2 First-Order Equations: Method of Characteristics

In this section, we describe a general technique for solving first-order equations. We begin with linear equations and work our way through the semilinear, quasilinear, and fully nonlinear cases. We start by looking at the case when u is a function of only two variables as that is the easiest to picture geometrically. Towards the end of the section, we show how this technique extends to functions u of n variables.

2.1 Linear Equation

We start with the simplest case: the linear equation. The ideas here will extend to the more complicated cases. Consider the following first-order, linear equation,

$$a(x,y)u_x + b(x,y)u_y = c(x,y). \tag{2.1}$$

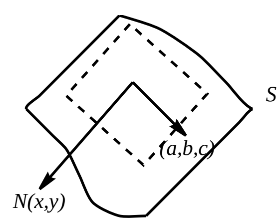
Suppose we can find a solution $u(x,y)$. Consider the graph of this function given by

$$S \equiv \{(x,y,u(x,y))\}.$$

If u is a solution of (2.1), we know that at each point (x,y),

$$(a(x,y), b(x,y), c(x,y)) \cdot (u_x(x,y), u_y(x,y), -1) = 0.$$

But, recall from calculus, the normal to the surface $S = \{(x,y,u(x,y))\}$ at the point $(x,y,u(x,y))$ is given by $N(x,y) = (u_x(x,y), u_y(x,y), -1)$. Therefore, if the vector $(a(x,y), b(x,y), c(x,y))$ is perpendicular to $(u_x(x,y), u_y(x,y), -1)$ at each point $(x,y,u(x,y))$, then the vector $(a(x,y), b(x,y), c(x,y))$ lies in the tangent plane to S.

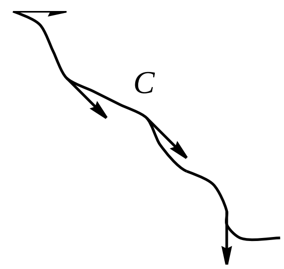


Consequently, to find a solution to (2.1), we will look for a surface S such that at each point (x,y,z) on S, the vector $(a(x,y), b(x,y), c(x,y))$ lies in the tangent plane. How do we construct such a surface?

We start by looking for a curve $\mathcal{C}$ which lies in S. We know we want the vector $(a(x,y), b(x,y), c(x,y))$ to lie in the tangent plane to our surface S at each point (x,y,z) on the surface. Therefore, let's start by constructing a curve $\mathcal{C}$ parametrized by s such that at each point on the curve $\mathcal{C}$, the vector $(a(x(s),y(s)), b(x(s),y(s)), c(x(s),y(s)))$ is tangent to the curve. In particular, the curve $\mathcal{C} = \{(x(s), y(s), z(s))\}$ will satisfy the following system of ODEs:

$$\begin{aligned} \frac{dx}{ds} &= a(x(s),y(s)) \\ \frac{dy}{ds} &= b(x(s),y(s)) \\ \frac{dz}{ds} &= c(x(s),y(s)). \end{aligned} \tag{2.2}$$

Such a curve $\mathcal{C}$ is known as an **integral curve** for the vector field $(a(x,y), b(x,y), c(x,y))$.



For a PDE of the form (2.1), we look for integral curves for the vector field $V = (a(x,y), b(x,y), c(x,y))$ associated with the PDE. These integral curves are known as the **characteristic curves** for (2.1). These characteristic curves are found by solving the system of ODEs (2.2). This set of equations is known as the set of **characteristic equations** for (2.1).

Once we have found the characteristic curves for (2.1), our plan is to construct a solution of (2.1) by forming a surface S as a union of these characteristic curves. A surface $S = \{(x,y,z)\}$ for which the vector field $V = (a(x,y), b(x,y), c(x,y))$ lies in the tangent plane to S at each point (x,y,z) on S is known as an **integral surface** for V.

In effect, by introducing these characteristic equations, we have reduced our partial differential equation to a system of ordinary differential equations. We can use ODE theory to solve the characteristic equations, then piece together these characteristic curves to form a surface. Such a surface will provide us with a solution to our PDE.

Example 1. Find a solution to the transport equation,

$$u_t + au_x = 0. \tag{2.3}$$

As described above, we look for a solution to (2.3) by introducing the characteristic equations,

$$\begin{aligned} \frac{dx}{ds} &= a \\ \frac{dt}{ds} &= 1 \\ \frac{dz}{ds} &= 0. \end{aligned} \tag{2.4}$$

Solving this system, we have

$$\begin{aligned} x(s) &= as + c_1 \\ t(s) &= s + c_2 \\ z(s) &= c_3. \end{aligned}$$

Eliminating the parameter s, we observe that these curves are lines in $\mathbb{R}^3$ given by $x - at = x_0$, $z = k$ for constants x_0 and k. Now let S be the integral surface formed from a union of these characteristic curves. In doing so, we see that $z(x,t)$ is constant along the lines $x - at = x_0$. That is, $z(x,t) = f(x - at)$. Letting $u(x,t) = z(x,t) = f(x - at)$ for any (smooth) function

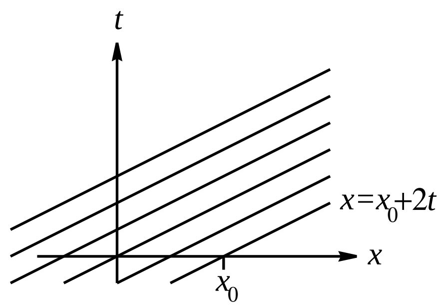
f, we have found the general solution of (2.3). An application of the chain rule shows that such a function does indeed satisfy the transport equation,

$$u_t + au_x = -af'(x - at) + af'(x - at) = 0.$$

Before discussing the initial-value problem for the transport equation, we make some remarks.

Remarks.

- The solution of (2.3) is constant along the lines $x = x_0 + at$. These lines are known as the **projected characteristic curves** for this equation. They are the projection of the characteristic curves onto the xt-plane. See the picture below for the projected characteristics for $a = 2$.



- The initial data is propagated along the projected characteristic curves. For this reason, (2.3) is known as the transport equation. Information is simply propagated along lines.

$\diamond$

In the above example, we found the general solution for the transport equation. In application, however, we are typically interested in finding a solution which not only satisfies a certain PDE, but also satisfies some auxiliary condition, i.e. - an initial or boundary condition.

Example 2. Before discussing the technique in generality, we consider the initial-value problem for the transport equation,

$$\begin{cases} u_t + au_x = 0 \\ u(x,0) = \phi(x). \end{cases} \tag{2.5}$$

As we saw in the previous example, the general solution of

$$u_t + au_x = 0$$

is given by $u(x,t) = f(x - at)$ for any smooth function f. Consequently, by letting $u(x,t) = \phi(x - at)$, we have a function which not only satisfies our PDE, but also satisfies our initial condition, and thus our initial-value problem (2.5).

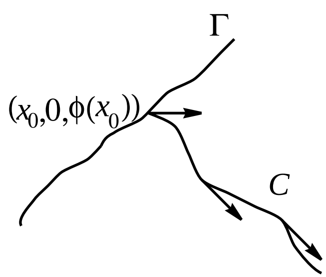
For a moment, however, suppose we did not know that. Let's think how we would construct our solution. As described earlier, the plan is to find a solution of our PDE by constructing an integral surface S as a union of characteristic curves. Now, we impose the extra condition that this surface contains the curve $\{(x, 0, \phi(x))\}$. As we will see, requiring this extra condition does not pose a problem.

Geometrically, the intuition is the following. Let $\Gamma \equiv \{(x, 0)\}$ be the curve in $\mathbb{R}^2$ on which we are prescribing our data. Let (Γ, ϕ) be the curve in $\mathbb{R}^3$ given by $\{(x, 0, \phi(x))\}$. To construct our solution, start by picking a point $(x_0, 0, \phi(x_0))$ on (Γ, ϕ). Now, let's construct a characteristic curve emanating from $(x_0, 0, \phi(x_0))$. That is, look for a solution of the following characteristic equations

$$\frac{dx}{ds} = a$$
$$\frac{dt}{ds} = 1$$
$$\frac{dz}{ds} = 0,$$

which satisfies the initial conditions

$$x(0) = x_0$$
$$t(0) = 0$$
$$z(0) = \phi(x_0).$$



If we construct a characteristic curve from each point on (Γ, ϕ) and take the union of these characteristic curves, we can find an integral surface S for the vector field $(a, 1, 0)$ which contains the curve (Γ, ϕ). This integral surface will give us a solution to (2.5).

Algebraically, we proceed as follows. Let's parametrize our curve Γ by r. In this case, $\Gamma = \{(r, 0)\}$. (We could have just parametrized it by x_0 but for consistency later, we will choose the parameter r.) Now for each r, we need to solve the following system

$$\frac{dx}{ds}(r, s) = a$$
$$\frac{dt}{ds}(r, s) = 1$$
$$\frac{dz}{ds}(r, s) = 0,$$

with initial conditions

$$x(r, 0) = r$$
$$t(r, 0) = 0$$
$$z(r, 0) = \phi(r).$$

First, solving our system, we see that

$$x(r, s) = as + c_1(r)$$
$$t(r, s) = s + c_2(r)$$
$$z(r, s) = c_3(r).$$

Now looking at our initial conditions, we see that

$$x(r, 0) = c_1(r) = r$$
$$t(r, 0) = c_2(r) = 0$$
$$z(r, 0) = c_3(r) = \phi(r).$$

Consequently, our solution is given by

$$x(r, s) = as + r$$
$$t(r, s) = s$$
$$z(r, s) = \phi(r).$$

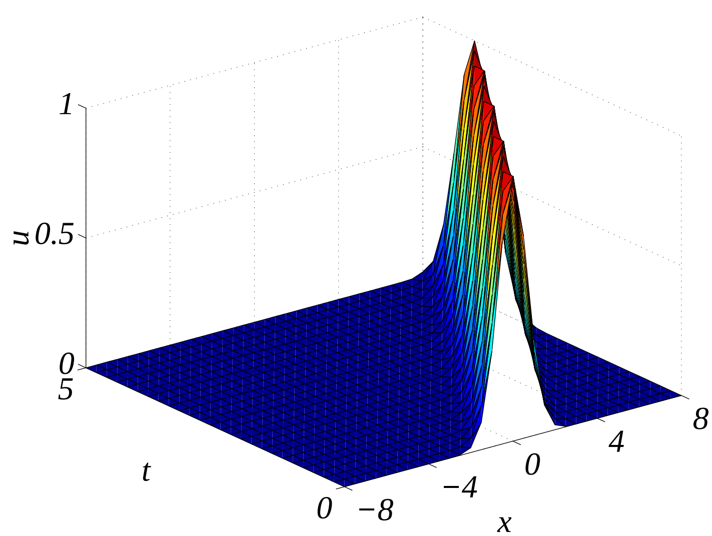
Now we solve for r, s in terms of x, t. In particular, we have

$$r(x, t) = x - at$$
$$s(x, t) = t.$$

Consequently, we can write our solution $z(r, s)$ as $z(r(x, t), s(x, t)) = \phi(x - at)$.

Now
$$u(x, t) = z(r(x, t), s(x, t)) = \phi(x - at),$$
gives us a solution of (2.5).

See the picture below for the solution of (2.5) with $a = 2$ and initial data $\phi(x) = e^{-x^2}$.



◇

Noncharacteristic Boundary Data. In solving the above problem, we specified a boundary condition on the curve $\{(x,0)\}$. Could we have specified data on any curve Γ in the xt-plane? As we will see below, the answer is no.

Consider the following problem

$$\begin{cases} u_t + a u_x = 0 \\ u|_\Gamma = \phi \end{cases} \tag{2.6}$$

where Γ is a curve in the xt-plane. This is an example of a *Cauchy problem.* More generally, for an $(n-1)$-dimensional *manifold* Γ in $\mathbb{R}^n$, finding a function u which satisfies

$$\begin{cases} F(\vec{x}, u, Du) = 0 \\ u|_\Gamma = \phi \end{cases}$$

is a first-order **Cauchy problem**.

Remark. An $(n-1)$-dimensional *manifold* in $\mathbb{R}^n$ is a surface which can be represented locally by the graph of a function $F : \mathbb{R}^{n-1} \to \mathbb{R}$. Here we list a couple of examples.

1. If $u = u(x_1, \ldots, x_{n-1})$, $\Gamma = \{(x_1, \ldots, x_{n-1}, u(x_1, \ldots, x_{n-1}))\}$ is an $(n-1)$-dimensional manifold in $\mathbb{R}^n$

2. $\Gamma = \{(x,y) : x^2 + y^2 = 1\}$, the unit circle in $\mathbb{R}^2$, is a one-dimensional manifold in $\mathbb{R}^2$.

We will now show that the Cauchy problem (2.6) is not well-posed for all curves Γ in $\mathbb{R}^2$. Consider the curve $\Gamma = \{(x,t) : x - at = 0\}$ in the xt-plane, parametrized by r such that $\Gamma = \{(r, ar)\}$. Now we prescribe a boundary condition on Γ such that we ask that $u|_\Gamma = \phi$. We claim this problem is ill-posed. Why? Recall that solutions to the transport equation are constant along the lines $\{(x,t) : x - at = C\}$, which are the *projected characteristic curves* for this PDE. However, we are trying to prescribe data on one of these projected characteristic curves. Consequently, two problems arise.

- As the solution must be constant along these projected characteristics, if we prescribe data along the line $x - at = 0$, such that $u|_\Gamma = \phi$, this means we need ϕ to be constant along $x - at = 0$.

- We don't have enough information to determine the values of u at points off the line $x - at = 0$.

The problem with the above choice of Γ is that it is itself a projected characteristic curve. In order to construct an integral surface (and thus a solution of our PDE), we construct characteristic curves emanating from each point $(x, t, \phi(x,t))$ for each $(x,t) \in \Gamma$. In order to guarantee that we can do so, we ask that the curve Γ be nowhere tangent to the vector field $(a, 1)$. If Γ satisfies this condition, we say Γ is *noncharacteristic.*

More generally, we say Γ is **noncharacteristic** for the Cauchy problem

$$\begin{aligned} &a(x,y)u_x + b(x,y)u_y = c(x,y) \\ &u|_\Gamma = \phi \end{aligned}$$

if Γ is nowhere tangent to the projected characteristics $(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r)))$. That is,

$$(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r)) \cdot (-\gamma_2'(r), \gamma_1'(r)) \neq 0. \tag{2.7}$$

As we will show later, as long as Γ is noncharacteristic, then we can find a unique solution of our Cauchy problem (at least near Γ).

Linear Equation Revisited. Let us summarize what we have discussed so far and look to make our statements more precise. Suppose Γ is a curve in $\mathbb{R}^2$ parametrized by r such that $\Gamma = \{(\gamma_1(r), \gamma_2(r))\}$. On the curve Γ, we prescribe the *boundary condition* $u|_\Gamma = \phi$. Consider the problem

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = c(x,y) \\ u|_\Gamma = \phi \end{cases} \tag{2.8}$$

We attempt to find a solution by constructing an integral surface S as a union of characteristic curves. Remember the characteristic curves are curves which are tangent to the vector field $V = (a(x,y), b(x,y), c(x,y))$. The characteristic curves can be found by solving the system of characteristic equations:

$$\begin{aligned} \frac{dx}{ds}(r,s) &= a(x,y) \\ \frac{dy}{ds}(r,s) &= b(x,y) \\ \frac{dz}{ds}(r,s) &= c(x,y) \end{aligned}$$

with initial conditions

$$\begin{aligned} x(r,0) &= \gamma_1(r) \\ y(r,0) &= \gamma_2(r) \\ z(r,0) &= \phi(r). \end{aligned}$$

By the theory of ODEs, this system has a unique solution

$$\begin{aligned} x &= x(r,s) \\ y &= y(r,s) \\ z &= z(r,s) \end{aligned}$$

which satisfies the initial conditions.

In defining the characteristic equations, we defined $z(r,s) = u(x(r,s), y(r,s))$. If we can find some function H such that $(r,s) = H(x,y)$, then we will have found the unique solution of (2.8),

$$u(x,y) = z(r,s) = z(H(x,y)).$$

In order to determine when we can do so, we make use of the Inverse Function Theorem. First, we recall the Jacobian function. Let $G : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Therefore,

$G = (G^1, \ldots, G^n)$ where $G^1, \ldots, G^n$ are the n component functions of G. We define the Jacobian of G at the point $\vec{x}_0 \in \mathbb{R}^n$ as

$$JG(\vec{x}_0) = \det \begin{bmatrix} G^1_{x_1}(\vec{x}_0) & G^1_{x_2}(\vec{x}_0) & \ldots & G^1_{x_n}(\vec{x}_0) \\ G^2_{x_1}(\vec{x}_0) & G^2_{x_2}(\vec{x}_0) & \ldots & G^2_{x_n}(\vec{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ G^n_{x_1}(\vec{x}_0) & G^n_{x_2}(\vec{x}_0) & \ldots & G^n_{x_n}(\vec{x}_0) \end{bmatrix}. \tag{2.9}$$

Theorem 3. (Inverse Function Theorem) *Assume $G : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function and $JG(\vec{x}_0) \neq 0$ where $JG(\vec{x}_0)$ is the Jacobian of G at the point $\vec{x}_0$. Then there exists an open set $V \subset U$ with $\vec{x}_0 \in V$ and an open set $W \subset \mathbb{R}^n$, with $\vec{z}_0 = G(\vec{x}_0) \in W$ such that*

$$G : V \to W$$

is one-to-one and onto, and the inverse function

$$G^{-1} : W \to V$$

is C^1.

Proof. Omitted.

Claim 4. *If Γ is noncharacteristic* (2.7)*, then we can find a local inverse of the function $G(r, s) = (x(r, s), y(r, s))$ near $s = 0$. That is, for each r_0, we can find an open set V containing the point $(r_0, 0)$ and an open set W containing the point $G(r_0, 0) = (x(r_0, 0), y(r_0, 0)) = (\gamma_1(r_0), \gamma_2(r_0))$, such that*

$$G : V \to W$$

is one-to-one and onto, and the inverse function

$$G^{-1} : W \to V$$

is C^1.

Proof. We make use of the Inverse Function Theorem. Fix r_0. By definition, $G(r, s) = (x(r, s), y(r, s))$. Therefore,

$$\begin{aligned} JG(r, s) &= \det \begin{bmatrix} x_r(r, s) & x_s(r, s) \\ y_r(r, s) & y_s(r, s) \end{bmatrix} \\ &= x_r y_s - x_s y_r. \end{aligned}$$

At the point $(r_0, 0)$,

$$JG(r_0, 0) = x_r(r_0, 0) y_s(r_0, 0) - x_s(r_0, 0) y_r(r_0, 0).$$

By definition, our functions $x(r, s)$, $y(r, s)$ satisfy the characteristic ODEs. Therefore, for each r,

$$\begin{aligned} x_s(r, s) &= a(x(r, s), y(r, s)) \\ y_s(r, s) &= b(x(r, s), y(r, s)) \end{aligned}$$

which implies

$$
\begin{aligned}
x_s(r_0, 0) &= a(x(r_0, 0), y(r_0, 0)) = a(\gamma_1(r_0), \gamma_2(r_0)) \\
y_s(r_0, 0) &= b(x(r_0, 0), y(r_0, 0)) = b(\gamma_1(r_0), \gamma_2(r_0)).
\end{aligned}
$$

Further,

$$
\begin{aligned}
x_r(r_0, 0) &= \gamma_1'(r_0) \\
y_r(r_0, 0) &= \gamma_2'(r_0).
\end{aligned}
$$

Therefore, for the point $(r_0, 0)$,

$$
JG(r_0, 0) = \gamma_1'(r_0) b(\gamma_1(r_0), \gamma_2(r_0)) - \gamma_2'(r_0) a(\gamma_1(r_0), \gamma_2(r_0)). \tag{2.10}
$$

By the Inverse Function Theorem, as long as $JG(r_0, 0) \neq 0$, we can find an inverse of G near $(r_0, 0)$. Recalling condition (2.7) on Γ we see that $JG(r_0, 0) \neq 0$ exactly when Γ is noncharacteristic. Therefore, as long as Γ is noncharacteristic, we will be able to find a local inverse for $G(r, s)$. $\square$

Let's work out another example.

Example 5. Consider

$$
\begin{cases}
u_t + x u_x = 0 \\
u(x, 0) = \phi(x).
\end{cases} \tag{2.11}
$$

We parametrize our curve Γ by r such that $\Gamma = \{(r, 0)\}$. Our characteristic equations are given by

$$
\begin{aligned}
\frac{dt}{ds}(r, s) &= 1 \\
\frac{dx}{ds}(r, s) &= x \\
\frac{dz}{ds}(r, s) &= 0
\end{aligned}
$$

with initial conditions

$$
\begin{aligned}
t(r, 0) &= 0 \\
x(r, 0) &= r \\
z(r, 0) &= \phi(r).
\end{aligned}
$$

We solve this system as follows:

$$
\begin{aligned}
t(r, s) &= s + c_1(r) \\
x(r, s) &= c_2(r) e^s \\
z(r, s) &= c_3(r).
\end{aligned}
$$

Imposing the initial condition, we see that

$$\begin{aligned}
t(r,s) &= s \\
x(r,s) &= re^s \\
z(r,s) &= \phi(r).
\end{aligned}$$

Now we want to write r, s in terms of x, t. We note that $\Gamma = \{(x,0)\}$ is noncharacteristic. Consequently, we can solve for r, s in terms of x, t. In particular, we have

$$\begin{aligned}
r(x,t) &= xe^{-t} \\
s(x,t) &= t.
\end{aligned}$$

Letting $u(x,t) = z(r(x,t), s(x,t)) = \phi(xe^{-t})$, we have found our unique solution to (2.11). $\diamond$

2.2 Semilinear Equation

Consider the Cauchy problem for a first-order semilinear equation in two variables,

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = c(x,y,u) \\ u|_\Gamma = \phi. \end{cases} \tag{2.12}$$

As before, we say that $\Gamma(r) = (\gamma_1(r), \gamma_2(r))$ is noncharacteristic if Γ is nowhere tangent to the projected characteristic curves. As before, this means that

$$(a(\gamma_1(r), \gamma_2(r)), b(\gamma_1(r), \gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) \neq 0.$$

We assume Γ is noncharacteristic.

Just as in the linear case, if u is a solution of (2.12), then at each point on the surface $S = \{(x, y, u(x,y))\}$, the vector $(a(x,y), b(x,y), c(x,y,u(x,y))$ must lie in the tangent plane to the surface. Consequently, to find our solution u, letting $z(r,s) = u(x(r,s), y(r,s))$ we reduce our PDE to a system of characteristic ODEs:

$$\begin{aligned}
\frac{dx}{ds}(r,s) &= a(x,y) \\
\frac{dy}{ds}(r,s) &= b(x,y) \\
\frac{dz}{ds}(r,s) &= c(x,y,z)
\end{aligned}$$

with initial conditions

$$\begin{aligned}
x(r,0) &= \gamma_1(r) \\
y(r,0) &= \gamma_2(r) \\
z(r,0) &= \phi(r).
\end{aligned}$$

We are able to find a unique solution to this system from ODE theory and then use the fact that the curve Γ is noncharacteristic to get a unique solution of (2.12) (at least near Γ),

$$u(x,y) = z(r(x,y), s(x,y))$$

Let's try a couple of examples.

Example 6. Consider the following semilinear equation,

$$\begin{cases} u_t + au_x = u^2 \\ u(x,0) = \cos(x). \end{cases} \tag{2.13}$$

Our characteristic ODEs are given by

$$\begin{aligned} \frac{dt}{ds}(r,s) &= 1 \\ \frac{dx}{ds}(r,s) &= a \\ \frac{dz}{ds}(r,s) &= z^2 \end{aligned}$$

with initial conditions

$$\begin{aligned} t(r,0) &= 0 \\ x(r,0) &= r \\ z(r,0) &= \cos(r). \end{aligned}$$

Note: The first two equations do not depend on z, thus, allowing us to solve this system separately. This will not be the case for quasilinear equations.

We solve our system as follows.

$$\begin{aligned} t(r,s) &= s + c_1(r) \\ x(r,s) &= as + c_2(r) \\ \frac{-1}{z(r,s)} &= s + c_3(r). \end{aligned}$$

Now using our initial condition, we see that

$$\begin{aligned} t(r,s) &= s \\ x(r,s) &= as + r \\ \frac{-1}{z(r,s)} &= s + \frac{-1}{\cos(r)}. \end{aligned}$$

From the top equation, we see that $t = s$, and, therefore, $r = x - at$. Consequently, letting $u(x,t) = z(r,s)$, we have found a solution to (2.13) given by

$$\frac{-1}{u(x,t)} = t - \frac{1}{\cos(x-at)},$$

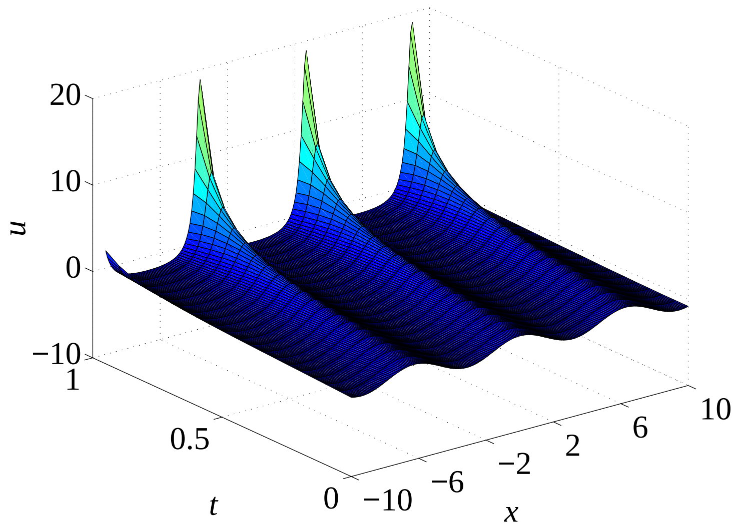
which implies

$$\frac{1}{u(x,t)} = \frac{1 - t\cos(x-at)}{\cos(x-at)}$$

so that

$$u(x,t) = \frac{\cos(x-at)}{1 - t\cos(x-at)}.$$

We have found a solution for (x,t) near Γ. *Note:* We have found a local solution. However, this solution will blow up when $1 - t\cos(x - at) = 0$. In particular, at the time t such that $\cos(x - at) = \frac{1}{t}$. This blow-up will first occur when $t = 1$ at those points x such that $\cos(x - a) = 1$. That is, $x = a + 2n\pi$. See figure below for $a = 2$.



$\diamond$

Example 7. Consider

$$\begin{cases} u_x + xu_y = u \\ u(1,y) = h(y). \end{cases} \tag{2.14}$$

Our characteristic ODEs are given by

$$\begin{aligned} \frac{dx}{ds} &= 1 \\ \frac{dy}{ds} &= x \\ \frac{dz}{ds} &= z \end{aligned}$$

with initial conditions

$$\begin{aligned} x(r,0) &= 1 \\ y(r,0) &= r \\ z(r,0) &= h(r). \end{aligned}$$

First, we see that

$$x(r,s) = s + c_1(r).$$

But, $x(r,0) = 1$ implies $x(r,s) = s + 1$. Therefore,

$$\frac{dy}{ds} = s + 1$$

which implies

$$y(r,s) = \frac{s^2}{2} + s + c_2(r).$$

Then, using our prescribed data, we have

$$y(r,s) = \frac{s^2}{2} + s + r.$$

Next,

$$\frac{dz}{ds} = z$$

implies

$$z(r,s) = c_3(r)e^s.$$

Now using the prescribed data, we have

$$z(r,s) = h(r)e^s.$$

Now we need to solve for r and s in terms of x and y. In particular, $x = s+1$ implies that $s = x-1$. Therefore, $y = \frac{(x-1)^2}{2} + (x-1) + r$ which implies that $r = y - \frac{(x-1)^2}{2} - (x-1)$. And as a result, we have found a solution of (2.14) given by

$$u(x,y) = h\left(y - \frac{(x-1)^2}{2} - (x-1)\right) e^{x-1}.$$

$\diamond$

2.3 Quasilinear Equation

Consider the first-order quasilinear equation of a function of two variables with data prescribed on a curve Γ in the xy-plane.

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ u|_\Gamma = \phi. \end{cases} \tag{2.15}$$

We parametrize Γ by r such that $\Gamma = \{(\gamma_1(r), \gamma_2(r))\}$. As before, we define the characteristic equations as

$$\begin{aligned} \frac{dx}{ds} &= a(x,y,z) \\ \frac{dy}{ds} &= b(x,y,z) \\ \frac{dz}{ds} &= c(x,y,z) \end{aligned}$$

with initial conditions

$$\begin{aligned} x(r,0) &= \gamma_1(r) \\ y(r,0) &= \gamma_2(r) \\ z(r,0) &= \phi(r). \end{aligned}$$

Using ODE theory, we will be able to find a unique solution to this system. As long as we can invert the function $G(r,s) \equiv (x(r,s), y(r,s))$ (at least near $s = 0$), we can find a solution of (2.15) given by

$$u(x,y) = z(r(x,y), s(x,y)).$$

Extending Claim 4, we will be able to find a local inverse for $G(r,s)$ as long as Γ is nowhere tangent to the projected characteristic curves; that is, as long as

$$(a(\gamma_1(r), \gamma_2(r), \phi(r)), b(\gamma_1(r), \gamma_2(r), \phi(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) \neq 0. \tag{2.16}$$

Notice that the projected characteristic curves now depend on z as well as x and y. In particular, we see that (2.16) depends not only on Γ, but also on ϕ. Consequently, we say the boundary data (Γ, ϕ) is **noncharacteristic boundary data** if (2.16) holds.

The main difference between the semilinear and quasilinear equation is that the first two equations in the system of characteristic ODEs decouples from the third equation in the semilinear case, but it does not in the quasilinear case. Consequently, in the quasilinear case, we may get projected characteristic curves crossing themselves. Let us see how that affects the existence of solutions.

Example 8. Consider the initial-value problem for Burgers' equation

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x). \end{cases} \tag{2.17}$$

Let Γ be parametrized by r, such that $\Gamma = \{(r,0)\}$. Now Γ will be noncharacteristic as long as

$$\gamma_1'(r) - \gamma_2'(r)\phi(r) \neq 0.$$

But, $\gamma_1'(r) = 1$ and $\gamma_2'(r) = 0$. Therefore, Γ is noncharacteristic. Our characteristic equations are given by

$$\frac{dt}{ds} = 1$$
$$\frac{dx}{ds} = z$$
$$\frac{dz}{ds} = 0$$

with initial conditions

$$t(r,0) = 0$$
$$x(r,0) = r$$
$$z(r,0) = \phi(r).$$

We solve this system as follows. First,

$$t(r,s) = s + c_1(r)$$
$$z(r,s) = c_3(r)$$

and, therefore,

$$x(r, s) = c_3(r)s + c_2(r).$$

Now, using the initial conditions, we have

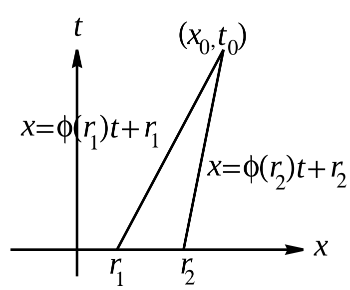
$$\begin{aligned} t(r, s) &= s \\ z(r, s) &= \phi(r) \\ x(r, s) &= \phi(r)s + r. \end{aligned}$$

Now $x = \phi(r)s + r$ and $t = s$ implies $r = x - \phi(r)s = x - \phi(r)t = x - zt$. Therefore, letting $u(x, t) = z(r, s)$, we have

$$u(x, t) = \phi(x - ut),$$

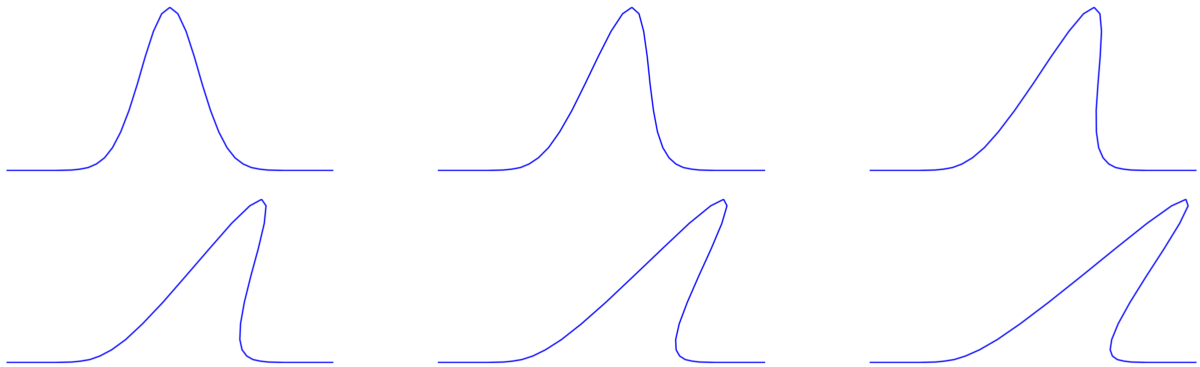
an implicit formula for a solution to (2.17).

Let's look at this solution a little more closely. First, we consider the projected characteristic curves, which are given by $x = \phi(r)s + r$ and $t = s$, which implies $x = \phi(r)t + r$. Now suppose the initial data ϕ satisfies the following. Suppose there is an $r_1 < r_2$ such that $\phi(r_1) > \phi(r_2)$. Then the projected characteristic curves intersect at some point (x_0, t_0). What does this mean? Look again at the characteristic equations. In particular, the solution u satisfies $\frac{du}{ds} = 0$. This means u is constant along characteristic curves. Therefore, $u(x_0, t_0) = u(r_1, 0) = \phi(r_1)$. But, also $u(x_0, t_0) = u(r_2, 0) = \phi(r_2)$. But, by assumption $\phi(r_1) > \phi(r_2)$. Therefore, we get a contradiction! We get a singularity formation at some time t.

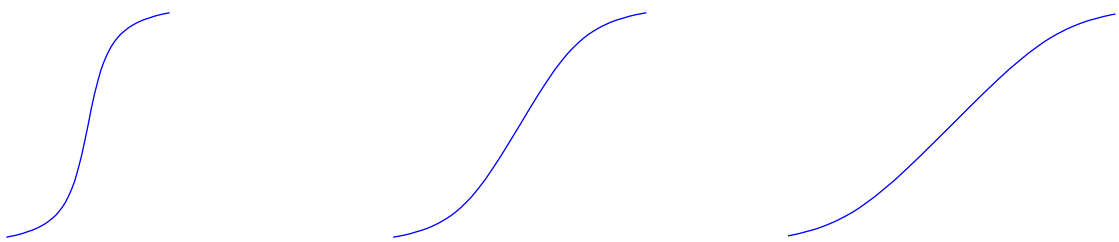


$\phi'(r)<0$

For example, consider (2.17) with initial data $\phi(x) = e^{-x^2}$. For $x > 0$, $\phi'(x) < 0$. Therefore, $\phi(r_1) > \phi(r_2)$ for $0 < r_1 < r_2$. Consequently, as described above, the projected characteristics will cross. What does the solution look like? As shown in the "movie" below, the taller part of the wave will overtake the shorter part of the wave, causing the wave to break. At the time the wave breaks, the solution u will take on multiple values, thus leading to a singularity in the solution. Of course, a function cannot take on multiple values. Consequently, the wave pictured below cannot be a solution of (2.17) after the time t when the wave breaks. We will discuss this issue in more detail later when we discuss conservation laws.



If $\phi'(r) \geq 0$, projected characteristic curves will not intersect, so there will be no conflict in defining u. For example, consider (2.17) with initial data $\phi(x) = 5 \arctan(x)$. In the movie below, we see that the wave does not break and we have a smooth solution.



Remark. In the case when projected characteristic curves do not intersect, we will not have a conflict in defining our solution u. However, it is possible that we will not have *enough* information to define u. We will discuss this idea as well as how to overcome the problem of characteristics crossing more when we talk about conservation laws.

◇

2.4 Fully Nonlinear Equation

Now we consider an initial-value problem for a fully nonlinear, first-order equation. We will continue to consider the case when the unknown function u is a function of two variables and will extend this to the general case later. We are interested in a problem of the form

$$\begin{cases} F(x, y, u, u_x, u_y) = 0 \\ u|_\Gamma = \phi. \end{cases} \tag{2.18}$$

Example 9. (Examples of Fully Nonlinear Equations)

- $u_x^2 + u_y^2 = 1$
- $u_x u_y - 1 = 0$.

◇

In the linear/semilinear/quasilinear cases, we were able to find the solution $u(x, y)$ as an integral surface by taking a union of characteristic curves. The characteristic curves were constructed by using the *characteristic direction vectors* $(a(x, y, z), b(x, y, z), c(x, y, z))$. In particular, for a Cauchy problem of the form

$$\begin{cases} a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \\ u|_\Gamma = \phi \end{cases} \tag{2.19}$$

by letting $z(s) = u(x(s), y(s))$ and by defining the characteristic equations as

$$\begin{aligned} \frac{dx}{ds} &= a(x, y, z) \\ \frac{dy}{ds} &= b(x, y, z) \\ \frac{dz}{ds} &= c(x, y, z), \end{aligned}$$

we were able to find a solution of (2.19) by constructing an integral surface as a union of characteristic curves emanating from each point on the curve (Γ, ϕ) in $\mathbb{R}^3$.

More generally, by introducing the variables

$$\begin{aligned} p(s) &= u_x(x(s), y(s)) \\ q(s) &= u_y(x(s), y(s)) \end{aligned}$$

and writing the equation from (2.19) as

$$F(x, y, z, p, q) = a(x, y, z)p + b(x, y, z)q - c(x, y, z) = 0,$$

we see that the characteristic equations defined above satisfy

$$\begin{aligned} \frac{dx}{ds} &= F_p(x, y, z, p, q) \\ \frac{dy}{ds} &= F_q(x, y, z, p, q) \\ \frac{dz}{ds} &= pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q). \end{aligned} \tag{2.20}$$

Using this motivation, we would like to define the characteristic equations for the fully nonlinear equation (2.18) by (2.20). However, in general (2.20) may be an underdetermined system. (For the previous cases, this was a complete system.) Therefore, we need to find equations for $p(s)$ and $q(s)$. Recall we have defined $p(s)$ and $q(s)$ as $u_x(x(s), y(s))$ and $u_y(x(s), y(s))$ respectively. Therefore, by the chain rule, along the curves $(x(s), y(s))$ defined in (2.20), we have

$$\begin{aligned} \frac{dp}{ds} &= \frac{du_x}{ds} = u_{xx}x'(s) + u_{xy}y'(s) = p_x x'(s) + p_y y'(s) = p_x F_p + p_y F_q \\ \frac{dq}{ds} &= \frac{du_y}{ds} = u_{yx}x'(s) + u_{yy}y'(s) = q_x x'(s) + q_y y'(s) = q_x F_p + q_y F_q. \end{aligned}$$

We want to add these equations to (2.20) to get a complete system. However, we don't want these equations to depend on p_x, p_y, etc. By making use of the PDE, however, we see that

$$\begin{aligned} \frac{dF}{dx} &= F_x + F_z u_x + F_p u_{xx} + F_q u_{xy} \\ &= F_x + F_z p + F_p p_x + F_q p_y = 0, \end{aligned}$$

using the fact that $q_x = (u_y)_x = u_{xy} = (u_x)_y = p_y$. Therefore, we have

$$p_x F_p + p_y F_q = -F_x - pF_z.$$

Similarly, we see that

$$\begin{aligned} \frac{dF}{dy} &= F_y + F_z u_y + F_p u_{xy} + F_q u_{yy} \\ &= F_y + F_z q + F_p q_x + F_q q_y = 0. \end{aligned}$$

Therefore, we have

$$q_x F_p + q_y F_q = -F_y - qF_z.$$

Consequently, we define our equations for dp/ds and dq/ds as

$$\begin{aligned} \frac{dp}{ds} &= -F_x - pF_z \\ \frac{dq}{ds} &= -F_y - qF_z. \end{aligned}$$

In summary, we define the characteristic equations for (2.18) by the system

$$\begin{aligned} \frac{dx}{ds} &= F_p(x, y, z, p, q) \\ \frac{dy}{ds} &= F_q(x, y, z, p, q) \\ \frac{dz}{ds} &= pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q) \\ \frac{dp}{ds} &= -F_x(x, y, z, p, q) - pF_z(x, y, z, p, q) \\ \frac{dq}{ds} &= -F_y(x, y, z, p, q) - qF_z(x, y, z, p, q). \end{aligned} \tag{2.21}$$

With a system of five ODEs, we need to prescribe five pieces of initial data. For an initial-value problem of the form (2.18), we have prescribed data for u on Γ. Therefore, as usual, our initial conditions for $x(r, 0), y(r, 0), z(r, 0)$ will be given by $\gamma_1(r), \gamma_2(r), \phi(r)$ respectively. In order to solve this system of five ODEs, however, we need to prescribe initial conditions $p(r, 0)$ and $q(r, 0)$. In so doing, we replace the initial curve $(\Gamma, \phi) = (\gamma_1(r), \gamma_2(r), \phi(r))$ with an *initial strip*. Initial values for p and q cannot be prescribed arbitrarily, however. In particular, the initial values for p and q must satisfy the following conditions. First, they must satisfy the PDE. Letting $\psi_1(r) = p(r, 0)$, $\psi_2(r) = q(r, 0)$, we need

$$F(\gamma_1(r), \gamma_2(r), \phi(r), \psi_1(r), \psi_2(r)) = 0. \tag{2.22}$$

Second, using the fact that

$$\frac{du}{dr}(r) = \frac{du}{dx}x'(r) + \frac{du}{dy}y'(r),$$

we need ψ_1, ψ_2 to satisfy

$$\phi'(r) = \psi_1(r)\gamma_1'(r) + \psi_2(r)\gamma_2'(r). \tag{2.23}$$

Initial data $(\Gamma, \phi, \Psi) = (\gamma_1, \gamma_2, \phi, \psi_1, \psi_2)$ which satisfies (2.22) and (2.23) is said to be **admissible**.

Remark. There may be no functions $\psi_1(r)$, $\psi_2(r)$ which satisfy these equations, or there may not be unique functions. If we are able to find functions $\psi_1(r)$ and $\psi_2(r)$ which satisfy these equations, however, we can find a unique solution of (2.21) with initial data $x(r,0) = \gamma_1(r)$, $y(r,0) = \gamma_2(r)$, $z(r,0) = \phi(r)$ and that choice $p(r,0) = \psi_1(r)$ and $q(r,0) = \psi_2(r)$.

Extending Claim 4, we will be able to invert the function $G(r,s) = (x(r,s), y(r,s))$ as long as

$$(F_p(\gamma_1(r), \gamma_2(r), \phi(r), \psi_1(r), \psi_2(r)), F_q(\gamma_1(r), \gamma_2(r), \phi(r), \psi_1(r), \psi_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) \neq 0. \tag{2.24}$$

If (2.24) holds, we say the boundary data $(\Gamma, \phi, \psi_1, \psi_2)$ is *noncharacteristic.*

Summary of Method for Fully Nonlinear Equations. To summarize, consider the Cauchy problem for a fully nonlinear first-order equation in two spatial variables,

$$\begin{cases} F(x, y, u, u_x, u_y) = 0 \\ u|_\Gamma = \phi. \end{cases}$$

We look for a solution u as a union of characteristic curves. For initial data

$$(\Gamma, \phi) = (\gamma_1(r), \gamma_2(r), \phi(r)),$$

we need to prescribe initial values for p and q by looking for functions $\psi_1(r)$ and $\psi_2(r)$ which satisfy

$$\begin{aligned} &F(\gamma_1(r), \gamma_2(r), \phi(r), \psi_1(r), \psi_2(r)) = 0 \\ &\phi'(r) = \psi_1(r)\gamma_1'(r) + \psi_2(r)\gamma_2'(r). \end{aligned}$$

If we can find functions $\psi_1(r)$ and $\psi_2(r)$ satisfying these conditions, then we can find a unique integral surface (for that choice of functions) so that we can find a solution of our PDE (at least near the initial curve Γ.) This integral surface is found as a solution to the following system of characteristic ODEs:

$$\begin{aligned} \frac{dx}{ds}(r,s) &= F_p \\ \frac{dy}{ds}(r,s) &= F_q \\ \frac{dz}{ds}(r,s) &= pF_p + qF_q \\ \frac{dp}{ds}(r,s) &= -F_x - pF_z \\ \frac{dq}{ds}(r,s) &= -F_y - qF_z \end{aligned}$$

for each r, with the initial conditions,

$$\begin{aligned}
x(r,0) &= \gamma_1(r)\\
y(r,0) &= \gamma_2(r)\\
z(r,0) &= \phi(r)\\
p(r,0) &= \psi_1(r)\\
q(r,0) &= \psi_2(r).
\end{aligned}$$

Let's look at some examples.

Example 10.

$$\begin{cases} u_t + au_x = 0 \\ u(x,0) = \sin x. \end{cases} \tag{2.25}$$

Note: Of course, we could solve this without using this more general method, but this method will work just as well for the simpler cases!

Our equation can be written as

$$F(x,t,u,u_x,u_t) = au_x + u_t = 0.$$

Now by letting $z = u$, $p = u_x$ and $q = u_t$, our equation can be written as:

$$F(x,t,z,p,q) = ap + q.$$

Our characteristic equations are the following:

$$\begin{aligned}
&\frac{dx}{ds}(r,s) = F_p = a\\
&\frac{dt}{ds}(r,s) = F_q = 1\\
&\frac{dz}{ds}(r,s) = pF_p + qF_q = ap + q = 0\\
&\frac{dp}{ds}(r,s) = -F_x - pF_z = 0\\
&\frac{dq}{ds}(r,s) = -F_y - qF_z = 0.
\end{aligned}$$

On our curve $\Gamma = \{(r,0)\}$, we prescribe initial data $u|_\Gamma = \sin(r)$. Therefore,

$$\begin{aligned}
x(r,0) &= r\\
t(r,0) &= 0\\
z(r,0) &= \sin(r).
\end{aligned}$$

To prescribe initial data for p and q, we need to find functions $\psi_1(r)$ and $\psi_2(r)$ such that

$$\begin{aligned}
&F(\gamma_1(r),\gamma_2(r),\phi(r),\psi_1(r),\psi_2(r)) = 0\\
&\phi'(r) = \psi_1(r)\gamma_1'(r) + \psi_2(r)\gamma_2'(r).
\end{aligned}$$

From the first equation, this means $a\psi_1(r) + \psi_2(r) = 0$. From the second equation, we have

$$\phi'(r) = \psi_1(r)$$

which implies

$$\psi_1(r) = \cos(r).$$

Therefore,

$$\psi_2(r) = -a\cos(r).$$

In summary, we have

$$\begin{aligned}
&\frac{dx}{ds}(r,s) = a \qquad && x(r,0) = r \\
&\frac{dt}{ds}(r,s) = 1 \qquad && t(r,0) = 0 \\
&\frac{dz}{ds}(r,s) = 0 \qquad && z(r,0) = \sin(r) \\
&\frac{dp}{ds}(r,s) = 0 \qquad && p(r,0) = \cos(r) \\
&\frac{dq}{ds}(r,s) = 0 \qquad && q(r,0) = -a\cos(r).
\end{aligned}$$

Now we can solve this system as follows:

$$\begin{aligned}
&x(r,s) = as + r \\
&t(r,s) = s \\
&z(r,s) = \sin(r) \\
&p(r,s) = \cos(r) \\
&q(r,s) = -a\cos(r).
\end{aligned}$$

Now solving for r and s in terms of x and t, we have $z(r(x,t), s(x,t)) = \sin(x - at)$. Therefore, we arrive at the solution

$$u(x,t) = z(r,s) = \sin(x - at),$$

as expected.

$\diamond$

Example 11. Consider

$$\begin{cases} u_x^2 + u_y^2 = 1 \\ u|_\Gamma = 0. \end{cases}$$

where Γ is the circle of radius one in the xy-plane. We can write the equation as

$$F(x, y, z, p, q) = p^2 + q^2 - 1 = 0.$$

The characteristic equations are given by

$$\begin{aligned}
\frac{dx}{ds}(r,s) &= F_p = 2p \\
\frac{dy}{ds}(r,s) &= F_q = 2q \\
\frac{dz}{ds}(r,s) &= pF_p + qF_q = 2p^2 + 2q^2 \\
\frac{dp}{ds}(r,s) &= -F_x - pF_z = 0 \\
\frac{dq}{ds}(r,s) &= -F_y - qF_z = 0.
\end{aligned}$$

We parametrize our curve Γ by r such that $\Gamma = \{(\cos(r), \sin(r))\}$. On Γ, we prescribe $u|_\Gamma = 0$. Therefore,

$$\begin{aligned}
x(r,0) &= \cos(r) \\
y(r,0) &= \sin(r) \\
z(r,0) &= 0
\end{aligned}$$

Now we need to find initial conditions for p and q. In particular, we need functions $\psi_1(r)$ and $\psi_2(r)$ satisfying

$$\begin{aligned}
&F(\gamma_1(r), \gamma_2(r), \phi(r), \psi_1(r), \psi_2(r)) = 0 \\
&\phi'(r) = \psi_1(r)\gamma_1'(r) + \psi_2(r)\gamma_2'(r).
\end{aligned}$$

We need

$$\psi_1^2 + \psi_2^2 = 1,$$

and

$$0 = \psi_1(r)(-\sin(r)) + \psi_2(r)(\cos(r)).$$

There are two possible sets of solutions. Either

$$\begin{aligned}
&(1) \quad \psi_1(r) = \cos(r) \quad \psi_2(r) = \sin(r) \\
\text{or} \qquad &(2) \quad \psi_1(r) = -\cos(r) \quad \psi_2(r) = -\sin(r).
\end{aligned}$$

Let's try choice (1) first. Summarizing, we have

$$\begin{aligned}
\frac{dx}{ds}(r,s) &= 2p & x(r,0) &= \cos(r) \\
\frac{dy}{ds}(r,s) &= 2q & y(r,0) &= \sin(r) \\
\frac{dz}{ds}(r,s) &= 2p^2 + 2q^2 & z(r,0) &= 0 \\
\frac{dp}{ds}(r,s) &= 0 & p(r,0) &= \cos(r) \\
\frac{dq}{ds}(r,s) &= 0 & q(r,0) &= \sin(r).
\end{aligned}$$

We solve this system as follows. First,

$$p(r,s) = \cos(r)$$
$$q(r,s) = \sin(r).$$

Therefore,

$$\begin{aligned} x(r,s) &= 2\cos(r)s + \cos(r) = \cos(r)[2s+1] \\ y(r,s) &= 2\sin(r)s + \sin(r) = \sin(r)[2s+1] \\ z(r,s) &= 2[\cos^2(r) + \sin^2(r)]s = 2s. \end{aligned}$$

Therefore,

$$x^2 + y^2 = [2s+1]^2 = [z+1]^2.$$

Therefore, letting $u(x,y) = z(r,s)$ we have found an implicit function for u. Namely,

$$[u+1]^2 = x^2 + y^2.$$

This implies that $u = -1 \pm \sqrt{x^2+y^2}$, but the function involving the negative square root will not satisfy our boundary condition. Therefore, our solution is given by $u(x,y) = -1 + \sqrt{x^2+y^2}$.

Now, suppose we chose alternative (2) for ψ_1 and ψ_2. How does this affect our solution? In this case, the only change is that $p(r,0) = -\cos(r)$ and $q(r,0) = -\sin(r)$. Therefore,

$$p(r,s) = -\cos(r)$$
$$q(r,s) = -\sin(r).$$

Consequently,

$$\begin{aligned} x(r,s) &= -2\cos(r)s + \cos(r) = \cos(r)[-2s+1] \\ y(r,s) &= -2\sin(r)s + \sin(r) = \sin(r)[-2s+1] \\ z(r,s) &= 2[\cos^2(r) + \sin^2(r)]s = 2s. \end{aligned}$$

Therefore,

$$x^2 + y^2 = [-2s+1]^2 = [-z+1]^2.$$

Consequently, our solution is given implicitly by

$$[1-u]^2 = x^2 + y^2.$$

This implies that $u(x,y) = 1 \pm \sqrt{x^2+y^2}$. Now, the function involving the positive square root will not satisfy our initial condition. Therefore, our solution is given by $u(x,y) = 1 - \sqrt{x^2+y^2}$.

◇

2.5 General Method of Characteristics for First-Order Equations.

We finish by describing the general method for solving a first-order equation in n variables. Consider the first-order, nonlinear equation,

$$F(\vec{x}, u, Du) = 0 \qquad x \in \mathbb{R}^n.$$

In the case of two spatial variables, we prescribed initial data on a curve Γ in $\mathbb{R}^2$. We now must prescribe data on an $(n-1)$-dimensional *manifold* Γ in R^n.

Remark. An m-dimensional manifold is a surface which can be represented locally as the graph of a function.

Our Cauchy problem is

$$\begin{cases} F(\vec{x}, u, Du) = 0 & x \in \mathbb{R}^n \\ u|_\Gamma = \phi. \end{cases} \tag{2.26}$$

First, we parametrize Γ by the vector $\vec{r} = (r_1, \ldots, r_{n-1}) \in \mathbb{R}^{n-1}$, so that $\Gamma = (\gamma_1(\vec{r}), \ldots, \gamma_n(\vec{r}))$. By letting $z(s) = u(\vec{x}(s))$, $p_i(s) = u_{x_i}(\vec{x}(s))$, we rewrite our equation as

$$F(\vec{x}, z, \vec{p}) = 0.$$

We define $2n+1$ characteristic equations by

$$\begin{aligned} \frac{dx_i}{ds} &= F_{p_i} \\ \frac{dz}{ds} &= \sum_{i=1}^n p_i F_{p_i} \\ \frac{dp_i}{ds} &= -F_{x_i} - F_z p_i, \end{aligned} \tag{2.27}$$

for $i = 1, \ldots, n$. Our initial conditions are given by

$$\begin{aligned} &x_i(\vec{r}, 0) = \gamma_i(\vec{r}) \\ &z(\vec{r}, 0) = \phi(\vec{r}) \\ &p_i(\vec{r}, 0) = \psi_i(\vec{r}), \qquad \vec{r} \in \mathbb{R}^{n-1}, \end{aligned} \tag{2.28}$$

where the functions ψ_i, $i = 1, \ldots, n$ are determined by solving the following equations. First, we need,

$$F(\gamma_1(\vec{r}), \ldots, \gamma_n(\vec{r}), \phi(\vec{r}), \psi_1(\vec{r}), \ldots, \psi_n(\vec{r})) = 0.$$

Second, we need

$$\frac{\partial u}{\partial r_i}(\vec{r}, 0) = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial r_i}$$

for $i = 1, \ldots, n-1$. But, $u(\vec{r}, 0) = \phi(\vec{r})$, $x_i(\vec{r}, 0) = \gamma_i(\vec{r})$, and $u_{x_i} = p_i$. Therefore, this equation becomes

$$\frac{\partial \phi}{\partial r_i} = \psi_1 \frac{\partial \gamma_1}{\partial r_i} + \ldots + \psi_n \frac{\partial \gamma_n}{\partial r_i}.$$

Therefore, our system of n equations for the n unknown functions $\psi_1(\vec{r}), \ldots, \psi_n(\vec{r})$ are given by

$$\begin{aligned}
&\phi_{r_i} = \psi_1(\vec{r})\frac{\partial \gamma_1}{\partial r_i} + \ldots + \psi_n(\vec{r})\frac{\partial \gamma_n}{\partial r_i} \qquad i = 1, \ldots, n-1 \\
&F(\gamma_1(\vec{r}), \ldots, \gamma_n(\vec{r}), \phi(\vec{r}), \psi_1(\vec{r}), \ldots, \psi_n(\vec{r})) = 0.
\end{aligned}$$

Again, functions $\psi_1, \ldots, \psi_n$ may not exist or may not be unique, but if they do exist, we can find a unique solution of (2.27) satisfying the initial conditions (2.28) for that choice of $\psi_1, \ldots, \psi_n$.

In order to guarantee that we can invert the function $\vec{x} = (\vec{x}(\vec{r}, s))$ near the manifold Γ we will assume our initial data is noncharacteristic. That is, defining $\Psi(\vec{r}) = (\psi_1(\vec{r}), \ldots, \psi_n(\vec{r}))$, we say $(\Gamma(\vec{r}), \phi(\vec{r}), \Psi(\vec{r}))$ is *noncharacteristic* if

$$\nabla_{\vec{p}} F(\Gamma(\vec{r}), \phi(\vec{r}), \Psi(\vec{r})) \cdot N(\Gamma(\vec{r})) \neq 0,$$

where $N(\Gamma(\vec{r}))$ is the normal vector to the $(n-1)$-dimensional manifold Γ.

In summary, for noncharacteristic boundary data (Γ, ϕ, Ψ), we can find a local solution of (2.26) by solving the characteristic equations (2.27) with initial conditions (2.28) and letting $u(\vec{x}) = z(\vec{r}, s)$.

We state this more precisely as follows. For each $\vec{r} \in \mathbb{R}^{n-1}$, let $(\vec{x}(\vec{r}, s), z(\vec{r}, s), \vec{p}(\vec{r}, s))$ be the unique solution of (2.27), (2.28). By the noncharacteristic assumption on the initial data, we can invert the function $\vec{x}(\vec{r}, s)$ near $s = 0$. That is, we can find functions $\vec{R}$, S such that $\vec{r} = \vec{R}(\vec{x})$ and $s = S(\vec{x})$. Now, let

$$u(\vec{x}) \equiv z(\vec{r}, s) = z(\vec{R}(\vec{x}), S(\vec{x})) \tag{2.29}$$

for $\vec{x}$ near Γ.

Theorem 12. (Local Existence Theorem) *Ref: Evans, Chap. 3. Let u be defined as above. Then u satisfies the initial-value problem*

$$\begin{cases} F(\vec{x}, u, Du) = 0 \\ u|_\Gamma = \phi \end{cases} \tag{2.30}$$

for $\vec{x}$ near Γ.

Remarks on proof. Suppose $(\vec{x}(\vec{r}, s), z(\vec{r}, s), \vec{p}(\vec{r}, s))$ is the solution of (2.27), (2.28). Let u be as defined in (2.29). To show that u satisfies

$$F(\vec{x}, u, Du) = 0$$

for $\vec{x}$ near Γ relies fundamentally on two steps: showing

(a) $F(\vec{x}(\vec{r}, s), z(\vec{r}, s), \vec{p}(\vec{r}, s)) = 0$ for s near 0.

(b) $u_{x_i}(\vec{x}) = p_i(\vec{x})$.

Assuming we can prove (a), then using the definition of u, we can conclude that

$$F(\vec{x}, u, \vec{p}) = 0$$

for $\vec{x}$ near Γ. Then, proving (b), we conclude that

$$F(\vec{x}, u, Du) = 0,$$

as desired.