_2 \in G_2''$. By Lemma 2.3, $G_1' \times G_2''$ is $\mathfrak{M}$-generic. By construction, $p \in G_1' \times G_2''$. Thus, by our hypothesis on p,

$$\mathfrak{M}[G_1'] \vDash \Phi \,.$$

This is absurd, since we know that $\mathfrak{M}[G_1'] \vDash \neg\, \Phi$. This contradiction completes the proof.

2.5. LEMMA. *Let* $\mathfrak{M}, \mathscr{P}_1, \mathscr{P}_2$ *be as in* § 2.4. *Let* $G = G_1 \times G_2$ *be an* $\mathfrak{M}$-*generic filter on* $\mathscr{P} = \mathscr{P}_1 \times \mathscr{P}_2$. *Let* $a \subseteq \omega$ *lie in* $\mathfrak{M}[G_1] \cap \mathfrak{M}[G_2]$. *Then* $a \in \mathfrak{M}$.

PROOF. By § 1.7, we can find a formula $\Psi(x, y, z)$ of $\mathscr{L}$, and elements x_1, x_2 of $\mathfrak{M}$ such that

$$a = \{ n \in \omega \mid \mathfrak{M}[G_i] \vDash \Psi(\underline{x}_i, \underline{G}, \underline{n}) \} \,, \qquad\qquad i = 1, 2 \,.$$

Hence there is a condition $p = \langle p_1, p_2 \rangle$ of $\mathscr{P}_1 \times \mathscr{P}_2$ which forces the following:

(α) $\qquad (\forall n \in \omega)\big(\mathfrak{M}[G_1] \vDash \Psi(\underline{x}_1, \underline{G}, \underline{n}) \longleftrightarrow \mathfrak{M}[G_2] \vDash \Psi(\underline{x}_2, \underline{G}, \underline{n}) \} \,.$

We claim that for all n, p decides $\mathfrak{M}[G_1] \vDash \Psi(\underline{x}_1, \underline{G}, \underline{n})$. Granting this,

$$a = \{ n \in \omega \mid p \Vdash \text{``} \mathfrak{M}[G_1] \vDash \Psi(\underline{x}_1, \underline{G}, \underline{n}) \text{''} \} \,,$$

so $a \in \mathfrak{M}$ (since "forcing is expressible in the ground model").

Suppose then that p does not decide the statement,

(β) $\qquad\qquad\qquad\qquad \mathfrak{M}[G_1] \vDash \Psi(x_1, \underline{G}, \underline{n}) \,,$

for some $n \in \omega$. Let p', p'' be extensions of p such that $p' \Vdash (\beta)$ and $p'' \Vdash \neg\, (\beta)$. We have, say, $p' = \langle p_1', p_2' \rangle$ and $p'' = \langle p_1'', p_2'' \rangle$. Also let $p = \langle p_1, p_2 \rangle$.

By Lemma 2.4, we have $\langle p_1', p_2 \rangle \Vdash (\beta)$. Since p'' extends p, and p forces (α), we see that p'' forces

(γ) $\qquad\qquad\qquad\qquad \mathfrak{M}[G_2] \vDash \neg\, \Psi(\underline{x}_2, \underline{G}, \underline{n}) \,.$

Hence, by Lemma 2.4, $\langle p_1, p_2'' \rangle \Vdash (\gamma)$.

Consider now $\langle p_1', p_2'' \rangle$. As a common extension of p, $\langle p_1', p_2 \rangle$ and $\langle p_1, p_2'' \rangle$,

it forces (α), (β), and (γ). Since (α) contradicts "(β) and (γ)", we have a contradiction. Thus p does decide (β), and the lemma is proved.

One can in fact show that $\mathfrak{M}[G_1] \cap \mathfrak{M}[G_2] = \mathfrak{M}$. (Otherwise pick a counter-example a of minimal rank. We have $a \subseteq \mathfrak{M}$, and the proof of Lemma 2.5 adapts to show that $a \in \mathfrak{M}$.)

3. Description of the model

3.1. The model used to prove Theorem 2 is due to Azriel Levy. In this section, we describe the model and prove some of its elementary properties. The results of this section are due to Levy and are included here with his permission.

Let $\mathfrak{M}$ be a countable transitive model of **ZFC** + "There is a strongly inaccessible cardinal". Let $\Omega \in \mathfrak{M}$ be strongly inaccessible in $\mathfrak{M}$.

3.2. Let λ be an ordinal. Let $\mathscr{P}^\lambda$ be the following set: $f \in \mathscr{P}^\lambda$ if

(1) f is a function;
(2) domain(f) is a finite subset of $\lambda \times \omega$;
(3) range$(f) \subseteq \lambda$;
(4) $f(\langle \alpha, n \rangle) < \alpha$ whenever $\langle \alpha, n \rangle \in$ domain(f).

We order $\mathscr{P}^\lambda$ by $\subseteq$. Note that if $\mathfrak{M} \subseteq \mathfrak{N}$ are transitive models of **ZF**, and $\lambda \in \mathfrak{M}$, $(\mathscr{P}^\lambda)_{\mathfrak{M}} = (\mathscr{P}^\lambda)_{\mathfrak{N}} = \mathscr{P}^\lambda$.

LEMMA. *Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}^\lambda$. Let $0 < \alpha < \lambda$. Define $f_\alpha \subseteq \omega \times \alpha$ by*

$$f_\alpha = \{\langle n, \beta \rangle : \{\langle\langle \alpha, n \rangle, \beta \rangle\} \in G\} .$$

Then f_α is a surjective map of ω onto α.

PROOF. Suppose first that $\langle n, \beta \rangle \in f_\alpha$, and $\langle n, \beta' \rangle \in f_\alpha$. Then

$$\{\langle\langle \alpha, n \rangle, \beta \rangle\} \cup \{\langle\langle \alpha, n \rangle, \beta' \rangle\}$$

for some $h \in \mathscr{P}^\lambda$. Since h is a function, $\beta = \beta'$. This proves f_α is a function.

Since $\{h \in \mathscr{P}^\lambda \mid \langle \alpha, n \rangle \in \text{dom}(h)\}$ is dense (since $\alpha > 0$), there is an $h \in G$, $h(\langle \alpha, n \rangle) = \beta$, say. But then $\{\langle\langle \alpha, n \rangle, \beta \rangle\} \in G$, by § (1.3.2), so $n \in$ domain(f_α). Thus domain$(f_\alpha) = \omega$. Now let $\beta < \alpha$. Since the set

$$\{h \in \mathscr{P}^\lambda \mid (\exists n < \omega)(h\langle \alpha, n \rangle) = \beta\}$$

is dense, one sees similarly that $\beta \in$ range(f_α). This proves the lemma.

COROLLARY. *Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}^\lambda$. Then $\lambda \leqq \aleph_1^{\mathfrak{M}[G]}$.*

PROOF. If $0 < \alpha < \lambda$, then there is a surjective map $f_\alpha: \omega \rightarrow \alpha$ in $\mathfrak{M}[G]$ (by the lemma just proved).

3.3. The model $\mathfrak{N}$ used to prove Theorem 2 is obtained as follows. Let

G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^\Omega$. Then $\mathfrak{N} = \mathfrak{M}[G]$.

We are going to show that $\Omega = \aleph_1^{\mathfrak{N}}$. We first prove the following lemma.

LEMMA. *Let* $\mathcal{F} \in \mathfrak{M}$, $\mathcal{F} \subseteq \mathcal{P}^\Omega$. *Suppose that any two distinct elements* $\mathcal{F}$ *are incompatible. Then, in* $\mathfrak{M}$, $\mathcal{F}$ *has cardinality less than* Ω. *In fact, there is a* $\xi < \Omega$ *such that* $\mathcal{F} \subseteq \mathcal{P}^\xi$.

PROOF. We work inside $\mathfrak{M}$. By Zorn's lemma we may assume that $\mathcal{F}$ is a maximal pairwise incompatible family of elements of $\mathcal{P}^\Omega$.

We define a sequence of ordinals $\{\xi_i, i < \omega\}$. Put $\xi_0 = \omega$. Suppose then that ξ_i has been defined, and $\xi_i < \Omega$. Then since Ω is inaccessible, $\mathcal{P}^{\xi_i}$ has cardinality less than Ω.

Let $h \in \mathcal{P}^{\xi_i}$. By the maximality of $\mathcal{F}$, there is an $f_h \in \mathcal{F}$ with f_h compatible with h. Since Ω is regular and $\operatorname{card}(\mathcal{P}^{\xi_i}) < \Omega$, we can find ξ_{i+1} with

$$\xi_i < \xi_{i+1} < \Omega$$

and $h \in \mathcal{P}^{\xi_i} \longrightarrow f_h \in \mathcal{P}^{\xi_{i+1}}$.

Let $\xi_\omega = \sup\{\xi_i, i \in \omega\}$. Then $\xi_\omega < \Omega$. We claim $\mathcal{F} \subseteq \mathcal{P}^{\xi_\omega}$.

Suppose not. Let $g \in \mathcal{F}$, $g \not\subseteq \mathcal{P}^{\xi_\omega}$. Let g' be the restriction of g to $\xi_\omega \times \omega$. Then since domain (g) is finite, $g' \in \mathcal{P}^{\xi_n}$ for some n. By construction, there is a $g'' \in \mathcal{P}^{\xi_{n+1}} \cap \mathcal{F}$ compatible with g'. But g'' is not compatible with g (since both lie in $\mathcal{F}$ and $g \notin \mathcal{P}^{\xi_\omega}$.) So there is an $\langle \alpha, n \rangle \in \operatorname{domain}(g) \cap \operatorname{domain}(g'')$ with

$$g(\langle \alpha, n \rangle) \neq g''(\langle \alpha, n \rangle).$$

Since $\langle \alpha, n \rangle \in \operatorname{domain}(g'')$, $\alpha < \xi_{n+1} < \xi_\omega$. By the definition of g', $g(\langle \alpha, n \rangle) = g'(\langle \alpha, n \rangle) \neq g''(\langle \alpha, n \rangle)$. But this contradicts the fact that g' and g'' are compatible.

So $\mathcal{F} \subseteq \mathcal{P}_{\xi_\omega}$. But then the cardinality estimate of the lemma is clear.

COROLLARY. $\aleph_1^n = \Omega$.

PROOF. In view of Corollary 3.2, we have to show that if $f \in \mathfrak{M}[G]$, $f: \omega \to \Omega$, then f is not onto. But this follows in a known way (cf. [1, p. 132]) from the lemma.

3.4. Let $G^\lambda = G \cap \mathcal{P}^\lambda$. By § 2.3, G^λ is an $\mathfrak{M}$-generic filter on $\mathcal{P}^\lambda$. We are going to prove the following lemma.

LEMMA. Let $f \in \mathfrak{N}$ be a function such that

$$f: \omega \to OR$$

(OR is the class of ordinals). Then for some $\xi < \Omega$,

$$f \in \mathfrak{M}[G^\xi].$$

PROOF. Let $\Phi(x, y)$ be a formula of $\mathcal{L}'$ such that

$$f = \{\langle x, y \rangle \colon \mathfrak{N} \models \Phi(x, y)\} \ .$$

Let $p_0 \in G$ force

"$\{\langle x, y \rangle \colon \Phi(x, y)\}$ is a function from ω to OR".

Let $n \in \omega$. We say that $p \geq p_0$ decides the value of $f(n)$ if for some ordinal λ of $\mathfrak{M}$

$$p \Vdash \Phi(\underline{n}, \underline{\lambda}) \ .$$

Since p extends p_0, the ordinal λ is uniquely determined by p and n. Since $p_0 \in G$, and $OR^{\mathfrak{M}} = OR^{\mathfrak{N}}$, for each $n \in \omega$, there is a $p \in G$ which decides $f(n)$.

We work inside $\mathfrak{M}$. Let

$$\mathcal{G}_n = \{p \in \mathcal{P}^{\Omega} \colon p \geq p_0 \text{ and } p \text{ decides } f(n)\} \ .$$

Let $\mathcal{F}_n$ be a maximal pairwise incompatible subfamily of $\mathcal{G}_n$. By Lemma 3.3, there is a $\xi < \Omega$ such that, for all $n \in \omega$,

$$\mathcal{F}_n \subseteq \mathcal{P}^{\xi} \ .$$

It follows that $p_0 \in \mathcal{P}^{\xi}$.

Claim. Let $n \in \omega$. Then there is a $p \in G^{\xi}$ such that p decides $f(n)$. In fact, let $X_n \subseteq \mathcal{P}^{\Omega}$ be the set of p such that

(1) if p is compatible with p_0, $p \geq p_0$;

(2) If $p \geq p_0$, then p decides $f(n)$;

(3) if $p \geq p_0$, then $p \geq q$ for some $q \in \mathcal{F}_n$.

One can verify easily that if $p \in X_n$, and $p \leq p'$, then $p' \in X_n$. Let $p \in \mathcal{P}^{\Omega}$. Then either p is incompatible with p_0 (and so $p \in X_n$), or there is a $p_1 \in \mathcal{P}^{\Omega}$ extending both p and p_0. Since $p_1 \geq p_0$, there is a $p_2 \geq p_1$ which decides $f(n)$. Thus $p_2 \in \mathcal{G}_n$. By the maximality of $\mathcal{F}_n$, there is a $q \in \mathcal{F}_n$ with p_2 compatible with q. Let p_3 be a common extension of p_2 and q. Then by construction, p_3 is an extension of p lying in X_n. Thus X_n is dense. Since X_n clearly lies in $\mathfrak{M}$, and G is an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\Omega}$, there is a $p \in G$, with $p \in X_n$.

Now $p_0 \in G$, and any two elements of G are compatible. By clause (1) of the definition of X_n, $p \geq p_0$. By clause (3) of the definition of X_n, there is a $q \in \mathcal{F}_n$, with $q \leq p$. Hence $q \in G$, since p is. Since $q \in G^{\xi}$, and q decides $f(n)$, our claim is clear.

But now

$$f = \{\langle n, \lambda \rangle \colon (\exists p \in G^{\xi})(p \Vdash \Phi(\underline{n}, \underline{\lambda}))\}$$

so that lemma is clear.

COROLLARY 1. *Let $s \in \mathfrak{N}$ be a subset of ω. Then $s \in \mathfrak{M}[G^{\xi}]$, for some $\xi < \Omega$.*

COROLLARY 2. *Let $s \in \mathfrak{N}$ be a subset of ω. Then Ω is inaccessible in*

$\mathfrak{M}[s]$. *The set $A_s = \{t \subseteq \omega \mid t \in \mathfrak{M}[s]\}$ is countable in $\mathfrak{M}$.*

PROOF. By the lemma, we can pick $\xi < \Omega$, such that $s \in \mathfrak{M}[G^\xi]$. It follows that

$$\mathfrak{M}[s] \subseteq \mathfrak{M}[G^\xi] \, .$$

By Lemma 1.11, Ω is inaccessible in $\mathfrak{M}[G^\xi]$. *A fortiori*, it is inaccessible in $\mathfrak{M}[s]$. Let α be the cardinal of A_s in $\mathfrak{M}[s]$. Then $\alpha < \Omega$, since Ω is inaccessible in $\mathfrak{M}[s]$. By Corollary 3.3, α is countable in $\mathfrak{M}$.

3.5. *Symmetry.* Let π be a permutation of ω lying in $\mathfrak{M}$. Define $\pi_*: \mathscr{P}^\Omega \longrightarrow \mathscr{P}^\Omega$ by

$$\pi_*(h)(\langle \alpha, n \rangle) = h(\langle \alpha, \pi(n) \rangle) \, .$$

Then π_* is an automorphism of $\mathscr{P}^\Omega$ lying in $\mathfrak{M}$. If G is an $\mathfrak{M}$-generic filter on $\mathscr{P}^\Omega$, so is $\pi_*[G]$, by Lemma 2.1. Clearly,

$$\mathfrak{M}[G] = \mathfrak{M}[\pi_*[G]] \, .$$

Let Φ be a statement of $\mathscr{L}'$, not involving G. We claim $p \Vdash \Phi$ if and only if $\pi_*(p) \Vdash \Phi$. To see this, we construct the following chain of equivalent statements.

(1) $p \Vdash \Phi$.

(2) For all $\mathfrak{M}$-generic filters G on $\mathscr{P}^\Omega$ which contain p,

$$\mathfrak{M}[G] \models \Phi \, .$$

Since Φ does not contain G, and $\mathfrak{M}[G] = \mathfrak{M}[\pi_*(G)]$, (2) is equivalent to

(3) For all $\mathfrak{M}$-generic filters G containing p,

$$\mathfrak{M}[\pi_*(G)] \models \Phi \, .$$

Now $p \in G$ if and only if $\pi_*(p) \in \pi_*(G)$. Moreover, as G ranges over the set of $\mathfrak{M}$-generic filters on $\mathscr{P}^\Omega$, so does $\pi_*(G)$. It follows that (3) is equivalent to

(4) For all $\mathfrak{M}$-generic filters G containing $\pi_*(p)$

$$\mathfrak{M}[G] \models \Phi \, .$$

But (4) just says that $\pi^*(p) \Vdash \Phi$.

LEMMA. *Let Φ be a sentence of $\mathscr{L}'$ not containing G. Let 0 be the minimal element of $\mathscr{P}^\Omega$. Then 0 decides Φ.*

PROOF. Otherwise there are $p_1, p_2 \in \mathscr{P}^\Omega$, with $p_1 \Vdash \Phi$, $p_2 \Vdash \neg \Phi$. We can find a permutation $\pi \in \mathfrak{M}$ such that $\pi_*(p_1)$ has domain disjoint from p_2. By our previous remark, $\pi_*(p_1) \Vdash \Phi$. But then $\pi_*(p_1)$ must be incompatible with p_2 since $p_2 \Vdash \neg \Phi$. This is absurd since $\pi_*(p_1)$ and p_2 have disjoint domains.

COROLLARY. *Let $a \subseteq \omega$, $a \in \mathfrak{M}$. Suppose that there is a formula $\Phi(x)$ of $\mathscr{L}'$ not containing G such that a is the unique z such that*

$$\mathfrak{N} \models \Phi(z) \ .$$

Then $a \in \mathfrak{M}$.

PROOF. There is a formula $\Psi(z)$ of $\mathscr{L}'$, not containing G, such that

$$a = \{n \in \omega \mid \mathfrak{N} \models \Psi(\underline{n})\} \ .$$

By the lemma,

$$a = \{n \in \omega \mid 0 \Vdash \Psi(\underline{n})\} \ .$$

Since "forcing is expressible in the ground model"; (cf. § 1.9), this shows that $a \in \mathfrak{M}$.

4. An important lemma

4.1. This section is devoted to the proof of the following result. It is the key technical fact we need about $\mathfrak{N}$.

THEOREM. *Let* $f\colon \omega \to OR$, $f \in \mathfrak{N}$. *Then there is an* $\mathfrak{M}[f]$-*generic filter* G' *on* $\mathscr{P}^{\alpha}$, *such that* $\mathfrak{N} = \mathfrak{M}[f][G']$.

The effect of this theorem (which is the "important lemma" of the section title) is to give us excellent control on the extension $\mathfrak{N}/\mathfrak{M}[f]$.

COROLLARY. *Let* $s \subseteq \omega$, $s \in \mathfrak{N}$. *Then there is an* $\mathfrak{M}[s]$-*generic filter* G' *on* $\mathscr{P}^{\alpha}$, *such that* $\mathfrak{N} = \mathfrak{M}[s][G']$.

4.2. We begin with some easy lemmas.

LEMMA 1. *Let* α *be an ordinal of* $\mathfrak{M}$. *Let* β *be the cardinal of* α *in* $\mathfrak{M}$. *Let* $F\colon \omega \to \alpha$ *be an* $\mathfrak{M}$-*generic collapsing map. Then there is an* $\mathfrak{M}$-*generic collapsing map* $G\colon \omega \to \beta$ *such that*

$$(1) \qquad\qquad\qquad \mathfrak{M}[F] = \mathfrak{M}[G] \ .$$

Conversely, let $G\colon \omega \to \beta$ *be an* $\mathfrak{M}$-*generic collapsing map. Then there is an* $\mathfrak{M}$-*generic collapsing map* $F\colon \omega \to \alpha$ *such that* (1) *holds.*

PROOF. Let $\psi\colon \alpha \to \beta$ be a bijection lying in $\mathfrak{M}$. Let $\mathscr{P}_{\alpha}$, $\mathscr{P}_{\beta}$ be as in § 1.12. Then ψ induces an order isomorphism of $\mathscr{P}_{\alpha}$ with $\mathscr{P}_{\beta}$ lying in $\mathfrak{M}$. The lemma now follows from Lemma 2.1.

LEMMA 2. *Let* α *be an ordinal. Let* $F\colon \omega \to \alpha$ *be a generic collapsing map. Define* F_1, $F_2\colon \omega \to \alpha$ *by*

$$F_1(n) = F(2n); \quad F_2(n) = F(2n + 1) \ .$$

Then F_1 *is an* $\mathfrak{M}$-*generic collapsing map, and* F_2 *is an* $\mathfrak{M}[F_1]$-*generic collapsing map.*

Conversely let F_1, $F_2\colon \omega \to \alpha$ *be respectively an* $\mathfrak{M}$-*generic and a* $\mathfrak{M}[F_1]$-*generic collapsing map. Then if we define* $F\colon \omega \to \alpha$ *by*

$$F(2n) = F_1(n); \quad F(2n + 1) = F_2(n) ,$$

then F is an $\mathfrak{M}$-generic collapsing map. In either case, we have

$$\mathfrak{M}[F] = \mathfrak{M}[F_1, F_2] .$$

PROOF. Define an isomorphism

$$\psi \colon \mathscr{P}_\alpha \longrightarrow \mathscr{P}_\alpha \times \mathscr{P}_\alpha$$

by $\psi(h) = \langle h_1, h_2 \rangle$, with $h_1(n) = h(2n)$, $h_2(n) = h(2n + 1)$. Let G be the $\mathfrak{M}$-generic filter on $\mathscr{P}_\alpha$ associated to F. By Lemma 2.1, $\psi_*(G)$ is an $\mathfrak{M}$-generic filter on $\mathscr{P}_\alpha \times \mathscr{P}_\alpha$. By Lemma 2.3,

$$\psi_*(G) = G_1 \times G_2$$

with G_1 an $\mathfrak{M}$-generic filter on $\mathscr{P}_\alpha$, and G_2 an $\mathfrak{M}[G_1]$-generic filter on $\mathscr{P}_\alpha$. It is clear from the definition of ψ that

$$\bigcup G_1 = F_1; \quad \bigcup G_2 = F_2 .$$

Thus the first half of the lemma is clear. The second half is proved similarly.

4.3. LEMMA. *Let α be an ordinal $\geqq \omega$ of $\mathfrak{M}$. Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}^{\alpha+1}$. Then there is an $\mathfrak{M}$-generic collapsing function $F \colon \omega \to \alpha$ such that $\mathfrak{M}[G] = \mathfrak{M}[F]$. Conversely, if $F \colon \omega \to \alpha$ is an $\mathfrak{M}$-generic collapsing map, there is an $\mathfrak{M}$-generic filter G on $\mathscr{P}^{\alpha+1}$ with $\mathfrak{M}[G] = \mathfrak{M}[F]$.*

PROOF. We first prove the lemma under the additional assumption that α is countable in $\mathfrak{M}$. We then show how to remove this assumption.

Since α is countable in $\mathfrak{M}$, there is a bijection $\psi \colon \omega \to (\alpha + 1 - \{0\}) \times \omega$ lying in $\mathfrak{M}$, such that $\psi(2n) = \langle \alpha, n \rangle$. Let $\varphi(n)$ be the first component of $\psi(n)$. Thus $\varphi \colon \omega \to \alpha + 1 - \{0\}$, $\varphi(2n) = \alpha$, all $n \in \omega$.

Let $\mathscr{P}'$ be the following collection of functions: $h \in \mathscr{P}'$ if and only if domain(h) is a finite subset of ω, range $(h) \subseteq \alpha$, and $h(n) < \varphi(n)$ for all $n \in$ domain(h). We order $\mathscr{P}'$ by $\subseteq$.

The map $\{h \to h \circ \psi\}$ is clearly an order isomorphism of $\mathscr{P}^{\alpha+1}$ with $\mathscr{P}'$.

Let $\mathscr{P}''$ be the following subset of $\mathscr{P}'$: $h \in \mathscr{P}''$ if and only if $h \in \mathscr{P}'$ and $(\forall n \in \omega) (2n \in$ domain $(n) \to 2n + 1 \in$ domain $(h))$. $\mathscr{P}''$ is clearly a cofinal subset of $\mathscr{P}'$. We are going to set up an isomorphism of $\mathscr{P}''$ with $\mathscr{P}_\alpha$, lying in $\mathfrak{M}$.

To describe this isomorphism, let

$$S = \{\langle \gamma_1, \gamma_2, \gamma_3 \rangle \colon \gamma_2 < \gamma_1 \leqq \alpha \text{ and } \gamma_3 < \alpha\} .$$

Let $\psi' \colon S \to \alpha$ be a map lying in $\mathfrak{M}$ such that

$$\langle \gamma_2, \gamma_3 \rangle \longrightarrow \psi'(\gamma_1, \gamma_2, \gamma_3)$$

is a bijection of $\gamma_1 \times \alpha$ with α whenever $0 < \gamma_1 \leqq \alpha$.

Define $\psi'': \mathscr{P}'' \to \mathscr{P}_\alpha$ as follows. $\psi''(h)$ is defined at $m \in \omega$ if and only if $h(2m)$ and $h(2m + 1)$ are defined. In that case,

$$\psi''(h)(m) = \psi'\big(\varphi(2m + 1),\, h(2m + 1),\, h(2m)\big) .$$

A moments reflection shows that ψ'' gives an isomorphism of $\mathscr{P}''$ with $\mathscr{P}_\alpha$.

Under the assumption that α is countable in $\mathfrak{M}$ the lemma is now clear. Suppose, for example, that G is an $\mathfrak{M}$-generic filter on $\mathscr{P}^{\alpha+1}$. By applying Lemma 2.1 and 2.2 we get a filter G' on $\mathscr{P}_\alpha$ such that $\mathfrak{M}[G] = \mathfrak{M}[G']$. It suffices to take $F = \bigcup G'$.

Now drop the assumption that α is countable in $\mathfrak{M}$. Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}^{\alpha+1}$. Writing $\mathscr{P}^{\alpha+1} \cong \mathscr{P}_\alpha \times \mathscr{P}^\alpha$, we see that there is an $\mathfrak{M}$-generic collapsing map $F_1 : \omega \to \alpha$ and an $\mathfrak{M}[F_1]$-generic filter, G_1, on $\mathscr{P}^\alpha$ such that

$$\mathfrak{M}[G] = \mathfrak{M}[F_1, G_1] .$$

We apply Lemma 4.2.2, to get collapsing maps $F_2, F_3 : \omega \to \alpha$, generic over $\mathfrak{M}$ and $\mathfrak{M}[F_2]$ respectively, with

$$\mathfrak{M}[F_1] = \mathfrak{M}[F_2][F_3] .$$

Again using the isomorphism $\mathscr{P}^{\alpha+1} \cong \mathscr{P}_\alpha \times \mathscr{P}^\alpha$, we can coalesce F_3 and G_1 into an $\mathfrak{M}[F_1]$-generic filter on $\mathscr{P}^{\alpha+1}$, G_2. So

$$\mathfrak{M}[G] = \mathfrak{M}[F_1][G_2] .$$

But α is countable in $\mathfrak{M}[F_1]$. By the special case of the lemma previously proved there is an $\mathfrak{M}[F_1]$-generic collapsing map, $F_4 : \omega \to \alpha$ with

$$\mathfrak{M}[G] = \mathfrak{M}[F_1][G_2] = \mathfrak{M}[F_1][F_4] .$$

But Lemma 4.2.2 allows us to coalesce F_1 and F_4 into a single generic collapsing map F, with

$$\mathfrak{M}[G] = \mathfrak{M}[F_1][F_4] = \mathfrak{M}[F] .$$

To prove the converse, run the argument backward.

4.4. The following lemma is the crucial step in the proof of Theorem 4.1.

LEMMA. *Let* $\alpha \in \mathfrak{M}$, $\alpha \geqq \omega$. *Let* $F_1, F_2 : \omega \to \alpha$ *be collapsing maps generic over* $\mathfrak{M}$ *and* $\mathfrak{M}[F_1]$ *respectively. Let* $s \subseteq OR$ *be a set of* $\mathfrak{M}[F_1]$. *Then there is a collapsing map* $F : \omega \to \alpha$, *generic over* $\mathfrak{M}[s]$ *with*

$$\mathfrak{M}[s][F] = \mathfrak{M}[F_1, F_2] .$$

PROOF. We begin by describing a certain cofinal subset $\mathscr{P}_1$ of $\mathscr{P}_\alpha \times \mathscr{P}_\alpha$. A pair $\langle h_1, h_2 \rangle$ lies in $\mathscr{P}_1$ if and only if $\mathrm{domain}(h_1) = \mathrm{domain}(h_2)$ and $\mathrm{domain}(h_1)$ is a finite initial segment of the integers. If $\langle h_1, h_2 \rangle \in \mathscr{P}_1$, then we put

$l(\langle h_1, h_2 \rangle) = \text{domain}(h_1)$. Thus if $x \in \mathscr{P}_1$, $l(x) \in \omega$. Let G' be a generic filter on $\mathscr{P}_1$. Then there is a pair of functions, F_1', F_2': $\omega \to \alpha$, say, and

$$G' = \{\langle F_1' \mid n, F_2' \mid n \rangle : n \in \omega\} .$$

We fix a definition of $s \in \mathfrak{M}[F_1]$. Thus $\lambda \in s$ if and only if $\mathfrak{M}[F_1] \vDash \Phi(\underline{\lambda}, \underline{\dot{F}}_1)$, and Φ is a formula involving only ε, S, and terms denoting elements of $\mathfrak{M}$.

By Lemma 2.3, F_1 and F_2 determine a generic filter, G_1, on $\mathscr{P}_\alpha \times \mathscr{P}_\alpha$. Let G be the associated filter on $\mathscr{P}_1$ given by Lemma 2.2: $G = G_1 \cap \mathscr{P}_1$.

Our next step is to define a certain subset Σ of $\mathscr{P}_1$. Roughly speaking, Σ has the following motivation: The fact that

$$s = \{\lambda \mid \mathfrak{M}[F_1] \vDash \Phi(\underline{\lambda}, \underline{F}_1)\}$$

gives a certain amount of information about G. This information is summed up in the fact that $G \subseteq \Sigma$.

Let $\Psi(x)$ be a formula involving only ε, S, and terms denoting elements of $\mathfrak{M}$ and G such that $\mathfrak{M}[G] \vDash \Psi(\underline{\lambda})$ if and only if $\mathfrak{M}[F_1] \vDash \Phi(\underline{\lambda}, F_1)$. Then, if Ψ is constructed in a reasonable manner, the following will be true (by Lemma 2.4):

If $\langle h_1, h_2 \rangle$ and $\langle h_1, h_3 \rangle \in \mathscr{P}_1$, then $\langle h_1, h_2 \rangle \Vdash \Psi(\underline{\lambda})$ if and only if $\langle h_1, h_3 \rangle \Vdash \Psi(\underline{\lambda})$.

We work in $\mathfrak{M}[s]$. Define a sequence of subsets of $\mathscr{P}_1$, $\{A_\alpha\}$, by transfinite induction.

(1) $p \in A_0$ if either $p \Vdash \Psi(\underline{\lambda})$ and $\lambda \notin s$ (for some $\lambda \in OR^{\mathfrak{M}}$) or $p \Vdash \neg \Psi(\underline{\lambda})$ and $\lambda \in s$).

(2) Let $\alpha = \beta + 1$. $p \in A_\alpha$ if for some dense subset X of $\mathscr{P}_1$, lying in $\mathfrak{M}$, every extension of p in X is in A_β.

(3) Let α be a limit ordinal. Then $A_\alpha = \bigcup_{\beta < \alpha} A_\beta$.

We note the following facts about $\{A_\alpha\}$.

(A1) If $p \in A_\alpha$, and $p \leq q$, then $q \in A_\alpha$.
(This is easily checked by induction on α.)

(A2) If $\alpha < \beta$, then $A_\alpha \subseteq A_\beta$.
(The crucial case is when $\beta = \alpha + 1$. Take the dense set X to be $\mathscr{P}_1$ itself and use A1.)

(A3) Let $p = \langle h_1, h_2 \rangle$, $q = \langle h_1, h_3 \rangle$, and suppose $p, q \in \mathscr{P}_1$. Then $p \in A_\alpha$ if and only if $q \in A_\alpha$.

Since $\mathfrak{M}[s]$ is a model for ZF, there is an ordinal δ such that $A_\delta = A_{\delta+1}$. We put

$$\Sigma = \mathscr{P}_1 - A_\delta .$$

We next list some properties of Σ.

(Σ1) $G \subseteq \Sigma$.

Otherwise, there is an $x \in G$ such that $x \in A_\beta$. Pick x, β so that β is

minimal. Clearly β is not zero, since in $\mathfrak{M}[F_1, F_2]$,

$$s = \{\lambda \mid \Psi(\underline{\lambda})\} \ .$$

Also, β is not a limit ordinal since for λ a limit ordinal $A_\lambda = \bigcup_{\gamma < \lambda} A_\gamma$. Thus $\beta = \gamma + 1$. Since $x \in A_\beta$, there is a dense set X with each extension of x in X lying in A_γ, and $X \in \mathfrak{M}$. Since G is $\mathfrak{M}$-generic, there is a $y \in G \cap X$. Let $z \in G$ be a common extension of x and y. Then $z \in X$, since y is; thus, $z \in A_\gamma$ (since z extends x). But this contradicts the minimality of β.

($\Sigma 2$) Let $p \in \Sigma$. Let X be a dense subset of $\mathscr{P}_1$ lying in $\mathfrak{M}$. Then there is a $p' \in \Sigma \cap X$ with p' extending p.

Proof. Since $p \notin A_{\delta+1}$, there is a $p' \geqq p$, with $p' \in X$, $p' \in A_\delta$; i.e., $p' \in \Sigma \cap X$.

($\Sigma 3$) Let $p \in \Sigma$. Let $q \leqq p$. Then $q \in \Sigma$. (This follows from (A2) since Σ is the complement of A_δ.)

($\Sigma 4$) Let $p \in \Sigma$. Then there is an $\mathfrak{M}$-generic filter, G', on $\mathscr{P}_1$ such that $p \in G'$ and

$$s = \{\lambda \mid \mathfrak{M}[G'] \models \Psi(\underline{\lambda})\} \ .$$

Proof. Since $\mathfrak{M}$ is countable, we can enumerate the dense subsets of $\mathscr{P}_1$ in a sequence: $\{X_i, i \in \omega\}$. Using ($\Sigma 2$), we can construct an increasing sequence of elements of Σ, $\{p_n\}$, with $p_0 = p$, and $p_{n+1} \in X_n$. Put $G' = \{x \in \mathscr{P}_1 \mid x \leqq p_n$ for some $n\}$. Then G' has the desired properties.

($\Sigma 5$) G is an $\mathfrak{M}[s]$-generic filter on Σ.

Proof. Clauses (1) and (2) of § 1.3 are clear. We turn to clause (3). Let X be a dense subset of Σ lying in $\mathfrak{M}[s]$. We must show that $X \cap G \neq \varnothing$. We assume the contrary and get a contradiction.

We fix a formula $\Phi_1(x, y)$ of $\mathscr{L}'$, not containing G, such that Φ_1 defines X from s in $\mathfrak{M}[s]$ (i.e., $\Phi_1(y, s)$ holds in $\mathfrak{M}[s]$ if and only if $y = X$). We now form a sentence Ψ_1 of $\mathscr{L}'$ such that for any generic filter G' on $\mathscr{P}_1$, we have $\mathfrak{M}[G'] \models \Psi_1$ if and only if

(1) if $s' = \{\lambda \in OR \mid \mathfrak{M}[G'] \models \Psi(\underline{\lambda})\}$ then s' is a set and there is a unique $X' \in \mathfrak{M}[s']$ such that

$$\Phi_1(X', s') \ .$$

(2) X' is a dense subset of Σ', where Σ' is the set obtained by applying our definition of Σ inside $\mathfrak{M}[s']$.

(3) $X' \cap G' = \varnothing$.

By our assumptions Ψ_1 holds in $\mathfrak{M}[G]$. Let $p \in G$ force Ψ_1. By ($\Sigma 1$), $p \in \Sigma$. Since X is dense in Σ, there is a $q \in X$, with $q \geqq p$. Let G' be an $\mathfrak{M}$-generic filter on $\mathscr{P}_1$ such that $q \in G'$ and $s' = \{\lambda \in OR \mid \mathfrak{M}[G'] \models \Psi(\underline{\lambda})\} = s$. ($G'$ exists by ($\Sigma 4$).) Then with notations as in our description of Ψ_1, we have

$\Sigma' = \Sigma$ and $X' = X$ (since Σ' and X' are defined in $\mathfrak{M}[s']$ by the definitions that yield Σ and X in $\mathfrak{M}[s]$, and $s' = s$). But $q \in G' \cap X'$. Thus Ψ_1 is false in $\mathfrak{M}[G']$. But this is absurd since $q \geqq p$, and $p \Vdash \Psi_1$.

Let $\mathscr{P}'_\alpha$ be the following cofinal subset of $\mathscr{P}_\alpha$: $h \in \mathscr{P}'_\alpha$ if and only if $h \in \mathscr{P}_\alpha$ and domain (h) is a finite initial segment of ω.

(Σ6) In $\mathfrak{M}[s]$, Σ is isomorphic to $\mathscr{P}'_\alpha$.

Proof. We work inside $\mathfrak{M}[s]$. Recall that if $\langle h_1, h_2 \rangle \in \mathscr{P}_1$, $l(\langle h_1, h_2 \rangle) = $ domain (h_1). It follows from (A3) and (Σ2) that if $p \in \Sigma$ and $l(p) = k$, then $\{q \in \Sigma \mid q \geqq p$ and $l(q) = k + 1\} = S_p$, has the same cardinality as α. Let ψ_p be a bijection of S_p onto α. Let now $p \in \Sigma$, with $l(p) = n$. We can find p_i, $0 \leqq i \leqq n$, with $p_i \leqq p$, and $l(p_i) = i$. Let $\chi(p): n \to \alpha$ be defined by $\chi(p)(j) = \psi_{p_j}(p_{j+1})$. Then χ is easily seen to be an isomorphism of Σ with $\mathscr{P}'_\alpha$.

We can now easily prove the lemma. Let $\chi[G]$ be the image of G in $\mathscr{P}'_\alpha$. By (Σ5), (Σ6), and Lemma 2.1, $\mathfrak{M}[G] = \mathfrak{M}[s][G] = \mathfrak{M}[s][\chi[G]]$. Moreover, $\chi[G]$ is an $\mathfrak{M}[s]$-generic filter on $\mathscr{P}'_\alpha$. Hence if we put

$$F = \bigcup \chi[G] ,$$

then F is an $\mathfrak{M}[s]$-generic collapsing map of ω onto α. Since clearly, $\mathfrak{M}[s][F] = \mathfrak{M}[s][\chi[G]] = \mathfrak{M}[G] = \mathfrak{M}[F_1, F_2]$, the lemma is proved.

4.5. We can now easily prove Theorem 4.1. Let $\mathfrak{N}$ be as in Theorem 4.1, and G a generic filter on $\mathscr{P}^\Omega$ such that $\mathfrak{N} = \mathfrak{M}[G]$. Let $f: \omega \to OR$, $f \in \mathfrak{N}$. By Lemma 3.4, we have $f \in \mathfrak{M}[G^\beta]$, where $G^\beta = G \cap \mathscr{P}^\beta$ and $\beta < \Omega$. We may as well suppose that $\omega \leqq \beta$; put $\alpha = \beta + 2$.

We have an obvious isomorphism

$$\mathscr{P}^\Omega = \mathscr{P}^\alpha \times \mathscr{R}^\alpha .$$

Here $\mathscr{R}^\alpha = \{f \in \mathscr{P}^\Omega \mid$ domain $(f) \cap \alpha \times \omega = \varnothing\}$. Hence, by Lemma 2.3, $\mathfrak{N} = \mathfrak{M}[G^\alpha][G_1]$, where $G^\alpha = G \cap \mathscr{P}^\alpha$ is $\mathfrak{M}$-generic and G_1 is an $\mathfrak{M}[G_1]$-generic filter on $\mathscr{R}^\alpha$.

We have $\mathscr{P}^\alpha = \mathscr{P}^{\beta+1} \times \mathscr{P}_{\beta+1}$, up to canonical isomorphism. Hence by Lemma 4.2.1 and Lemma 4.3, there are generic collapsing maps $F_1: \omega \to \beta$, $F_2: \omega \to \beta$ such that

(1) F_1 is $\mathfrak{M}$-generic and $\mathfrak{M}[F_1] = \mathfrak{M}[G^{\beta+1}]$. (Here $G^\xi = G \cap \mathscr{P}^\xi$.)

(2) F_2 is $\mathfrak{M}[F_1]$-generic and $\mathfrak{M}[F_1, F_2] = \mathfrak{M}[G^\alpha]$.

We now apply Lemma 4.4 with f in the role of s. We get an $\mathfrak{M}[f]$-generic collapsing map $F_3: \omega \to \beta$ such that $\mathfrak{M}[G^\alpha] = \mathfrak{M}[f][F_3]$.

We are now almost home. By Lemma 4.3 (and Lemma 4.2.1) there is an $\mathfrak{M}[f]$-generic filter, G_2, on $\mathscr{P}^\alpha$ such that

$$\mathfrak{M}[f][G_2] = \mathfrak{M}[f][F_3] = \mathfrak{M}[G^\alpha] .$$

Apply Lemma 2.3 to G_2, G_1 and the isomorphism

$$\mathscr{P}^\Omega = \mathscr{P}^\alpha \times \mathscr{R}^\alpha \,.$$

We get an $\mathfrak{M}[f]$-generic filter G' on $\mathscr{P}_\Omega$ such that $\mathfrak{M}[f][G'] = \mathfrak{M}[f, G_2, G_1] = \mathfrak{M}[G^\alpha, G_1] = \mathfrak{N}$. This proves Theorem 4.1.

4.6. LEMMA. *Let $\mathfrak{M}$, $\mathfrak{N}$, Ω be as above. Let $\omega \leqq \alpha < \Omega$. Let G_1 be an $\mathfrak{M}$-generic filter on $\mathscr{P}^{\alpha+1}$ with $G_1 \in \mathfrak{N}$. Then there is an $\mathfrak{M}$-generic filter G_2 on $\mathscr{P}^\Omega$ with*

$$\mathfrak{N} = \mathfrak{M}[G_2]$$

and $G_1 = G_2 \cap \mathscr{P}^{\alpha+1}$.

PROOF. By Lemma 4.3 and Theorem 4.1, there is an $\mathfrak{M}[G_1]$-generic filter G_3 on $\mathscr{P}^\Omega$ with $\mathfrak{N} = \mathfrak{M}[G_1, G_3]$.

We now write

(a) $\mathscr{P}^\Omega = \mathscr{P}^{\alpha+1} \times \mathscr{P}_{\alpha+1} \times \mathscr{R}_{\alpha+2} \,.$

Applying § 2.3, we see that G_3 determines filters G_4, G_5, G_6 on $\mathscr{P}^{\alpha+1}$, $\mathscr{P}_{\alpha+1}$, $\mathscr{R}_{\alpha+2}$ such that G_4, G_5, G_6 are generic over $\mathfrak{M}[G_1]$, $\mathfrak{M}[G_1, G_4]$, $\mathfrak{M}[G_1, G_4, G_5]$ respectively, and

$$\mathfrak{N} = \mathfrak{M}[G_1, G_4, G_5, G_6] \,.$$

By § 4.2 and § 4.3, there is an $\mathfrak{M}[G_1]$-generic filter G_7 on $\mathscr{P}_{\alpha+1}$ such that

$$\mathfrak{M}[G_1, G_7] = \mathfrak{M}[G_1, G_4, G_5] \,.$$

We now again apply § 2.3 to the isomorphism (a) and get an $\mathfrak{M}$-generic filter G_2 on $\mathscr{P}^\Omega$ with

$$\mathfrak{M}[G_2] = \mathfrak{M}[G_1, G_7, G_6] = \mathfrak{N}$$

and $G_2 \cap \mathscr{P}^{\alpha+1} = G_1$. This proves the lemma.

II. The Concept of a Random Real

We first discuss, in II.1, the relation between Borel sets of a countable transitive model $\mathfrak{M}$ and Borel sets of the real world. This is a preliminary to a study of the key concept of this paper, the concept of a random real. This is our main tool in adapting Cohen's method to measure theoretic problems.

1. Extending Borel sets

We let **DC** be the principle of dependent choices. A precise statement of **DC** will be given in III. For our present purposes it suffices to know the following:

(1) All the positive results of measure theory and point set topology on the real line (such as the existence of Lebesgue measure and the Baire category theorem) can be proved in $\mathbf{ZF} + \mathbf{DC}$.

(2) $\mathbf{DC}$ justifies a countable sequence of consecutive choices. In particular, it has the following corollary (known as $\mathbf{AC}_\omega$ or the countable axiom of choice):

Let $\{A_i : i \in \omega\}$ be a sequence of non-empty sets. Then there is a function f with domain ω such that $f(i) \in A_i$.

Throughout this section II.1 all theorems of mathematical nature (i.e., theorems not relating to models of set theory) will be theorems of $\mathbf{ZF} + \mathbf{DC}$. Therefore they will hold in any model $\mathfrak{M}$ of $\mathbf{ZF} + \mathbf{DC}$.

1.1. We use functions from ω to ω to code (or Gödel number) Borel subsets of $\mathbf{R}$.[6]

Let $\{r_i\}$ be an arithmetical enumeration of $\mathbf{Q}$; let J be the pairing function

$$J(a, b) = 2^a(2b + 1) .$$

The coding is defined recursively as follows:

Definition. (1) α codes $[r_i, r_j]$ if $\alpha(0) \equiv 0 \pmod 3$, $\alpha(1) = i$, and $\alpha(2) = j$.

(2) Suppose α_i codes B_i, $i = 0, 1, 2, \cdots$; then α codes $\bigcup_i B_i$ if $\alpha(0) \equiv 1 \pmod 3$ and

$$\alpha(J(a, b)) = \alpha_a(b) .$$

(3) Suppose β codes B, $\alpha(0) \equiv 2 \pmod 3$ and $\alpha(n + 1) = \beta(n)$. Then α codes the complement of B.

(4) α codes B only as required by (1)–(3).

LEMMA 1. *Suppose α codes B_1, and α codes B_2. Then $B_1 = B_2$.*

PROOF. Let $I = \{\langle \alpha, B \rangle : \alpha$ codes $B\}$. Let $I_1 = \{\langle \alpha, B \rangle : \alpha$ codes only $B\}$. Then I_1 is closed under (1)–(3) of Definition 1. By (4), $I = I_1$, q.e.d.

We write B_α for the Borel set coded by α. If $\alpha \in \mathfrak{M}$ and α codes a Borel set in $\mathfrak{M}$, we denote this set with $B_\alpha^{\mathfrak{M}}$.

LEMMA 2. *Every Borel set is coded by some α.*

PROOF. The family of sets coded by some α is closed under complements and countable unions ($\mathbf{DC}$!) and contains all sets $[r, s]$ with rational endpoints. Thus it contains all Borel sets.

LEMMA 3. *Every set coded by an α is a Borel set.*

PROOF. (Similar to proof of Lemma 1 and left to the reader.)

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[6] $\mathbf{R}$ is the field of real numbers; $\mathbf{Q}$ is the field of rational numbers.

1.2. THEOREM. *There are Π_1^1 predicates $A_1(\alpha)$, $A_2(\alpha, x)$, $A_3(\alpha, x)$ which are provably equivalent, (in **ZF** + **DC**) to the following concepts:*

(1) α *codes a Borel set;*
(2) α *codes a Borel set and $x \in B_\alpha$;*
(3) α *codes a Borel set and $x \notin B_\alpha$.*

PROOF. We let $\{s_n\}$ be a recursive enumeration without repetitions of the finite sequences of integers (including the void sequence), arranged so that if the sequence s_n is an initial segment of the sequence s_m, then $n \leqq m$. (Thus s_0 is the void sequence.) We define a function $\Phi(\alpha, n)$, taking values in ω^ω as follows:

(1) $n = 0$. Then $\Phi(\alpha, n) = \alpha$.

(2) $n > 0$. Then s_n is a non-empty sequence, of length k, say. Let s_m be the initial segment of s_n of length $k - 1$, and let r be the last element of s_n. (So $r \in \omega$.) Note that $m < n$.

Case 2.1. $\Phi(\alpha, m)(0) \equiv 0 \pmod 3$. Then put $\Phi(\alpha, n)$ equal to the identically zero function.

Case 2.2. $\Phi(\alpha, m)(0) \equiv 1 \pmod 3$. Then put

$$\Phi(\alpha, n)(x) = \Phi(\alpha, m)\big(J(r, x)\big) , \qquad\qquad (x \in \omega) .$$

(Here r, m are as in the preceding paragraph, and J is defined in § 1.1.)

Case 2.3. $\Phi(\alpha, m)(0) \equiv 2 \pmod 3$. Then put

$$\Phi(\alpha, n)(x) = \Phi(\alpha, m)(x + 1) .$$

($\Phi(\alpha, \cdot)$ allows us to recover those Borel sets from which B_α is constructed.)

The following lemma is easily checked by induction on m.

LEMMA 1. *Let α code a Borel set. Then for all m, $\Phi(\alpha, m)$ codes a Borel set.*

Let $\beta\colon \omega \to \omega$. Define a function $\bar{\beta}\colon \omega \to \omega$ by

$$s_{\bar{\beta}(n)} = \langle \beta(0), \cdots, \beta(n - 1) \rangle .$$

(Here the right hand side denotes the finite sequence consisting of the first n members of β.)

We can now define the Π_1^1 predicate, A_1

$$A_1(\alpha) \equiv (\beta)(\exists n)\Phi\big(\alpha, \bar{\beta}(n)\big) = 0 .$$

An argument similar to the proof of Lemma 1.1.1 shows that if α codes a Borel set, then $A_1(\alpha)$ holds. Conversely, suppose that α fails to code a Borel set. Then one can construct a function $\beta\colon \omega \to \omega$ by induction on n, so that for all n, $\Phi\big(\alpha, \bar{\beta}(n)\big)$ fails to code a Borel set. But then, for all n, $\Phi\big(\alpha, \bar{\beta}(n)\big)(0) \neq 0$ (since otherwise, $\Phi\big(\alpha, \bar{\beta}(n)\big)$ codes by Case 1 of the definition (§ 1.1)).

Now suppose that α codes a Borel set, and that x is a real. We define a function $\gamma \colon \omega \to \omega$, as follows: if x lies in the Borel set coded by $\Phi(\alpha, n)$, then $\gamma(n) = 1$. Otherwise, $\gamma(n) = 0$. (Lemma 1.2.1 states that $\Phi(\alpha, n)$ always codes a Borel set.)

LEMMA 2. *There is an arithmetic predicate, $A_4(\alpha, \beta, x)$ such that $A_4(\alpha, \beta, x)$ holds if and only if β is the function γ of the preceding paragraph.*

PROOF. We can describe A_4 as follows:

(1) Suppose that $\Phi(\alpha, n)(0) \equiv 0 \pmod 3$. Let $\Phi(\alpha, n)(1) = i$, $\Phi(\alpha, n)(2) = j$. Then $\beta(n) = 1$ if and only if $x \in [r_i, r_j]$.

(2) Suppose that $\Phi(\alpha, n)(0) \equiv 1 \pmod 3$. Let $\langle j \rangle$ be the length one sequence whose one element is j; let $s_{\varphi(n, j)}$ be the concatenation

$$s_n \frown \langle j \rangle \, .$$

Then $\beta(n) = 1$ if and only if for some j, $\beta\big(\varphi(n, j)\big) = 1$.

(3) Suppose that $\Phi(\alpha, n)(0) \equiv 2 \pmod 3$. Then $\beta(n) = 1$ if and only if $\beta\big(\varphi(n, 0)\big) = 0$.

(4) $\beta(n) = 0$ or 1 for all n.

It is clear that the predicate $A_4(\alpha, \beta, x)$ is arithmetic, and that if γ is as in the paragraph prior to the statement of Lemma 2, then $A_4(\alpha, \gamma, x)$ holds.

Now suppose that α codes a Borel set, that x is a real, that γ is as above, and that $\gamma' \colon \omega \to \omega$ is such that $A_4(\alpha, \gamma', x)$. We want to show $\gamma' = \gamma$. Suppose not. Then for some n, $\gamma'(n) \neq \gamma(n)$. Say s_n has length r. Then we can define a function $\delta \colon \omega \to \omega$ with the following property.

(1) $\bar{\delta}(r) = n$

(2) If $m \geqq r$,

$$\gamma\big(\bar{\delta}(m)\big) \neq \gamma'\big(\bar{\delta}(m)\big) \, .$$

(One defines $\delta(m)$ for $m \geq r$, by induction on m so that (2) holds. Indeed, if $\gamma\big(\bar{\delta}(m)\big) \neq \gamma'\big(\bar{\delta}(m)\big)$, we see first that

$$\Phi\big(\alpha, \bar{\delta}(m)\big)(0) \not\equiv 0 \pmod 3 \, .$$

(Otherwise $\gamma\big(\bar{\delta}(m)\big) = \gamma'\big(\bar{\delta}(m)\big)$ by clause (1) of the definition of A_4.) Moreover, by clauses (2) and (3), of the definition of A_4, we see that for at least one extension s_n of $\bar{\delta}(m)$, of length $m + 1$, we have $\gamma(n) \neq \gamma'(n)$. We now select $\delta(m)$ so that $\bar{\delta}(m + 1) = n$.)

We have already remarked that since

$$\gamma\big(\bar{\delta}(n)\big) \neq \gamma'\big(\bar{\delta}(n)\big)$$

for all $n \geqq r$, we have

$$\Phi\big(\alpha, \bar{\delta}(n)\big)(0) \neq 0 \, , \qquad \text{for } n \geqq r.$$

If $\Phi(\alpha, \bar{\delta}(n))(0) = 0$ for some n less than r, then it would also be zero for all larger n (cf. Case 2.1 of the definition of Φ). Thus

$$(\forall n)\Phi(\alpha, \bar{\delta}(n)) \neq 0 ,$$

i.e., $A_1(\alpha)$ is false. We have already shown that this implies that α fails to code a Borel set. This contradicts our assumption on α, and shows that $\gamma = \gamma'$. The proof of Lemma 2 is complete.

We now define $\Lambda_2(\alpha, x)$ as follows:

$$A_2(\alpha, x) \equiv (\beta)\big(A_4(\alpha, \beta, x) \rightarrow \beta(0) = 1\big) \wedge A_1(\alpha) .$$

Clearly A_2 is Π_1^1. Let α code a Borel set. Let x be a real, and let γ be as in the statement of Lemma 2. Then $A_2(\alpha, x)$ if and only if $\gamma(0) = 1$, by Lemma 2. Moreover, $\gamma(0) = 1$ if and only if x lies in the Borel set coded by $\Phi(\alpha, 0)$. But $\Phi(\alpha, 0) = \alpha$. Thus $A_2(\alpha, x)$ if and only if α codes a Borel set and x lies in the Borel set coded by α.

The treatment of A_3 is similar. We put

$$A_3(\alpha, x) \equiv (\beta)\big(\Phi(\alpha, \beta, x) \rightarrow \beta(0) = 0\big) \wedge A_1(\alpha) .$$

COROLLARY. *There are Π_1^1 predicates $A_4(\alpha, \beta)$, $A_5(\alpha, \beta)$ which are provably equivalent in* ZF + DC *to the following concepts*

(4) $B_\alpha \subseteq B_\beta$.

(5) $B_\alpha = B_\beta$.

PROOF. Put

$$A_4(\alpha, \beta) \equiv A_1(\alpha) \wedge A_1(\beta) \wedge (x)\big(A_3(\alpha, x) \vee A_2(\beta, x)\big)$$

and

$$A_5(\alpha, \beta) \equiv A_4(\alpha, \beta) \wedge A_4(\beta, \alpha) .$$

This suffices.

1.3. Kleene has shown that there is an extremely close relation between Π_1^1 relations and the concept of well-orderings (cf. [7]). Moreover, if $\mathfrak{M}$ is a transitive model of ZF, then the ordinals of $\mathfrak{M}$ are an initial segment of the ordinary ordinals (cf. [1, p. 94]). Putting these facts together, one has the following lemma (cf. [13, pp. 137–138]).

LEMMA. *Let $\Phi(\alpha)$ be a Π_1^1 predicate. Let $\mathfrak{M}$ be a transitive model of* ZF. *Let $\alpha: \omega \rightarrow \omega$, be an element of $\mathfrak{M}$. Then*

$$\mathfrak{M} \models \Phi(\alpha)$$

if and only if $\Phi(\alpha)$ holds in the real world.

1.4. We have two situations to consider simultaneously.

(a) $\mathfrak{M}$ and $\mathfrak{N}$ are transitive models of ZF + DC, and $\mathfrak{M} \subseteq \mathfrak{N}$;

(b) $\mathfrak{M}$ is a transitive model of $\mathbf{ZF} + \mathbf{DC}$, and $\mathfrak{N}$ is the universe of all sets (so the axiom of choice holds in $\mathfrak{N}$).

THEOREM. *Let $\mathfrak{M}$, $\mathfrak{N}$ be as in* (a) *or* (b) *above. Let $\alpha, \beta \in (\omega^\omega)_{\mathfrak{M}}$, and $x \in \mathbf{R}_{\mathfrak{M}}$. Then the following statements hold in $\mathfrak{M}$ if they hold in $\mathfrak{N}$.*

(1) α *codes a Borel set.*

(2) α *codes a Borel set, B_α, and $x \in B_\alpha$.*

(3) α, β *code Borel sets and $B_\alpha = B_\beta$.*

PROOF. Case (a) of the lemma follows easily from case (b). Case (b) follows from Lemma 1.3 and the results of § 1.2.

1.5. Theorem 1.4 implies that the assignment

(1) $$\{B_\alpha^{\mathfrak{M}} \longrightarrow B_\alpha^{\mathfrak{N}}\}$$

gives a 1-1 correspondence between the Borel sets of reals in $\mathfrak{M}$ and a certain subcollection of the Borel sets of reals in $\mathfrak{N}$. The map (1) is, in general, not surjective. For example, if $\mathfrak{M}$ is countable, and $\mathfrak{N}$ is the real world, (1) is certainly not surjective.

Let C be a Borel set in $\mathfrak{N}$. We say that C is *rational* over $\mathfrak{M}$, if $C = B_\alpha^{\mathfrak{N}}$ for some code α lying in $\mathfrak{M}$. By part (2) of Theorem 1.4 the Borel set in $\mathfrak{M}$ corresponding to C is then

$$C \cap \mathbf{R}_{\mathfrak{M}} \ .$$

Similarly, we say that a sequence of Borel sets in $\mathfrak{N}$, $\{C_i\}$, is rational over $\mathfrak{M}$ if there is a sequence of codes, $\{\alpha_i\}$, lying in $\mathfrak{M}$ such that

$$C_i = B_{\alpha_i}^{\mathfrak{N}} \ .$$

(Note carefully that we require not only that each α_i lie in $\mathfrak{M}$ but that the function $\{i \rightarrow \alpha_i\}$ also lie in $\mathfrak{M}$.) Since $\mathbf{DC}$ holds in $\mathfrak{M}$ it is equivalent to require that there is in $\mathfrak{M}$ a sequence of $\mathfrak{M}$-Borel sets, $\{B_i\}$, such that for each i, B_i corresponds to C_i under (1).

The correspondence just described clearly possesses the following naturality property. Let $\mathfrak{M} \subseteq \mathfrak{N}$ be transitive models of $\mathbf{ZF} + \mathbf{DC}$. Let V be the real world. Let B be a Borel set in $\mathfrak{M}$ and let $B_{\mathfrak{N}}$, B_V be the corresponding Borel sets in $\mathfrak{N}$ and V respectively. Then B_V is the Borel set in V corresponding to the Borel set $B_{\mathfrak{N}}$ of $\mathfrak{N}$.

1.6. We are going, eventually, to use the map (1) to identify the Borel sets of $\mathfrak{M}$ with certain of the Borel sets of $\mathfrak{N}$. As a temporary piece of notation, if B is a Borel set of $\mathfrak{M}$, we write B^* for the corresponding Borel set of

$\mathfrak{M}$. We proceed to verify that certain properties and operations are "absolute" with respect to the map $\{B \to B^{\sharp}\}$.

LEMMA 1. (1) *Boolean operations are absolute.*

(2) *Let $\{A_n\}$ be a sequence of Borel sets of $\mathfrak{M}$, with $\{A_n\} \in \mathfrak{M}$. Then*

$$(\bigcup_n A_n)^{\sharp} = \bigcup_n A_n^{\sharp}$$
$$(\bigcap_n A_n)^{\sharp} = \bigcap_n A_n^{\sharp} \, .$$

(3) *The relation $A \subseteq B$ is absolute.*

(4) *The relation $A = \varnothing$ is absolute.*

PROOF. (1) Consider for example, the intersection operation. Given A, B Borel in $\mathfrak{M}$ with codes α and β respectively. One constructs easily from α and β a code γ which codes $A \cap B$ in $\mathfrak{M}$ and $A^{\sharp} \cap B^{\sharp}$ in $\mathfrak{N}$. Thus

$$(A \cap B)^{\sharp} = A^{\sharp} \cap B^{\sharp} \, .$$

(2) Similar to the proof of (1).

(3) $A \subseteq B \Leftrightarrow A \cup B = B$. By Theorem 1.4 and (1) of this lemma, we have (3).

(4) Clear from Theorem 1.4.

LEMMA 2. *The following operations and notions are absolute.*

(1) *Interior* (int);

(2) *"Open";*

(3) *Closure;* (Cl)

(4) *"Closed";*

(5) *"Closed nowhere dense";*

(6) *"Compact".*

PROOF. (1) $A = \operatorname{int} B$ if and only if

$$A = \bigcup \{(r, s): r, s \in \mathbf{Q} \quad \text{and} \quad (r, s) \subseteq B\}$$

(2) A is open $\Leftrightarrow A = \operatorname{int} A$.

(3) $\operatorname{Cl}(A) = \mathbf{R} - \operatorname{int}(\mathbf{R} - A)$.

(4) A is closed $\Leftrightarrow A = \operatorname{Cl}(A)$.

(5) A is closed nowhere dense $\Leftrightarrow A$ is closed and $\operatorname{int}(A) = \varnothing$.

(6) A is compact if and only if A is closed and for some $N \in \omega, A \subseteq [-N, N]$.

LEMMA 3. *Let r, s be reals of $\mathfrak{M}$. Then $(r, s)^{\sharp} = (r, s)$; $[r, s]^{\sharp} = [r, s]$; $\{r\}^{\sharp} = \{r\}$.*

PROOF.

$$(r, s) = \bigcup \{[t, u]: r < t \leq u < s; t, u \in \mathbf{Q}\}$$
$$[r, s] = \bigcap \{(r - 1/n, s + 1/n): n \in \omega\}$$
$$\{r\} = [r, r] \, .$$

LEMMA 4. *Let μ be Lebesgue measure. Let B be a Borel set in $\mathfrak{M}$. Then $\mu_{\mathfrak{M}}(B) = \mu_{\mathfrak{N}}(B^{*})$.*

Case 1. B is the union of a finite number of disjoint open intervals with rational endpoints.

Say $r_1 < s_1 \leqq r_2 < s_2 \leqq \cdots \leqq r_n < s_n$, and $B = \bigcup_{i=1}^{n} (r_i, s_i)$ in both $\mathfrak{M}$ and $\mathfrak{N}$. Then $\mu(B) = \sum_{i=1}^{n} (s_i - r_i)$, which is absolute.

There are clearly only countably many sets of the sort considered in case 1; let $\{W_n\}$ be an enumeration of these sets in $\mathfrak{M}$.

Case 2. B compact.

We have $\mu(B) = \inf \{\mu(W_n): B \subseteq W_n\}$ which proves μ is absolute in this case.

Case 3. B open.

Clear since $\mu(B) = \sup \{\mu(W_n): W_n \subseteq B\}$.

Case 4. B arbitrary.

$\mu_{\mathfrak{M}}(B) = \sup \{\mu(K): K$ compact, $K \subseteq B$, and K rational over $\mathfrak{M}\} \leqq \sup \{\mu(K):$ K compact, $K \subseteq B^{*}$, and K rational over $\mathfrak{N}\} = \mu_{\mathfrak{N}}(B^{*})$.

Similarly $\mu_{\mathfrak{M}}(B) = \inf \{\mu(U): U$ open, $B \subseteq U$ and U rational over $\mathfrak{M}\} \geqq$ $\inf \{\mu(U):$ open, $B^{*} \subseteq U$ and U rational over $\mathfrak{N}\} = \mu_{\mathfrak{N}}(B^{*})$. These two inequalities show $\mu_{\mathfrak{M}}(B) = \mu_{\mathfrak{N}}(B^{*})$.

COROLLARY. *"Set of measure zero" is an absolute notion.*

Recall that the symmetric difference of two sets, A and B, (notation: $A \triangle B$) is

$$(A - B) \cup (B - A) .$$

We recall some elementary constructions from the theory of boolean algebras. This material is all contained in Halmos [4].

Let $\mathcal{I}$ be an ideal of subsets of $\mathbf{R}$. This means that if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$, and if $A \in \mathcal{I}$, and $B \subseteq A$, then $B \in \mathcal{I}$. We say that two sets A and B are equal mod $\mathcal{I}$ if

$$A \triangle B \in \mathcal{I} .$$

Equality mod $\mathcal{I}$ is an equivalence relation. The set of equivalence classes of Borel sets is, in a natural way, a boolean algebra, since the boolean operations "pass to quotients".

If $\mathcal{I}$ is a σ-ideal (i.e., is closed under countable unions), then the boolean algebra of equivalence classes mod $\mathcal{I}$ is a σ-algebra.

The two basic examples of σ-ideals in the power set of $\mathbf{R}$ are:

(1) the σ-ideal $\mathcal{I}_1$ of sets of Lebesgue measure zero;

(2) the σ-ideal $\mathcal{I}_2$ of sets of the first category.

Two Borel sets equal mod $\mathcal{I}_1$ are said to be equal "almost everywhere"; two Borel sets equal mod $\mathcal{I}_2$ are "equal modulo a set of the first category".

The following lemma will be useful in a moment. For the proof see Halmos [4, p. 58].

LEMMA 5. *Let B be Borel. Then B is equal to an open set U modulo a set of the first category.*

LEMMA 6. *"First category" is an absolute notion.*

PROOF. Using Lemma 1 parts (2) and (3), and Lemma 2 part (5), we see that if B is first-category in $\mathfrak{M}$, B is first-category in $\mathfrak{N}$.

Suppose now that B is not first category in $\mathfrak{M}$. By Lemma 5 in $\mathfrak{M}$, there exists U open, rational over $\mathfrak{M}$ such that $B \triangle U$ is first-category in $\mathfrak{M}$. By the preceding paragraph, $B \triangle U$ is first-category in $\mathfrak{N}$. If $U = \varnothing$, $B = B \triangle U$ and so B is first category in $\mathfrak{M}$. Thus $U \neq \varnothing$, and by the Baire category theorem U is not first category in $\mathfrak{N}$. Since

$$U \subseteq B \cup (B \triangle U),$$

B is not first category in $\mathfrak{N}$. The proof is complete.

The following lemma is not needed in the present paper but will be used in another paper of the author [15].

LEMMA 7. *The following notions are absolute.*
(1) *X has at least two points.*
(2) *X has exactly one point.*
(3) *X is perfect.*
(4) *X is countable.*

PROOF. (1) X has at least two points if and only if there exist rationals: $r < s < t < u$ such that $X \cap (r, s) \neq \varnothing$; $X \cap (t, u) \neq \varnothing$.

(2) Immediate from (1).

(3) X is perfect if and only if X is closed, $X \neq \varnothing$, and for all $r, s \in \mathbf{Q}$, $X \cap (r, s) \neq \varnothing \Rightarrow X \cap (r, s)$ has at least two points.

(4) By (2) and (2) of Lemma 1, X countable in $\mathfrak{M}$ implies X countable in $\mathfrak{N}$. If X is not countable in $\mathfrak{M}$, X contains a perfect subset K. In $\mathfrak{N}$, K is perfect (by (3)) and $K \subseteq X$. Thus X is uncountable in $\mathfrak{N}$.

1.7. The following concept will be needed in §2 below.

Let $C = \{\alpha \mid \alpha \text{ codes a real}\}$. Let

$$\lambda \colon C \to OR$$

be defined as follows.

(1) If α codes by case 1 of Definition 1.1, then $\lambda(\alpha) = 0$.

(2) If α codes by case 2, then

$$\lambda(\alpha) = \sup \{\lambda(\alpha_i) + 1\}$$

(notation as in case 2 of Definition 1.1).

(3) If α codes by case 3,

$$\lambda(\alpha) = \lambda(\beta) + 1 .$$

It is easy to see that domain $\lambda = C$ by an argument similar to the proof of Lemma 1.1.1.

We write $\lambda^{\mathfrak{M}}$ for the relativization of λ to $\mathfrak{M}$.

1.8. Our results would apply to any of the standard spaces, such as 2^ω, *mutatis mutandis*.

1.9. Let $j = 1$ or 2. Let $\mathcal{I}_j$ be the ideal described in § 1.6. Let $\mathcal{B}_j$ be the quotient algebra of the σ-algebra of Borel sets associated to the ideal $\mathcal{I}_j$. Then the following facts are proved in [4]: $\mathcal{B}_j$ is a complete boolean algebra satisfying the countable chain condition.

2.1. Let $\mathfrak{M}$ be a fixed transitive model of **ZFC**. A real x is *random* over $\mathfrak{M}$ if it lies in no Borel set of measure zero rational over $\mathfrak{M}$. Similarly a subset of ω is *random* over $\mathfrak{M}$ if it lies in no Borel set of measure zero of 2^ω rational over $\mathfrak{M}$.

Notice that if x is random over $\mathfrak{M}$, then $x \notin \mathfrak{M}$. (*Proof.* $x \in \mathfrak{M} \Rightarrow \{x\}$ is a Borel set of measure zero, rational over $\mathfrak{M}$, and containing x.) This definition is in accord with the usual intuitive requirements for a random real. For example if we let $\xi(x, N)$ be the number of 1's in the first N digits of the decimal expansion of x, then for x random the limit as $N \to \infty$ of $\xi(x, N)/N$ exists and equals $1/10$. (A proof would show that the set of reals x for which this is false form a Borel set of measure zero rational over $\mathfrak{M}$.)

The following lemma provides for the existence of many random reals (if, for example, $\mathfrak{M}$ is countable):

LEMMA. *If $(2^{\aleph_0})_{\mathfrak{M}}$ is countable, then almost all reals are random over $\mathfrak{M}$. (In fact, the non-random reals form a Borel set of measure zero.)*

PROOF. If $(2^{\aleph_0})_{\mathfrak{M}}$ is countable, we can enumerate the Borel sets of measure zero rational over $\mathfrak{M}$ in a sequence $N_0, N_1, \cdots$. Then x is random over $\mathfrak{M}$ if and only if $x \notin \bigcup_i N_i$. But $\bigcup_i N_i$ is a Borel set of measure zero.

2.2. There is an analogous notion of a real (or set of integers) being generic over $\mathfrak{M}$. A real x is *generic* over $\mathfrak{M}$ if it lies in no Borel set of the

first category rational over $\mathfrak{M}$. We shall see below that this is essentially the same as the notion introduced by Cohen. (Cohen worked with sets of integers; for reals, the conditions analogous to Cohen's have the form $r < x < s$, where r, s are rationals.)

It is true that no real x is both generic and random over $\mathfrak{M}$. We shall not stop to prove this here. (For example, the set of reals x in which 1 has the frequency 1/10 in the decimal expansion of x form a set of first category, rational over $\mathfrak{M}$, and containing all random reals.)

All the results we prove for random reals in this section have analogues for generic reals with "the same proofs". The translation consists in replacing "random" by "generic" and "measure zero" by "first category". We leave this translation to the reader.

2.3. We are going eventually to show that the random reals are in natural one-one correspondence with the generic filters on a certain partially ordered set $\mathcal{P} \in \mathfrak{M}$. The following discussion is a heuristic motivation for the correct choice of $\mathcal{P}$.

Let x be a real random over $\mathfrak{M}$. An observer stationed in $\mathfrak{M}$ cannot have total knowledge about x (since x is not in $\mathfrak{M}$). However, he can have partial knowledge about x. For example, if B is a Borel set rational over $\mathfrak{M}$, then a natural question the observer can ask about x is "Is $x \in B$?" If $\mu(B) = 0$, then the answer is certainly no. On the other hand, if $\mu(B) > 0$, it is possible for x to be in B (cf. Lemma 2.1). A similar discussion shows that if B_1 and B_2 are Borel sets rational over $\mathfrak{M}$, and $\mu(B_1 \bigtriangleup B_2) = 0$ (i.e., B_1 and B_2 are equal almost everywhere), then for x random over $\mathfrak{M}$, the questions "Is $x \in B_1$?" and "Is $x \in B_2$?" are equivalent.

We therefore make the following definition.

Definition 1. $\mathcal{P}$ is the set of equivalence classes of non-null Borel sets of reals (in $\mathfrak{M}$). Two sets B_1 and B_2 are equivalent if and only if $B_1 \bigtriangleup B_2$ has measure zero. (Let $[B]$ be the equivalence class of B. We think of the condition $[B]$ as telling us $x \in B$.)

We order $\mathcal{P}$ by an order $\leqq$ as follows. $[B] \leqq [B']$ if and only if $B' \subseteq B$ almost everywhere (i.e., $B' - B$ is a set of measure zero).

2.4. Let $\mathfrak{B} \in \mathfrak{M}$. Then clearly $\mathfrak{B}$ is a boolean algebra in $\mathfrak{M}$ if and only if $\mathfrak{B}$ is a boolean algebra in the real world. However, $\mathfrak{B}$ can be complete in $\mathfrak{M}$ (cf. [4, p. 25]) without being complete in the real world, since there may be subsets $S \subseteq \mathfrak{B}$ such that $\sup S$ is not defined, but $S \notin \mathfrak{M}$. We say that $\mathfrak{B}$ is $\mathfrak{M}$-*complete* if and only if $\mathfrak{M} \models \mathfrak{B}$ is complete.

Let now $\mathfrak{B} \in \mathfrak{M}$ be a boolean algebra, and

$$h: \mathscr{B} \longrightarrow 2$$

be a homomorphism. (We *do not* require that h lie in $\mathfrak{M}$.) We say that h is $\mathfrak{M}$-*completely additive* if whenever $S \subseteq \mathscr{B}$, $S \in \mathfrak{M}$, and $\sup S$ exists in $\mathscr{B}$, then

$$h(\sup S) = \sup \{h(s): s \in S\} \, .$$

Let $\mathscr{B}$ be a boolean algebra lying in $\mathfrak{M}$. Let $\leq$ be the usual order on $\mathscr{B}$: $b_1 \leq b_2$ if and only if $b_1 \vee b_2 = b_2$. Let $\mathscr{P}$ be the set of non-zero elements of $\mathscr{B}$. We provide $\mathscr{P}$ with the order $\prec$ inverse to $\leq$: $b_1 \prec b_2$ if and only if $b_2 \leq b_1$.

LEMMA. *Let* $h: \mathscr{B} \longrightarrow 2$ *be an* $\mathfrak{M}$-*completely additive homomorphism. Then*

(1) $$G = \{b \mid h(b) = 1\}$$

is an $\mathfrak{M}$-*generic filter on* $\mathscr{P}$. *Conversely, if* G *is an* $\mathfrak{M}$-*generic filter on* $\mathscr{P}$, *there is a unique homomorphism*

$$h: \mathscr{B} \longrightarrow 2$$

such that (1) *holds.*

PROOF. Let $h: \mathscr{B} \longrightarrow 2$ be a homomorphism. Then if G is defined by (1), then G satisfies clauses (1) and (2) of Definition I.1.3. Conversely, if $G \subseteq \mathscr{P}$ satisfies clauses (1) and (2) of Definition I.1.3, then G is, in the usual terminology of the theory of boolean algebras, a filter on $\mathscr{B}$. If G also satisfies (3) of Definition I.1.3 then G is an ultrafilter on $\mathscr{B}$. (For any $b_0 \in \mathscr{B}$, the set $\{b \in \mathscr{P}: b \leq b_0$ or $b \cdot b_0 = 0\}$ is dense in $\mathscr{P}$.) Hence there is a unique homomorphism $h: \mathscr{B} \longrightarrow 2$ such that (1) holds.

Now let X be dense in $\mathscr{P}$, $X \in \mathfrak{M}$. We say $\sup_{\mathscr{B}}(X) = 1_{\mathscr{B}}$. Otherwise there is a $b > 0$ with $b \cdot x = 0$, for all $x \in X$. But X is dense, so for some $x_0 \in X$, $x_0 \leq b$. But then $b \cdot x_0 = x_0 \neq 0$. This contradiction proves our claim.

It follows that if $h: \mathscr{B} \longrightarrow 2$ is $\mathfrak{M}$-completely additive and X is as above, we have $h(x) = 1$ for some $x \in X$. Thus $x \in G \cap X$, and G satisfies clause 3 of Definition I.1.3.

Conversely, suppose that G is $\mathfrak{M}$-generic. Let $S \subseteq \mathscr{B}$, $S \in \mathfrak{M}$, $\sup S = s_0$. We want to show

(2) $$h(s_0) = \sup \{h(s): s \in S\} \, .$$

We may as well suppose that $s_0 = 1_{\mathscr{B}}$. (Otherwise, replace S by $S \cup \{1_{\mathscr{B}} - s_0\}$.) Let

$$X = \{a \in \mathscr{P} \mid a \leq b \text{ for some } b \in S\} \, .$$

Then X is dense in $\mathscr{P}$ (cf. Definition I.1.2). Indeed, clause (1) of the definition of "dense" is obvious; we verify clause (2). Let $x \in \mathscr{P}$. Then $x \neq 0$. Since

$\sup S = 1$, $x \wedge s \neq 0$ for some $s \in S$. Hence $x \wedge s \in X$, and $x \prec x \wedge s$. This verifies clause (2).

Since X is dense in $\mathscr{P}$, and X lies in $\mathfrak{M}$, we have $G \cap X \neq \varnothing$. Hence there is an $a \in G$ and a $b \in S$ with $a \leqq b$. Since $h(a) = 1$, we have $h(b) = 1$. Since $b \in S$, and $s_0 = 1$, (2) is proved.

2.5. We now suppose in addition that $\mathscr{B}$ is $\mathfrak{M}$-complete, and that only countably many subsets of $\mathscr{B}$ lie in $\mathfrak{M}$.

LEMMA. *Let Φ be a sentence of $\mathscr{L}'$. Then there is a $b_0 \in \mathscr{B}$ such that if G is an $\mathfrak{M}$-generic filter on $\mathscr{P}$, and $h: \mathscr{B} \to 2$, then $\mathfrak{M}[G] \models \Phi$ if and only if $h(b_0) = 1$. Moreover, b_0 is uniquely determined by Φ.*

PROOF. Let

$$S = \{b \in \mathscr{P} \mid b \Vdash \Phi\} \, .$$

Since forcing is expressible in $\mathfrak{M}$, $S \in \mathfrak{M}$. Let $b_0 = \sup S$. (If $S = \varnothing$, we take $b_0 = 0$, and the lemma is clear. So we may assume $S \neq \varnothing$.) We maintain $b_0 \Vdash \Phi$. Otherwise, there is a $c \in \mathscr{P}$ with

$$0 < c \leqq b_0 \, ,$$

i.e., $b_0 \prec c$, and $c \Vdash \neg \, \Phi$. Since $b_0 = \sup S$, there is a $b_1 \in S$ with $b_1 \wedge c \neq 0$. But then

$$c \prec b_1 \wedge c$$

so $b_1 \wedge c \Vdash \neg \, \Phi$. On the other hand, $b_1 \Vdash \Phi$, since $b_1 \in S$; since $b_1 \prec b_1 \wedge c$, $b_1 \wedge c \Vdash \Phi$. This contradiction shows $b_0 \Vdash \Phi$.

If $h(b_0) = 1$, then $b_0 \in G$, so $\mathfrak{M}[G] \models \Phi$ (since $b_0 \Vdash \Phi$.) Conversely, if $\mathfrak{M}[G] \models \Phi$, there is a $b_1 \in S \cap G$, so $h(b_1) = 1$, so $h(b_0) = 1$. Thus b_0 has the desired properties.

Suppose now that b_1 has the same relation to Φ as b_0. We show $b_1 = b_0$. Otherwise, let $b_2 = b_1 \bigtriangleup b_0$. Then if h is $\mathfrak{M}$-completely additive, $h(b_0) = h(b_1) =$ truth value of Φ, so $h(b_2) = 0$. If $b_1 \neq b_0$, then $b_2 \neq 0$. By I.1.8, there is an $\mathfrak{M}$-generic filter G with $b_2 \in G$. But then, $h(b_2) = 1$, contradicting our remark that $h(b_2) = 0$. This shows $b_1 = b_0$.

2.6. The following theorem provides the link between the abstract material of § 2.4–5 and the concept of a random real introduced in § 2.1.

Let $\mathscr{B}_1$ (resp. $\mathscr{B}_2$) be the algebra of $\mathfrak{M}$-Borel sets modulo the ideal of sets of measure zero (resp. of first category). By § 1.9, these algebras are $\mathfrak{M}$-complete.

THEOREM. *There is a canonical 1–1 correspondence between the reals random over $\mathfrak{M}$ and the $\mathfrak{M}$-completely additive homomorphisms of $\mathscr{B}_1$.*

PROOF. Let h be an $\mathfrak{M}$-completely additive homomorphism of $\mathfrak{B}_1 \to \{0, 1\}$. Set

$$x_h = x = \{r \in \mathbf{Q} \colon h((r, \infty)) = 1\} \, .$$

(To abbreviate, we sometimes use the same symbol to denote an $\mathfrak{M}$-Borel set and its image in $\mathfrak{B}_1$.)

LEMMA 1. *x is an irrational left Dedekind cut in* $\mathbf{Q}$.

PROOF. (1) $x \neq \varnothing$: Since $h((-\infty, \infty)) = 1$, for some $n \in \omega$, we have $h((-n, \infty)) = 1$, so $-n \in x$.

(2) $x \neq \mathbf{Q}$: Since $h(\varnothing) = 0$ and $\varnothing = \bigcap_{n=1}^{\infty}(n, \infty)$, we have $h((n, \infty)) = 0$, for some $n \in \omega$. But then $n \notin x$.

(3) $r_2 < r_1 \in x \Rightarrow r_2 \in x$: Since $(r_1, \infty) \subseteq (r_2, \infty)$, $h((r_1, \infty)) = 1 \Rightarrow h((r_2, \infty)) = 1$.

(4) x is irrational: Let $r \in \mathbf{Q}$. Since $\{r\} = \bigcap_n (r - 1/n, \ r + 1/n)$, $h((r - 1/n, \ r + 1/n)) = 0$ for some n. But then $r - 1/n \in x \Leftrightarrow r + 1/n \in x$, so x is not the Dedekind cut centered at r. (Similarly, we see that $x \notin \mathfrak{M}$.)

LEMMA 2. *Let A be a Borel set rational over $\mathfrak{M}$. Then $x \in A$ if and only if $h(A) = 1$.*

PROOF. Let α be a code for A, lying in $\mathfrak{M}$. The proof proceeds by induction on $\lambda^{\mathfrak{M}}(\alpha)$ and is straightforward (cf. §1.7). (Note that $\lambda^{\mathfrak{M}}(\alpha) \in OR^{\mathfrak{M}} \subseteq OR$, so the induction is legitimate, even though $h \notin \mathfrak{M}$.)

It is now easy to show that x is random over $\mathfrak{M}$. Let N be a set of measure zero rational over $\mathfrak{M}$. Then $x \in N$ if and only if $h([N]) = 1$. But $[N] = 0$ in $\mathfrak{B}_1$, so $h([N]) = 0$.

Now suppose that x is random over $\mathfrak{M}$. Define $h_x \colon \mathfrak{B}_1 \to \{0, 1\}$, by $h_x([A]) = 1$ if and only if $x \in A$. (A rational over M.) (To see that h_x is well defined, let A_1 and A_2 be Borel sets rational over $\mathfrak{M}$ such that $[A_1] = [A_2]$. Then $\mu(A_1 \bigtriangleup A_2) = 0$ so $x \notin A_1 \bigtriangleup A_2$ (since x is random over $\mathfrak{M}$). It follows that $x \in A_1 \equiv x \in A_2$.)

It is not hard to check that h_x is $\mathfrak{M}$-countably additive. The proof that h_x is $\mathfrak{M}$-completely additive is based on the following lemma (Halmos [4, p. 61]).

LEMMA 3. *Let $\mathfrak{B}$ be a complete boolean algebra satisfying the countable chain condition. Let $S \subseteq \mathfrak{B}$. Then S has a countable subset S_0 such that*

$$\bigvee S = \bigvee S_0 \, .$$

Let $S \subseteq \mathfrak{B}_1$, $S \in \mathfrak{M}$. Since h_x is finitely-additive, $h_x(\bigvee S) \geqq \bigvee \{h_x(s) \colon s \in S\}$.

To get the reverse inequality, we apply Lemma 3, within $\mathfrak{M}$, to $\mathfrak{B}_1$. Let $S_0 \subseteq S$, S_0 countable in $\mathfrak{M}$, such that $\bigvee S_0 = \bigvee S$. Since h_x is $\mathfrak{M}$-countably additive

$$h_x(\bigvee S) = h_x(\bigvee S_0) = \bigvee \{h_x(s) \colon s \in S_0\} \leqq \bigvee \{h_x(s) \colon s \in S\} \, .$$

The reverse inequality has already been proved. Thus h_x is $\mathfrak{M}$-completely additive.

We have shown that if h is $\mathfrak{M}$-completely additive, x_h is random over $\mathfrak{M}$. Lemma 2 shows that h can be recovered from x_h and the discussion just completed shows that every x random over $\mathfrak{M}$ arises in this way. The theorem is proved. The theorem has an analogue for random elements of 2^ω. (In that case, Lemma 2 is unnecessary.) There is a corresponding theorem, of course, identifying reals generic over $\mathfrak{M}$ with the $\mathfrak{M}$-completely additive homomorphisms of $\mathfrak{B}_2$.

2.7. Let $\mathfrak{M}$, $\mathfrak{B}_1$, be as above. Let x be a real random over $\mathfrak{M}$, and G the associated filter on $\mathfrak{B}_1$. It is clear from the discussion in § 2.6 that $x \in \mathfrak{M}[G]$. Moreover, in view of Lemma 2.6.2 and Lemma 2.4, it is clear that if $\mathfrak{N}$ is any transitive model of **ZFC** with $x \in \mathfrak{N}$ and $\mathfrak{M} \subseteq \mathfrak{N}$, then $G \in \mathfrak{N}$. By I.1.4, we then have $\mathfrak{M}[G] \subseteq \mathfrak{N}$.

Notations being as in the preceding paragraph, we write $\mathfrak{M}[x]$ for $\mathfrak{M}[G]$. Thus the discussion of the preceding paragraph shows that $\mathfrak{M}[x]$ is the minimal transitive model, $\mathfrak{N}$, of **ZFC** such that $\mathfrak{M} \subseteq \mathfrak{N}$ and $x \in \mathfrak{N}$.

Because we know that $\mathfrak{M}[x] = \mathfrak{M}[G]$, it is clear that $\mathfrak{M}[x]$ has the same ordinals as $\mathfrak{M}$. Moreover, using the fact that $\mathfrak{B}_1$ satisfies the countable chain condition in $\mathfrak{M}$, it would be easy to show that $\mathfrak{M}$ and $\mathfrak{M}[x]$ have the same cardinals. (This is true for all reals random over $\mathfrak{M}$, and hence for almost all reals.)

2.8. We can now prove our fundamental result about random reals. We alter the language $\mathscr{L}'$ of Chapter I slightly, by replacing the constant G by a constant $\boldsymbol{x}$. Call the resulting language $\mathscr{L}''$. If x is a real random over $\mathfrak{M}$, we interpret $\mathscr{L}''$ in $\mathfrak{M}[x]$ in the obvious way. In particular, we let $\boldsymbol{x}$ denote x.

THEOREM. *Let Φ be a sentence of $\mathscr{L}''$. Then there is a Borel set A rational over $\mathfrak{M}$ such that for all x random over $\mathfrak{M}$, we have*

$$(1) \qquad\qquad \mathfrak{M}[x] \models \Phi \longleftrightarrow x \in A .$$

PROOF. Let $h_x \colon \mathfrak{B}_1 \to 2$, G_x be the homomorphism and filter determined by x. Since $\mathfrak{M}[x] = \mathfrak{M}[G_x]$ and x is definable in $\mathfrak{M}[x]$ from $\mathfrak{B}_1$, G_x, we can find a sentence Φ' of $\mathscr{L}'$ such that for all x random over $\mathfrak{M}$,

$$\mathfrak{M}[x] \models \Phi \longleftrightarrow \mathfrak{M}[G_x] \models \Phi' .$$

By Lemma 2.5, there is an element b_0 of $\mathfrak{B}_1$, not depending on x, such that for all x random over $\mathfrak{M}$,

$$\mathfrak{M}[G_x] \models \Phi' \longleftrightarrow h_x(b_0) = 1.$$

Let $\alpha \in \omega^\omega$ code a Borel set in $\mathfrak{M}$ whose equivalence class lies in b_0. Let A be the Borel set coded by α in the real world. Then by Lemma 2.6.2,

$$h_x(b_0) = 1 \longleftrightarrow x \in A \ .$$

The theorem is proved.

2.9. We show that the notion of a generic subset of ω introduced by Cohen is the same as the notion of "generic" introduced in § 4.2. The result will not be used in this paper, but is included for its historical interest.

We recall the precise definition of "generic" given by Levy in [8]. Let $\mathscr{P}_0$ be the set of Cohen conditions: an element $p \in \mathscr{P}_0$ is a function with domain a finite subset of ω and range a subset of $\{0, 1\}$.

Let $\mathfrak{M}$ be a transitive model of $\mathbf{ZF} + \mathbf{DC}$. A subset D of $\mathscr{P}_0$ is dense if $(\forall p \in \mathscr{P}_0)(\exists p' \in D)(p \subseteq p')$. An element $f \in 2^\omega$ is I-generic over $\mathfrak{M}$ if for each dense $D \in \mathfrak{M}$, there exists $p \in D$ such that $p \subseteq f$. (This definition is essentially that of Levy [8]. It comes, *via* Easton [3], from an idea of the author.) An element $f \in 2^\omega$ is II-generic over $\mathfrak{M}$ if it lies in no first category Borel set rational over $\mathfrak{M}$. We shall prove that the following properties of f are equivalent:

(1) f is I-generic over $\mathfrak{M}$;

(2) f lies in each dense open set rational over $\mathfrak{M}$;

(3) f is II-generic over $\mathfrak{M}$.

Proof. The essential point is that Lemma I.2.2 allows us to relate generic filters on the two different partially ordered sets implicit in the notions "I-generic" and "II-generic". The details are as follows.

Let $\mathscr{B}_2$ be the boolean algebra of $\mathfrak{M}$ defined (in $\mathfrak{M}$) as the quotient of the σ-algebra of Borel subsets of 2^ω modulo the σ-ideal of first category Borel sets.

We shall need an alternative description of $\mathscr{B}_2$. We recall that an open set U is *regular* if and only if U is the interior of the closure of U. Then each element of $\mathscr{B}_2$ is the representative of a unique regular open set. In this way, we get a bijective correspondence between $\mathscr{B}_2$ and the regular open sets rational over $\mathfrak{M}$. This correspondence is order preserving if we order the regular open sets by inclusion.

Let $\mathscr{P}_2$ be the set of non-zero elements of $\mathscr{B}_2$ equipped with the order, $\prec$, inverse to that of $\mathscr{B}_2$. Thus $\mathscr{P}_2$ is canonically isomorphic to the set of non-void regular open sets rational over $\mathfrak{M}$. Call this latter set $\mathscr{P}_2'$.

Now let G be a generic filter on $\mathscr{P}_2$, x the II-generic element of 2^ω determined by G (cf. § 2.6). Let $G' \subseteq \mathscr{P}_2'$ correspond to G. Using the analogue of Lemma 2.6.2, one sees that

$$\{x\} = \bigcap G' \ .$$

We map $\mathscr{P}_0$ into $\mathscr{P}_2'$ as follows: if $p \in \mathscr{P}_0$, put $W_p = \{f \in 2^\omega: p \subseteq f\}$.

Then W_p is open-closed and hence regular. It is clearly rational over $\mathfrak{M}$. The map

$$\{p \longrightarrow W_p\}$$

is order-preserving. ($\mathscr{P}_2'$ carries the order induced by the given order on $\mathscr{P}_2$.) Moreover, the image, $\mathscr{P}_0'$, of $\mathscr{P}_0$ in $\mathscr{P}_2'$ is cofinal since sets of the form W_p form a basis for the open sets in 2^ω.

By definition, an element $x \in 2^\omega$ is I-generic if and only if $\{p \in \mathscr{P}_0: p \subseteq x\}$ is a generic filter on $\mathscr{P}_0$. It follows that the I-generic elements of 2^ω are obtained as follows: take an $\mathfrak{M}$-generic filter G on $\mathscr{P}_0$, copy it onto a filter G' on $\mathscr{P}_0'$, and take $\bigcap G'$.

Now if G_1 is an $\mathfrak{M}$-generic filter on $\mathscr{P}_0$, there is an $\mathfrak{M}$-generic filter G_2 on $\mathscr{P}_2$ such that $G_1 = \{p: W_p \in G_2'\}$ (cf. Lemma I.2.2). Moreover, all generic filters on $\mathscr{P}_0$ arise in this way. Thus x is I-generic if and only if $x = \bigcap G_1'$ for a generic filter G_1 on $\mathscr{P}_0$ if and only if $x = \bigcap G_2'$ for some generic filter G_2 on $\mathscr{P}_2$ if and only if x is II-generic.

We can now drop the prefixes I, and II. Let x be generic over $\mathfrak{M}$. If U is a dense open set rational over $\mathfrak{M}$, then the complement of U, U', is first category. So $x \notin U'$. So $x \in U$.

Conversely, suppose that x lies in each dense open subset of 2^ω rational over $\mathfrak{M}$. We prove that x is generic. Let N be a first category set rational over $\mathfrak{M}$, and $N^{\mathfrak{M}}$ the corresponding Borel set of $\mathfrak{M}$. By Lemma 1.6.6, $N^{\mathfrak{M}}$ is first category in $\mathfrak{M}$. Hence, in $\mathfrak{M}$, there is a countable sequence $\{F_i^{\mathfrak{M}}\}$ of closed nowhere dense sets, with

$$N^{\mathfrak{M}} \subseteq \bigcup_{i \in \omega} F_i^{\mathfrak{M}} .$$

Let F_i be the Borel set of the real world corresponding to $F_i^{\mathfrak{M}}$ (by § 1). Then results in § 1 imply that F_i is closed nowhere dense and

$$N \subseteq \bigcup_i F_i .$$

Our assumption on x implies $x \notin F_i$, for any i. Hence $x \notin N$. So x is generic.

<div align="center">III. Proof of Theorems 1–3</div>

<div align="center">1. Proof of Theorem 2</div>

1.1. Let $\mathfrak{M}$ be a countable transitive model of **ZFC** + "There exists an inaccessible cardinal". Let Ω be inaccessible in $\mathfrak{M}$. Let $\mathscr{P}^\Omega$ be as in I § 3.2. Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}^\Omega$. We put $\mathfrak{N} = \mathfrak{M}[G]$.

1.2. Let t be a real of $\mathfrak{N}$.

LEMMA. *Almost all reals of $\mathfrak{N}$ are random over $\mathfrak{M}[t]$. Precisely, there*

is a $B \in \mathfrak{N}$ such that $\mathfrak{N} \models B$ is a Borel set of reals of measure zero and $x \in \mathfrak{N} \cap \mathbf{R}$ is random over $\mathfrak{M}$ if and only if $x \notin B$.

PROOF. By Corollary I.3.4.2 $(2^{\aleph_0})_{\mathfrak{M}[t]}$ is countable in $\mathfrak{N}$. The lemma follows from Lemma II.2.1 applied inside $\mathfrak{N}$.

1.3. A set $x \in \mathfrak{N}$ is $\mathfrak{M}$-$\mathbf{R}$-definable if and only if there is a real $t \in \mathfrak{N}$ and an element $y \in \mathfrak{M}$ and a formula $\Phi(v_1, v_2, v_3)$ of $\mathfrak{L}$ such that x is the unique $z \in \mathfrak{N}$ for which

$$\mathfrak{N} \models \Phi(t, y, z) \ .$$

An argument of Scott and Myhill [11] shows that there is a predicate $\Psi(v_1)$ of $\mathfrak{L}$ such that

$$\mathfrak{N} \models \Psi(y)$$

if and only if y is $\mathfrak{M}$-$\mathbf{R}$-definable.

(There is a similar notion of $\mathfrak{M}$-definable, which is obtained by omitting all mention of the real t in the definition of "$\mathfrak{M}$-$\mathbf{R}$-definable".)

1.4. LEMMA. *Let U be a set of reals in $\mathfrak{N}$ which is $\mathfrak{M}$-$\mathbf{R}$-definable. Then $\mathfrak{N} \models U$ is Lebesgue measurable.*

PROOF. We fix a set-theoretical formula $\Phi_1(v_1, v_2, v_3)$, an element $x \in \mathfrak{M}$, and a real $t \in \mathfrak{N}$ such that for $y \in \mathfrak{N}$ we have $\mathfrak{N} \models \Phi_1(x, t, y)$ if and only if $y = U$.

Using Φ_1, we construct a set-theoretical formula $\Phi_2(v_1, v_2, v_3)$ such that for $y \in \mathfrak{N}$ we have $\mathfrak{N} \models \Phi_2(x, t, y)$ if and only if $y \in U$.

Let $\mathfrak{M}_1 = \mathfrak{M}[t]$. Then by Corollary I.3.4., Ω is inaccessible in $\mathfrak{M}_1$. By Theorem I.4.1, there is an $\mathfrak{M}_1$-generic filter G_1 on $\mathcal{P}^\Omega$ such that $\mathfrak{N} = \mathfrak{M}_1[G_1]$. We put $x_1 = \langle x, t \rangle$. Then $x_1 \in \mathfrak{M}_1$ and there is a set-theoretical formula $\Phi_3(v_1, v_2)$ such that for all $y \in \mathfrak{N}$,

(1) $$\mathfrak{N} \models \Phi_3(x_1, y) \qquad \text{if and only if } y \in U.$$

Thus all our assumptions about the pair $\langle \mathfrak{N}; \mathfrak{M} \rangle$ hold for $\langle \mathfrak{N}; \mathfrak{M}_1 \rangle$ as well. All the results of I § 3–4 can be applied to $\langle \mathfrak{N}; \mathfrak{M}_1 \rangle$.

1.5. Let $t \in \mathfrak{N}$ be random over $\mathfrak{M}_1$. By Theorem I.4.1, there is an $\mathfrak{M}_1[t]$-generic filter, G_t on $\mathcal{P}^\Omega$ such that

$$\mathfrak{N} = \mathfrak{M}_1[t][G_t] \ .$$

Also, by Corollary I.3.4., Ω is inaccessible in $\mathfrak{M}_1[t]$.

We now apply Lemma I.3.5., considering $\mathfrak{N}$ as a Cohen extension of $\mathfrak{M}_1[t]$. We thus have

(2) $$\mathfrak{N} \models \Phi_3(x_1, t) \qquad \text{if and only if } 0 \Vdash \Phi_3(\boldsymbol{x}_1, \boldsymbol{t}) \ .$$

Since "forcing is expressible in the ground model", there is a set-theoretical formula $\Phi_4(v_1, v_2)$ and an element x_2 (which we can take to be $\langle x_1, \Omega \rangle$) such that

(3) $0 \Vdash \Phi_3(\boldsymbol{x}_1, \boldsymbol{t})$ if and only if $\mathfrak{M}[t] \models \Phi_4(x_2, t)$.

We now invoke Theorem II.2.8. There is a Borel set B, rational over $\mathfrak{M}_1$, such that for any y random over $\mathfrak{M}_1$, $\mathfrak{M}_1[y] \models \Phi_4(x_2, y)$ if and only if $y \in B$.

Let B_1 be the Borel set of $\mathfrak{N}$ corresponding to B. Then $B_1 = B \cap \mathfrak{N}$. Hence if $t \in \mathfrak{N}$ is random over $\mathfrak{M}_1$, we have

(4) $\mathfrak{M}_1[t] \models \Phi_4(x_2, t)$ if and only if $t \in B_1$.

If we string (1)–(4) together, we get, for all reals t random over $\mathfrak{M}_1$,

(5) $t \in U$ if and only if $t \in B_1$.

Let $U \triangle B_1$ be the symmetric difference of U and B_1. Then (5) says

(6) $U \triangle B_1 \subseteq \{t \in \mathfrak{N} \mid t \text{ is not random over } \mathfrak{M}_1\}$.

By Lemma 1.2, the right hand side of (6) is a Borel set of measure zero of $\mathfrak{N}$. So U differs from the Borel set B_1 by a subset of a Borel set of measure zero, i.e., U is Lebesgue measurable in $\mathfrak{N}$.

1.6. Let U be a set of reals of $\mathfrak{N}$ which is $\mathfrak{M}$-$\mathbf{R}$-definable. By an argument similar to the proof of Lemma 1.4 (obtained by replacing "random" by "generic" and "measure zero" by "first category" we can show that every $\mathfrak{M}$-definable set of reals is equal to a Borel set modulo a set of the first category. We can in fact do slightly better. It is known (cf. [4, p. 58]) that every Borel set is equal to an open set modulo a set of the first category. Thus every $\mathfrak{M}$-$\mathbf{R}$-definable set is equal to an open set off a set of the first category.

1.7. Now let U be a set of reals, in $\mathfrak{N}$, which is $\mathfrak{M}$-$\mathbf{R}$-definable and which is uncountable in $\mathfrak{N}$. We are going to show that U contains a perfect subset. Before giving details, we outline the proof.

(1) By extending $\mathfrak{M}$ if necessary, we may assume that U is $\mathfrak{M}$-definable.

(2) We pick $s_1 \in U - \mathfrak{M}$. (We can do this since U is uncountable and $\mathfrak{M} \cap \mathbf{R}$ is countable.)

(3) s_1 lies in $\mathfrak{M}[G_\xi]$, for some $\xi < \Omega$. Exploiting the connection between forcing and truth, we can find $f \in \mathscr{P}_\xi$ such that for any $F: \omega \to \xi$ extending f which is $\mathfrak{M}$-generic, $\mathfrak{M}[F] \cap (U - \mathfrak{M}) \neq \varnothing$. In fact, we will construct an explicit $s(F) \in U - \mathfrak{M}$, with $s(F) \in \mathfrak{M}[F]$.

(4) We show that $s(F_1) \neq s(F_2)$ if F_1 is $\mathfrak{M}[F_2]$-generic.

(5) We construct a perfect set K of generic collapsing maps of ξ, and show that

$$F \longmapsto s(F)$$

maps K homeomorphically into U.

We turn to the details. Defining $\mathfrak{M}_1$ as in § 1.4, and replacing $\mathfrak{M}$ by $\mathfrak{M}_1$ if necessary, we may assume that U is $\mathfrak{M}$-definable.

By Corollary I.3.4.2, the reals of $\mathfrak{M}$ are countable in $\mathfrak{N}$. Since U is uncountable in $\mathfrak{N}$, we can select a real s_1 of U, not lying in $\mathfrak{M}$.

Let G_1 be an $\mathfrak{M}$-generic filter on $\mathscr{P}^\Omega$ such that $\mathfrak{N} = \mathfrak{M}[G_1]$. By Lemma I.3.4, there is a $\xi < \Omega$ such that $s_1 \in \mathfrak{M}[G_1^{\xi+1}]$. Here $G_1^{\xi+1} = G_1 \cap \mathscr{P}^{\xi+1}$ is an $\mathfrak{M}$-generic filter on $\mathscr{P}^{\xi+1}$. We may assume that $\omega \leqq \xi$.

By Lemma I.4.3, there is an $\mathfrak{M}$-generic collapsing map $F_1 : \omega \to \xi$ such that

$$\mathfrak{M}[F_1] = \mathfrak{M}[G_1^{\xi+1}] \ .$$

By I.1.7, there is a set-theoretical formula $\psi(v_1, v_2, v_3)$ and an element x_1 of $\mathfrak{M}$ such that

$$s_1 = \{q \in \mathbf{Q} : \mathfrak{M}[F_1] \models \psi(x_1, F_1, q)\} \ .$$

LEMMA. *There is an $f_1 \in \mathscr{P}_\xi$ with the following property. Let $F : \omega \to \xi$ be an $\mathfrak{M}$-generic collapsing map. Suppose $F \in \mathfrak{N}$, $f_1 \subseteq F$, and put*

$$s = s(F) = \{q \in \mathbf{Q} : \mathfrak{M}[F] \models \psi(x_1, F, q)\} \ .$$

Then s is a real, $s \in U$, and $s \notin \mathfrak{M}$.

PROOF. We can construct a formula $\psi_1(v_1, v_2)$ of $\mathscr{L}$ and an element x_2 of $\mathfrak{M}$ such that

$$\mathfrak{N} \models \psi_1(x_2, F)$$

if and only if s has the stated properties. Moreover, ψ_1 and x_2 do not depend on F.

By Theorem I.4.1 and Lemma I.3.5, there is a formula $\psi_2(v_1, v_2)$ and an element x_3 of $\mathfrak{M}$ (not depending on F) such that $\mathfrak{N} \models \psi_1(x_2, F)$ if and only if $\mathfrak{M}[F] \models \psi_2(x_3, F)$ (cf. § 1.5).

Suppose now that $F = F_1$. Then $s = s_1$, so s has the stated properties and

$$\mathfrak{M}[F_1] \models \psi_2(x_3, F_1) \ .$$

Hence, by the connection between forcing and truth, there is an $f_1 \subseteq F_1$, $f_1 \in \mathscr{P}_\xi$, such that

$$f_1 \Vdash \psi_2(x_3, F_1) \ .$$

The lemma is now clear.

1.8. Let F_1, F_2 be $\mathfrak{M}$-generic collapsing maps of ω onto ξ, lying in $\mathfrak{N}$. Suppose $f_1 \subseteq F_i$, $i = 1, 2$. Suppose further that the pair $\langle F_1, F_2 \rangle$ is generic over $\mathfrak{M}$ (i.e., if G_i is the $\mathfrak{M}$-generic filter on $\mathscr{P}_\xi$ associated to F_i, then $G_1 \times G_2$ is an $\mathfrak{M}$-generic filter on $\mathscr{P}_\xi \times \mathscr{P}_\xi$). Then

$$s(F_1) \neq s(F_2) \ .$$

In fact, if $s(F_1) = s(F_2)$, then $s(F_1) \in \mathfrak{M}$, by Lemma I.2.5. But this contradicts Lemma 1.7.

1.9. Lemma 1.7 and § 1.8 indicate how to manufacture many elements of U. We are going to construct a "perfect set" K of $\mathfrak{M}$-generic collapsing maps of ξ. We shall also arrange that if F_1, F_2 lie in K and $F_1 \neq F_2$, then $\langle F_1, F_2 \rangle$ is $\mathfrak{M}$-generic. Finally, we shall arrange that if $F \in K$, then $f_1 \subseteq F$. It will then be shown that the map

$$\{F \rightarrow s(F)\}$$

maps K onto a perfect subset of U.

1.10. Since Ω is strongly inaccessible in $\mathfrak{M}$ and equals $\aleph_1^{\mathfrak{M}}$, we can enumerate the dense subsets of $\mathscr{P}_\xi$ lying in $\mathfrak{M}$ in a sequence $\{X_n\}$ within $\mathfrak{M}$. Similarly, let $\{W_n\}$ be an enumeration, in $\mathfrak{M}$, of the dense subsets of $\mathscr{P}_\xi \times \mathscr{P}_\xi$ lying in $\mathfrak{M}$.

Let Σ be the set of finite sequences of zeros and ones. Thus $f \in \Sigma$ if and only if f is a function, domain $f \in \omega$, and range $(f) \subseteq \{0, 1\}$. We partially order Σ by inclusion.

LEMMA.   *There is in $\mathfrak{M}$ a function*

$$\psi \colon \Sigma \longrightarrow \mathscr{P}_\xi$$

with the following properties:

   (1) $\psi(\varnothing) = f_1$.
   (2) *If f, f' are elements of Σ and $f \subseteq f'$, then $\psi(f) \subseteq \psi(f')$.*
   (3) *If f, f' in Σ are incompatible, then $\psi(f)$ and $\psi(f')$ are incompatible.*
   (4) *If domain $(f) = n$, and $f \in \Sigma$, then domain$(\psi(f)) \supseteq n$, and $\psi(f) \in X_m$, for $m < n$.*
   (5) *If f, $f' \in \Sigma$, and domain $(f) = $ domain$(f') = n$, and $f \neq f'$, then $\langle \psi(f), \psi(f') \rangle \in W_m$, for $m < n$.*

PROOF.   We define $\psi(f)$ by induction on the domain of f. So put $\psi(\varnothing) = f_1$. Suppose $\psi(h)$ is defined for domain$(f) \leqq n$. We provisionally pick $\psi(f)$ for domain$(f) = n + 1$ so that

   (a) $\psi(f)$ extends $\psi(f \mid n)$. ($f \mid n$ is the restriction of f to n.)
Replacing $\psi(f)$ (for f of length $n + 1$) by an extension, and relabeling, we may assume

   (b) $n \in \mathrm{domain}(\psi(f))$,
   (c) $\psi(f) \in X_n$ (since X_n is dense).
Continuing to extend $\psi(f)$ (for f of length $n+1$) and relabeling, we may assume

   (d) if f, f' are sequences of length $n + 1$, and $f \neq f'$, then $\psi(f) \neq \psi(f')$

and

$$\langle \psi(f), \psi(f')\rangle \in W_m \qquad\qquad (m \leqq n) \ .$$

(We use here that W_m is dense.)

We can now "freeze" our definition of $\psi(f)$ for length $(f) = n + 1$, and turn to f of length $n + 2 \cdots$.

It is clear that ψ has the properties stated in the lemma, and that ψ can be constructed inside $\mathfrak{M}$.

1.11. We now define, inside $\mathfrak{M}$, a map

$$\psi_* : 2^\omega \to U \ .$$

Let $h: \omega \to 2$. Then from Lemma 1.10,

$$\bigcup_{n \in \omega} \psi(h \mid n)$$

is a function mapping ω into ξ. We denote it by $\chi(h)$. It follows from clause (4) of Lemma 1.10 that $\chi(h)$ is an $\mathfrak{M}$-generic collapsing map of ω onto ξ. Moreover, it is clear that $f_1 \subseteq \psi(h)$ from (1) of Lemma 1.10. We put $\psi_*(h) = s(\chi(h))$. (It is important to realise that ψ_* can be defined inside $\mathfrak{M}$, but this is clear) (cf. Lemma 1.7 for $s(\cdot)$). By Lemma 1.7, $\psi_*(h)$ is an element of U.

We show next that ψ_* is one-one. Indeed, if $h_1, h_2 \in 2^\omega \cap \mathfrak{M}$ and $h_1 \neq h_2$, then $\langle \chi(h_1), \chi(h_2)\rangle$ is an $\mathfrak{M}$-generic pair of collapsing functions. Hence by § 1.8, $\psi_*(h_1) \neq \psi_*(h_2)$.

We show next that ψ_* is continuous. Let $N \in \omega$, $N \geqq 1$. Consider the set, X, of $p \in \mathscr{P}_\xi$ such that

(a) if p is compatible with f_1, then $p \supseteq f_1$.

(b) if $p \supseteq f_1$, then for some $q \in Q$, p forces $|s(F) - q| < 1/(2N)$.

Using Lemma 1.7, we see that X is dense, Hence, since $X \in \mathfrak{M}$, $X = X_m$ for some m. It follows that if h_1, h_2 are functions in $2^\omega \cap \mathfrak{M}$, and $h_1 \mid m + 1 = h_2 \mid m + 1$, then

$$|\psi_*(h_1) - \psi_*(h_2)| < 1/N \ .$$

(In fact, let $g = h_i \mid m + 1$. Then, by Lemma 1.10 (4), $\psi(g) \in X$. It follows that for some $q \in Q$,

$$\psi(g) \Vdash |s(F) - q| < 1/(2N) \ .$$

Hence $|\psi^*(h_i) - q| < 1/(2N)$, $i = 1, 2$.) It is now clear that ψ_* is continuous.

So ψ_* is, in $\mathfrak{M}$, a continuous one-one map of 2^ω into U. Since 2^ω is compact, ψ_* is a homeomorphism. Hence U contains the perfect set

$$\psi_*[2^\omega] \ .$$

(Our proof that every $\mathfrak{M}$-**R**-definable subset of $\mathfrak{N}$ is countable or contains

a perfect subset is, essentially, a slight refinement of the following result of Levy [9]: Every uncountable $\mathfrak{M}$-$\mathbf{R}$-definable subset of $\mathfrak{N}$ has power $2^{\aleph_0}$.)

1.12. We now wish to consider the following situation. Let $A \subseteq \mathbf{R}^2$, in $\mathfrak{N}$. Suppose that

$$\forall x \exists y \langle x, y \rangle \in A$$

holds in $\mathfrak{N}$; here x, y range over $\mathbf{R}$. Suppose finally that A is $\mathfrak{M}$-$\mathbf{R}$-definable.

We shall show that there is a Borel function, $h: \mathbf{R} \to \mathbf{R}$ in $\mathfrak{N}$ such that

$$\langle x, h(x) \rangle \in A$$

for almost all x. Thus h is a choice function, which selects a y in

$$A_x = \{ y \mid \langle x, y \rangle \in A \} \,,$$

for almost all x.

Since the axiom of choice holds in $\mathfrak{N}$, there is a choice function h in $\mathfrak{N}$ defined for all x. Later, we will give an example of an A for which there is no $\mathfrak{M}$-$\mathbf{R}$-definable h such that for *all* x, $h(x) \in A_x$.

We now give an outline of the proof.

(1) We construct a provisional h which is $\mathfrak{M}$-$\mathbf{R}$-definable and is defined almost everywhere. Using the fact that $\mathfrak{M}$-$\mathbf{R}$-definable subsets of $\mathbf{R}$ are Lebesgue measurable, it will then be easy to alter h on a set of measure zero to make h Borel.

(2) We may reduce ourselves to the case that A is $\mathfrak{M}$-definable.

(3) Since almost all reals in $\mathfrak{N}$ are random over $\mathfrak{M}$, we need only define $h(x)$ for x random over $\mathfrak{M}$.

(4) Using an argument similar to that of Lemma 1.7, we show that there is an $\mathfrak{M}$-definable function $\varphi(x, y)$ and an ordinal $\xi < \Omega$ such that whenever x is random over $\mathfrak{M}$, and $F: \omega \to \xi$ is an $\mathfrak{M}[x]$-generic collapsing map, then $\varphi(x, F) \in A_x$.

(5) To complete the proof, we show that there is an $\mathfrak{M}$-$\mathbf{R}$-definable function $\psi(x)$ such that whenever x is random over $\mathfrak{M}$, $\psi(x)$ is an $\mathfrak{M}[x]$-generic collapsing map mapping ω onto ξ. (We then put $h(x) = \varphi(x, \psi(x))$.)

We turn to the details.

LEMMA 1. *Let $h \in \mathfrak{N}$ map $\mathbf{R}$ into $\mathbf{R}$; suppose that h is $\mathfrak{M}$-$\mathbf{R}$-definable. Then there is a Borel function h_1 such that*

$$\{ x \mid h(x) = h_1(x) \}$$

has measure zero.

PROOF. For $r \in \mathbf{Q}$, let

$$U_r = \{ x \mid h(x) < r \} \,.$$

Since U_r is $\mathfrak{M}$-$\mathbf{R}$-definable, there is a Borel set B_r and a Borel set of measure zero N_r such that

$$U_r \triangle B_r \subseteq N_r .$$

Let $N = \bigcup_r N_r$. Then N is a Borel set of measure zero. Put $h_1(x) = h(x)$ for $x \notin N$; $h_1(x) = 0$ if $x \in N$.

If $r \leqq 0$, $r \in \mathbf{Q}$,

$$\{x \mid h_1(x) < r\} = U_r - N .$$

If $r > 0$, $r \in \mathbf{Q}$,

$$\{x \mid h_1(x) < r\} = U_r \cup N .$$

Thus h_1 is Borel. The lemma is now clear.

As usual, we may assume that A is $\mathfrak{M}$-definable (by extending $\mathfrak{M}$ if necessary).

We use a_1, a_2, etc. to denote parameters from $\mathfrak{M}$. Since A is $\mathfrak{M}$-definable, there is a set-theoretical formula, $\psi_1(a_1, x, y)$ such that

$$\mathfrak{M} \models \psi_1(a_1, x, y) \longleftrightarrow \langle x, y \rangle \in A .$$

Now let x_1 be random over $\mathfrak{M}$. Then we can find an $\mathfrak{M}[x_1]$-generic filter G_1 on $\mathscr{P}^\Omega$ such that $\mathfrak{M} = \mathfrak{M}[x_1][G_1]$. Select a $y_1 \in A_{x_1}$. By the results of I.3.4, $y_1 \in \mathfrak{M}[x_1][G_1^{\xi_1}]$ for some $\xi_1 < \Omega$. Apparently ξ_1 depends on our choice of x_1 and G_1. However, we have the following lemma.

LEMMA 2. *There is a $\xi < \Omega$ such that for all reals x random over $\mathfrak{M}$ and all filters G on $\mathscr{P}^\Omega$ generic over $\mathfrak{M}[x]$, the set*

$$A_x \cap \mathfrak{M}[x][G^\xi]$$

is non-empty.

PROOF. We extend $\mathscr{L}$ to a language $\mathscr{L}''$ as follows: for each $a \in \mathfrak{M}$, we adjoin a constant a; there are two additional constants x and G.

Let x be a real random over $\mathfrak{M}$ and G an $\mathfrak{M}[x]$-generic filter on $\mathscr{P}^\Omega$. We interpret $\mathscr{L}''$ in $\mathfrak{M}[x, G]$ in the obvious way. (Thus, variables range over $\mathfrak{M}[x, G]$; x denotes x; G denotes G, etc.)

Let $\Psi_2(\xi)$ be a formula of $\mathscr{L}''$ which expresses the following: ξ is an ordinal less than Ω, and

$$A_x \cap \mathfrak{M}[x][G^\xi] \neq \varnothing .$$

We now fix a real x random over $\mathfrak{M}$ and an $\mathfrak{M}[x]$-generic filter G on $\mathscr{P}^\Omega$. By Lemma I.3.4, there is a $\xi < \Omega$ such that

$$\mathfrak{M}[x][G] \models \Psi_2(\xi) .$$

Let $p \in G$ force $\Psi_2(\xi)$. (We are viewing $\mathfrak{M}[x][G]$ as a Cohen extension of $\mathfrak{M}[x]$.)
We say that in fact

$$0 \Vdash \Psi_2(\xi) .$$

Otherwise, there is a $p' \in \mathscr{P}^\Omega$ such that $p' \Vdash \neg \Psi_2(\xi)$. We select a bijection π of ω such that $\pi_*(p')$ is compatible with p. (Notation as in I §3.5.) A glance at the definition of π_* shows that

$$\mathfrak{M}[\pi_*(G)^\xi] = \mathfrak{M}[G^\xi] .$$

Hence an argument similar to the proof of Lemma I.3.5 will show that

$$\pi_*(p') \Vdash \neg \Psi_2(\xi) .$$

This is absurd since $p \Vdash \Psi_2(\xi)$ and p and $\pi_*(p')$ are compatible.

We can find a formula $\Psi_3(x, \xi)$ such that $\mathfrak{M}[x] \models \Psi_3(x, \xi)$ if and only if $\mathfrak{M}[x][G] \models$ "$0 \Vdash \Psi_2(\xi)$".

We know (viewing $\mathfrak{M}[x]$ as a Cohen extension of $\mathfrak{M}$) that the following are forced:

(1) $\exists \xi < \Omega \ \Psi_3(x, \xi)$,

(2) $\xi < \xi' < \Omega$ and $\Psi_3(x, \xi) \rightarrow \Psi_3(x, \xi')$.

By Zorn we pick inside $\mathfrak{M}$ a maximal family $\{\langle b_i, \xi_i \rangle : i \in I\}$ such that

(3) $\{b_i : i \in I\}$ is a pairwise disjoint family of non-zero elements of $\mathscr{B}_1$; $\xi_i < \Omega$.

(4) $b_i \Vdash \Psi_3(x, \xi_i)$.

Using (1) and (3), we see that $\sup \{b_i : i \in I\}$ is the unit of $\mathscr{B}_1$. Using the fact that $\mathscr{B}_1$ satisfies C.C.C., we see that I is countable. Hence if $\xi = \sup \{\xi_i : i \in I\}$, $\xi < \Omega$. By (2),

$$b_i \Vdash \Psi_3(x, \xi) , \qquad\qquad i \in I .$$

It follows that $\Vdash \Psi_3(x, \xi)$. (For example, from Theorem 2.8 of II.) Using the relation between Ψ_3 and Ψ_2 we see that ξ satisfies the requirements of the lemma.

We let ξ_0 have the property ascribed to ξ in Lemma 2. We may assume $\xi_0 = \xi'_0 + 1$.

LEMMA 3. *Let x be a real of $\mathfrak{N}$ random over $\mathfrak{M}$ and let G^{ξ_0} be an $\mathfrak{M}[x]$-generic filter $\mathscr{P}^{\xi_0}$. Then*

$$\mathfrak{M}[x][G^{\xi_0}] \cap A_x \neq \varnothing .$$

PROOF. Suppose not. Fix x, $G_1^{\xi_0}$ witnessing the fact that the lemma is false. By Lemma I.4.6, we can find an $\mathfrak{M}[x]$-generic filter G on $\mathscr{P}^\Omega$ with $G \cap \mathscr{P}^{\xi_0} = G_1^{\xi_0}$, and such that

$$\mathfrak{N} = \mathfrak{M}[x][G] .$$

But now our assumption on x and $G_i^{\xi_0}$ contradict Lemma 2.

LEMMA 4. *Let* $\lambda = 2^{\text{card}(\xi_0)}$, *as computed in* $\mathfrak{M}$. *Then* λ *is countable in* $\mathfrak{N}$. *Let* $F: \omega \to \lambda$ *be surjective. Then for any real* x *random over* $\mathfrak{M}$,

$$\mathfrak{M}[x, F] \cap A_x \neq \varnothing \ .$$

PROOF. Since Ω is inaccessible in $\mathfrak{M}$, $\lambda < \Omega$. Hence λ is countable in $\mathfrak{N}$. It is easy to see that

$$\text{card}\,(\mathscr{P}_{\xi_0}) = \text{card}\,(\xi_0)$$

in $\mathfrak{M}$. Moreover, standard arguments show that if x is random over $\mathfrak{M}$, then $\mathfrak{M}$ and $\mathfrak{M}[x]$ have the same cardinals, and that

$$(2^{\text{card}(\xi_0)})_{\mathfrak{M}} = (2^{\text{card}(\xi_0)})_{\mathfrak{M}[x]} \ .$$

(The essential point is that $\mathfrak{B}_1$ satisfies C.C.C. For details, cf. e.g. [12].)

Thus, in $\mathfrak{M}[F, x]$, we can enumerate the dense subsets of $\mathscr{P}^{\xi_0}$ lying in $\mathfrak{M}[x]$. It follows that there is an $\mathfrak{M}[x]$ generic filter on $\mathscr{P}^{\xi}$, G, lying in $\mathfrak{M}[F, x]$. The lemma now follows from Lemma 3.

The following lemma is standard, and we omit the proof. Let F be as in Lemma 4. Note that F is definable from a real, by I. 1.12.

LEMMA 5. *There is an* $\mathfrak{M}$-**R**-*definable function* $\psi(x)$ *such that for any real* x, $\psi(x)$ *is a well-ordering of the reals of* $\mathfrak{M}[F, x]$.

We now put it all together. Define $h: \mathbf{R} \to \mathbf{R}$ as follows: $h(x)$ is the $\psi(x)$-least member of $A_x \cap \mathfrak{M}[F, x]$ if this set is non-void. Otherwise, $h(x) = 0$. By Lemma 5, $h(x)$ is $\mathfrak{M}$-**R**-definable. By Lemma 4, $h(x) \in A_x$ for all x random over $\mathfrak{M}$. By Lemma II.2.1, it follows that $h(x) \in A_x$ for almost all x. By Lemma 1, we can alter h on a set of measure zero, so that it is Borel.

1.13. In a totally analogous way, we can prove that if A is as in 1.12, there is a Borel function h such that $h(x) \in A_x$ for all but a first category set of x's.

The following lemma is known.

LEMMA. *Let* $h: \mathbf{R} \to \mathbf{R}$ *be Borel. Then there is a set* N *of the first category such that* $h \mid \mathbf{R} - N$ *is continuous.*

The proof is similar to the proof of Lemma 1.12.1. We omit the details.

The lemma implies a similar property for $\mathfrak{M}$-**R**-definable functions.

1.14. It is now easy to complete the proof of Theorem 2. It follows from the results recalled in I.14 that $\mathfrak{N}$ is a model of **ZFC**.

From work of Gödel (cf. [1, Ch. 3]) it is known that if **ZFC + I** has a transitive model, then so does **ZFC + I + GCH**. We now sketch a proof that if **GCH** holds in $\mathfrak{M}$, it also holds in $\mathfrak{N}$. (Our proof would be slightly more

natural in terms of the concepts of [12].)

Since **GCH** holds in $\mathfrak{M}$, Ω is strongly inaccessible in $\mathfrak{M}$.

We let $\mathfrak{S}$ be the collection of subsets of $\mathscr{P}^{\Omega}$ in $\mathfrak{M}$ of cardinality less than Ω. $\mathfrak{S}$ has cardinality Ω. Let θ be a cardinal of $\mathfrak{N}$. Let $\mathfrak{S}_{\mathfrak{M}}^{\theta}$ be the collection of maps of θ into $\mathfrak{S}$ *lying in* $\mathfrak{M}$. We define a map $h \colon \mathfrak{S}_{\mathfrak{M}}^{\theta} \to P(\theta)_{\mathfrak{N}}$, in $\mathfrak{N}$, as follows: if $g \in \mathfrak{S}_{\mathfrak{M}}^{\theta}$, $h(g) = \{\alpha < \theta \colon g(\alpha) \cap G \neq \varnothing\}$. Using the **GCH** in $\mathfrak{M}$, the cardinality of $\mathfrak{S}_{\mathfrak{M}}^{\theta}$ is easily computed. We leave this to the reader. This computation shows that **GCH** holds in $\mathfrak{N}$ provided h is surjective.

To see that h is surjective, let $A \in P(\theta)_{\mathfrak{N}}$. We fix a definition Φ of A. Thus $A = \{\alpha \colon \mathfrak{N} \models \Phi(\underline{\alpha}, \underline{G})\}$. For each $\alpha < \theta$, let $\mathcal{F}_{\alpha}$ be a maximal pairwise incompatible family of conditions that decide $\Phi(\underline{\alpha}, \underline{G})$ and let $\mathcal{G}_{\alpha}$ be the subset of $\mathcal{F}_{\alpha}$ consisting of conditions that force $\Phi(\underline{\alpha}, \underline{G})$. Then Lemma I.3.3 shows that $\mathcal{G}_{\alpha} \in \mathfrak{S}$. Define $g \in \mathfrak{S}_{\mathfrak{M}}^{\theta}$ by $g(\alpha) = \mathcal{G}_{\alpha}$. We leave it to the reader to verify that $h(g) = A$ (cf. the proof of Lemma I.3.4). This completes our discussion of the **GCH** in $\mathfrak{N}$.

To complete the proof of Theorem 2, we must verify that the analogues of (2) to (5) of Theorem 1 hold in $\mathfrak{N}$. In view of the results of §§ 1.1–1.13, it suffices to cite the result proved in § 2.8 below, that every set of reals definable from a countable sequence of ordinals is $\mathfrak{M}$-**R**-definable.

1.15. We now give an example of an A which has no $\mathfrak{M}$-**R**-definable cross-section. We put

$$A = \{\langle x, y \rangle \colon y \text{ is not } \mathfrak{M}\text{-definable from } x\} \, .$$

It follows from the techniques of [11] that A is $\mathfrak{M}$-definable.

LEMMA. *Let $x \in \mathbf{R}$. Then there is a y not $\mathfrak{M}$-definable from x.*

PROOF. Using the techniques of [11], one shows that

$$A_x' = \{y \mid y \text{ is } \mathfrak{M}\text{-definable from } x\}$$

has an $\mathfrak{M}$-**R**-definable well-ordering. If $A_x' = \mathbf{R}$, then we would have an $\mathfrak{M}$-**R**-definable well-ordering of $\mathbf{R}$. Using this, one could construct an $\mathfrak{M}$-**R**-definable non-Lebesgue measurable set. This contradicts our result of 1.5. Thus $A_x \neq \varnothing$. q.e.d.

Suppose now that h is an $\mathfrak{M}$-**R**-definable function mapping $\mathbf{R}$ into $\mathbf{R}$. Say h is $\mathfrak{M}$-definable from $x \in \mathbf{R}$. Then $h(x)$ is $\mathfrak{M}$-definable from x, i.e., $h(x) \notin A_x$.

2. Proof of Theorem 1

2.1. The present method of presenting Theorems 1 and 2, in which Theorem 1 is essentially a corollary of Theorem 2, is due to Ken McAloon.

Our original approach was to prove Theorem 1 directly. (Theorem 2 is then an easy corollary.) Our original approach had the disadvantage that the verification of **DC** in the model for Theorem 1 was extremely delicate. With the present approach, it is a triviality.

2.2. Let $\mathfrak{N}$ be as in § 1. We say that x is *definable from a sequence of ordinals* (in $\mathfrak{N}$), there is an $f: \omega \to OR$, $f \in \mathfrak{N}$, and a set-theoretical formula $\Phi(v_1, v_2)$ such that, for any $y \in \mathfrak{N}$, $\mathfrak{N} \vDash \Phi(f, y)$ if and only if $y = x$.

2.3. Let x be a set. It is known that there is a minimal transitive set y such that $x \in y$. (The set y consists of x, the members of x, the members of the members of x, etc.) We call y the *transitive hull* of x. We say that x *hereditarily* possesses some property P if each member of the transitive hull of x has the property P.

2.4. Let $\mathfrak{N}_1$ be the set of elements hereditarily definable from a sequence of ordinals in $\mathfrak{N}$. (Thus $\mathfrak{N}_1 \subseteq \mathfrak{N}$.)

The methods of Myhill and Scott [11] allow one to prove the following lemma.

LEMMA. $\mathfrak{N}_1$ *is a transitive model of* **ZF**. *There is a single formula,* $\Phi_0(v_1, v_2)$, *of set-theory such that for any* $x \in \mathfrak{N}_1$, *there is an* $f \in \mathfrak{N}$, $f: \omega \to OR$, *and* x *is the unique* $y \in \mathfrak{N}$ *such that*

$$\mathfrak{N} \vDash \Phi_0(f, y) .$$

Thus the formula "$x \in \mathfrak{N}_1$" is expressible in $\mathfrak{N}$, by a set-theoretical formula, viz.,

$$(\exists f)\big(f: \omega \to OR \land (y)\big(y = x \longleftrightarrow \Phi_0(f, y)\big)\big) .$$

2.5. The following lemma is clear.

LEMMA. *Every real of* $\mathfrak{N}$, *and every sequence of ordinals of* $\mathfrak{N}$ *lies in* $\mathfrak{N}_1$.

2.6. LEMMA. *Let* $h: \omega \to \mathfrak{N}_1$, $h \in \mathfrak{N}$. *Then* $h \in \mathfrak{N}_1$.

PROOF. We work in $\mathfrak{N}$. Let $x \in \mathfrak{N}_1$. Define an ordinal, $\gamma(x)$, as follows: $\gamma(x)$ is the least ordinal λ such that for some $f: \omega \to \lambda$, x is the unique y such that $\Phi_0(f, y)$.

Let $\gamma = \sup \{\gamma(h(n)): n \in \omega\}$. Well-order the set $\{f: f \text{ maps } \omega \text{ into } \gamma\}$. Let $f_n: \omega \to \gamma$ be the least f (with respect to this well-ordering) such that $h(n)$ is the unique y such that $\Phi_0(f, y)$.

Define $g: \omega \to OR$ by:

$$g(2^m 3^n) = f_m(n) ;$$

otherwise $g(r) = 0$. Clearly h is definable from $\{f_m: m \in \omega\}$ and $\{f_m\}$ is definable

from g. Thus h is definable from a sequence of ordinals. Since, by assumption, $h \subseteq \mathfrak{N}_1$, it follows that $h \in \mathfrak{N}_1$.

2.7. We now state the principle of dependent choices, **DC**.

Let X be a set, R a binary relation on X. Suppose further that $X \neq \varnothing$. Finally, we assume that

$$(\forall x \in X)(\exists y \in X)(xRy) \ .$$

Then there is a map $h: \omega \to X$ such that, for all $n \in \omega$, $h(n)Rh(n + 1)$.

Note that **DC** follows easily from **AC** (the axiom of choice); one simply defines $h(n)$ by induction on n.

LEMMA. **DC** *holds in* $\mathfrak{N}_1$.

PROOF. Let $X, R \in \mathfrak{N}_1$ satisfy the hypotheses of **DC**. Since **AC** holds in $\mathfrak{N}$, there is an $h: \omega \to X$, $h \in \mathfrak{N}$, such that for all $n \in \omega$, $\langle h(n), h(n + 1) \rangle \in R$. By Lemma 2.5, $h \in \mathfrak{N}_1$. Thus **DC** holds in $\mathfrak{N}_1$.

2.8. LEMMA. *Let* $A \in \mathfrak{N}_1$. *Then, in* $\mathfrak{N}$, A *is* $\mathfrak{N}$-**R**-*definable*.

PROOF. We may as well assume that A is a map f of ω into OR.

By Lemma I.3.4, $f \in \mathfrak{N}[G^{\xi}]$ where $\omega \leqq \xi < \Omega$, and $\xi = \xi' + 1$. By Lemmas I.4.3 and I.1.12, there is a real s such that $f \in \mathfrak{N}[s]$. So the lemma is clear.

2.9. LEMMA. *In* $\mathfrak{N}_1$, *every set of reals is Lebesgue measurable*.

PROOF. Let A be a set of reals in $\mathfrak{N}_1$. By Lemma 2.8, A is $\mathfrak{N}$-**R**-definable in $\mathfrak{N}$. Thus by Lemma 1.4, A is Lebesgue measurable in $\mathfrak{N}$. Thus there is a Borel set B and a Borel set N of measure zero, in $\mathfrak{N}$, such that

(1) $B \bigtriangleup A \subseteq N \ .$

Let α_1, α_2 be codes for B and N in $\mathfrak{N}$. Trivially, α_1 and α_2 lie in $\mathfrak{N}_1$. (Lemma 2.5.) By Lemma 2.5, $\mathfrak{N}$ and $\mathfrak{N}_1$ have the same reals. Thus, by Theorem II 1.4, α_1 and α_2 code B and N also in $\mathfrak{N}$. Clearly (1) holds in $\mathfrak{N}$. By Lemma II.1.6.4, N has measure zero in $\mathfrak{N}_1$. Thus A is Lebesgue measurable in N_1.

2.10. The proof of the following lemma is totally analogous to that of Lemma 2.9.

LEMMA. *In* $\mathfrak{N}_1$, *every set of reals has the property of Baire*.

2.11. LEMMA. *In* $\mathfrak{N}_1$, *every uncountable set of reals contains a perfect subset*.

PROOF. Let A be a set of reals in $\mathfrak{N}_1$. By Lemma 2.8, A is $\mathfrak{N}$-**R**-definable in $\mathfrak{N}$.

Suppose first that A is countable in $\mathfrak{N}$. Then Lemma 2.6 shows that A is countable in $\mathfrak{N}_1$. On the other hand, suppose A is uncountable in $\mathfrak{N}$. Then, since A is $\mathfrak{M}$-**R**-definable in $\mathfrak{N}$, there is, in $\mathfrak{N}$, a perfect set K with $K \subseteq A$.

Let β be a code for K. Then $\beta \in \mathfrak{N}_1$ and β codes K in $\mathfrak{N}_1$ (cf. the proof of Lemma 2.9). By Lemma II.1.6.7, K is perfect in $\mathfrak{N}_1$, and the lemma is clear.

2.12. The following lemma is the key to verifying (5) in $\mathfrak{N}_1$.

LEMMA. *Let f be, in $\mathfrak{N}$, a Borel function mapping* **R** *into* **R**. *Then* $f \in \mathfrak{N}_1$.

PROOF. Using Lemma 2.5 and Theorem II.1.2 we see that every Borel set of reals of $\mathfrak{N}$ lies in $\mathfrak{N}_1$. Hence, by Lemma 2.6, the indexed family

$$\{f^{-1}((-\infty, q)): q \in \mathbf{Q}\}$$

lies in $\mathfrak{N}_1$. It follows easily that $f \in \mathfrak{N}_1$.

It is now easy to verify that (5) holds in $\mathfrak{N}_1$. Let $\{A_x: x \in \mathbf{R}\}$ be as in the statement of (5). Applying (5a) in $\mathfrak{N}$, we get a Borel function h and a Borel set of measure zero N such that

$$x \notin N \longrightarrow h(x) \in A_x .$$

Since h, N lie in $\mathfrak{N}_1$, this instance of (5a) holds in $\mathfrak{N}_1$. The verification of (5b) is similar.

2.13. The material in §§ 2.7–2.12 establishes Theorem 1.

3. Proof of Theorem 3

3.1. McAloon's idea of directly proving Theorem 2 allows one to prove Theorem 3 as well. (This fact was first noticed by McAloon.) We are going now to sketch the proof of Theorem 3. Using our sketch and the detailed proof of Theorem 2 given above, the reader should be able to fill in the details without trouble.

3.2. $\mathfrak{M}$ is a countable transitive model of **ZFC** + **GCH** + "There is an inaccessible cardinal". Ω is an inaccessible cardinal in $\mathfrak{M}$. Θ is a cardinal of $\mathfrak{M}$ with cofinality $\geqq \Omega$.

Let $\mathscr{P}'_\Theta$ be the partially ordered set appropriate to adding Θ generic sets of integers. Thus $\mathscr{P}'_\Theta$ is the set of all functions f such that

(1) domain(f) is a finite subset of $\Theta \times \omega$;
(2) range(f) $\subseteq \{0, 1\}$.

Let $\mathscr{P} = \mathscr{P}^\Omega \times \mathscr{P}'_\Theta$. Let G be an $\mathfrak{M}$-generic filter on $\mathscr{P}$. Let $\mathfrak{N}_2$ be $\mathfrak{M}[G]$.

3.3. LEMMA. *Let $X \in \mathfrak{M}$ be a pairwise incompatible family of elements of $\mathcal{P}$. Then, in $\mathfrak{M}$,*

$$\operatorname{card}(X) < \Omega .$$

(The proof is similar to that of Lemma I.3.3.)

This lemma has the following consequences.

(1) $\Omega = \aleph_1^{\mathfrak{N}_2}$ (cf. Corollary I.3.3).

(2) If $\lambda \geqq \Omega$, λ is a cardinal in $\mathfrak{M}$ if and only if λ is a cardinal in $\mathfrak{N}_2$.

By standard methods, one can compute $2^{\aleph_0}$ in $\mathfrak{N}_2$. One gets

(3) $2^{\aleph_0} = \Theta$ in $\mathfrak{N}_2$.

(*Example.* Suppose Θ is the least cardinal of $\mathfrak{M} > \Omega$. Then in $\mathfrak{N}_2$, $2^{\aleph_0} = \aleph_2$.)

3.4. Let $A \in \mathfrak{M}$, $A \subseteq \Theta$. Let $\mathcal{P}'_A = \{f \in \mathcal{P}'_\Theta : \operatorname{domain}(f) \subseteq A \times \omega\}$.

The following lemma is the analog of Lemma I.3.4 and has a similar proof.

LEMMA. *Let $f : \omega \to OR$, $f \in \mathfrak{N}_2$. Then there is a $\xi < \Omega$, and a subset A of Θ such that*:

(1) $A \in \mathfrak{M}$, *and in* $\mathfrak{M}$, $\operatorname{card}(A) < \Omega$.

(2) $f \in \mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$.

3.5. The following lemma is the analog of Lemma I.4.3 and has a similar proof.

LEMMA. *Let $A \in \mathfrak{M}$, $A \subseteq \Theta$, and suppose*

$$\operatorname{card}(A) \leqq \operatorname{card}(\xi) < \Omega$$

in $\mathfrak{M}$. *Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\xi+1} \times \mathcal{P}'_A$. Then there is an $\mathfrak{M}$-generic collapsing map $F : \omega \to \xi$ with $\mathfrak{M}[G] = \mathfrak{M}[F]$.*

3.6. Using Lemmas 3.4 and 3.5, one can adapt the proof of Theorem I.4.1 to prove

LEMMA. *Let $f \in \mathfrak{N}_2$, $f : \omega \to OR$. Then there is an $\mathfrak{M}[f]$-generic filter, G_1, on $\mathcal{P}$ such that*

$$\mathfrak{M}[f][G_1] = \mathfrak{N}_2 .$$

3.7. The following is the analog of Lemma I.3.5 and has a similar proof.

LEMMA. *Let Φ be a sentence of $\mathcal{L}'$ not containing $\mathbf{G}$. Let O be the minimal element of $\mathcal{P}$. Then O decides Φ.*

3.8. Using Lemmas 3.6 and 3.7, one can imitate the discussion of §1 and prove

LEMMA. *Let $A \in \mathfrak{N}_2$ be a set of reals which is $\mathfrak{M}$-$\mathbf{R}$-definable in $\mathfrak{N}_2$. Then A is Lebesgue measurable and has the Baire property. If A is uncountable, A contains a perfect subset.*

The proof of Theorem 3 is now clear.

3.9. Using the product lemma (Lemma I.2.3), we see that the model of Theorem 3 is obtained from the model of Theorem 2 by the (extremely well-understood) process of adding generic reals. Hence the possible cardinalities of $2^{\aleph}$ in the models provided by the proof of Theorem 3 are equally well understood.

4. An extension of Theorems 1 through 3

4.1. Theorems 1 through 3 state that certain subsets of the reals are well-behaved. In this section we replace **R** by an arbitrary complete separable metric space X, and Lebesgue measure by a totally σ-finite measure space μ. We shall discuss, very sketchily, a proof of the following theorem.

THEOREM. *The following is valid in* $\mathfrak{N}_1$: *Let* X, μ *be as above, and let* $A \subseteq X$. *Then* A *is* μ *measurable,* A *has the property of Baire, and* A *is either countable or contains a perfect subset.*

PROOF. One first re-does the material of II for the space X. (There are a few technical tricks needed to re-do II in this generality, which we shall not discuss.) One then works with, e.g., the μ-random elements of X in proving that A is Lebesgue measurable (first in $\mathfrak{N}$, and then in $\mathfrak{N}_1$). Similarly, one proves A has the Baire property.

To prove that if A is uncountable, it contains a perfect subset, we invoke the following theorem of **ZF** + **DC**: Any separable metric space imbeds homeomorphically into the Hilbert cube (cf. [6, p. 125]). This reduces the problem to the special case when X is the Hilbert cube. The argument given in § 1 in the case $X = $ **R** adapts easily to this case.

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