

FRACTIONAL HARMONIC MAPS INTO MANIFOLDS IN ODD DIMENSION $n > 1$

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Abstract

In this paper we consider critical points of the following nonlocal energy

$$\mathcal{L}_n(u) = \int_{\mathbb{R}^n} |(-\Delta)^{n/4}u(x)|^2 dx, \quad (1)$$

where $u \in \dot{H}^{n/2}(\mathbb{R}^n, \mathcal{N})$ $\mathcal{N} \subset \mathbb{R}^m$ is a compact k dimensional smooth manifold without boundary and $n > 1$ is an odd integer. Such critical points are called $n/2$ -harmonic maps into \mathcal{N} . We prove that $(-\Delta)^{n/4}u \in L^p_{loc}(\mathbb{R}^n)$ for every $p \geq 1$ and thus $u \in C^{0,\alpha}_{loc}(\mathbb{R}^n)$, for every $0 < \alpha < 1$. The local Hölder continuity of $n/2$ -harmonic maps is based on regularity results obtained in [4] for nonlocal Schrödinger systems with an antisymmetric potential and on some new *3-terms commutators* estimates.

Key words. Fractional harmonic maps, nonlinear elliptic PDE's, regularity of solutions, commutator estimates.

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1 Introduction

In the paper [6] the authors considered 1/2-harmonic maps in \mathbb{R} with values in a k -dimensional sub-manifold $\mathcal{N} \subset \mathbb{R}^m$ ($m \geq 1$), which is smooth, compact and without boundary. We recall that 1/2-harmonic maps are functions u in the space $\dot{H}^{1/2}(\mathbb{R}, \mathcal{N}) = \{u \in \dot{H}^{1/2}(\mathbb{R}, \mathbb{R}^m) : u(x) \in \mathcal{N}, \text{ a.e.}\}$, which are critical points for perturbation of the type $\Pi_{\mathcal{N}}^N(u + t\varphi)$, ($\varphi \in C_c^\infty$ and $\Pi_{\mathcal{N}}^N$ is the normal projection on \mathcal{N}) of the functional

$$\mathcal{L}_1(u) = \int_{\mathbb{R}} |(-\Delta)^{1/4}u(x)|^2 dx, \quad (2)$$

(see Definition 1.1 in [5]). The operator $(-\Delta)^{1/4}$ on \mathbb{R} is defined by means of the Fourier transform as follows

$$\widehat{(-\Delta)^{1/4}u} = |\xi|^{1/2}\hat{u},$$

(given a function f , both \hat{f} and $\mathcal{F}[f]$ denote the Fourier transform of f).

The Lagrangian (2) is invariant with respect to the Möbius group and it satisfies the following identity

$$\int_{\mathbb{R}} |(-\Delta)^{1/4}u(x)|^2 dx = \inf \left\{ \int_{\mathbb{R}_+^2} |\nabla \tilde{u}|^2 dx : \tilde{u} \in W^{1,2}(\mathbb{R}_+^2, \mathbb{R}^m), \text{ trace } \tilde{u} = u \right\}.$$

The Euler Lagrange equation associated to the nonlinear problem (2) can be written as follows :

$$(-\Delta)^{1/2}u \wedge \nu(u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}) \quad , \quad (3)$$

where $\nu(z)$ is the Gauss Map at $z \in \mathcal{N}$ taking values into the grassmannian $\tilde{G}r_{m-k}(\mathbb{R}^m)$ of oriented $m - k$ planes in \mathbb{R}^m , which to every point $z \in \mathcal{N}$ assigns the unit $m - k$ vector defining the oriented normal $m - k$ -plane to $T_z\mathcal{N}$. The $C_{loc}^{0,\alpha}$ regularity of 1/2 harmonic maps was deduced from a key result obtained in [6] concerning with nonlocal linear Schrödinger system in \mathbb{R} with an antisymmetric potential of the type:

$$\forall i = 1 \cdots m \quad (-\Delta)^{1/4}v^i = \sum_{j=1}^m \Omega_j^i v^j, \quad (4)$$

where $v = (v_1, \dots, v_m) \in L^2(\mathbb{R}, \mathbb{R}^m)$ and $\Omega = (\Omega_i^j)_{i,j=1 \dots m} \in L^2(\mathbb{R}, so(m))$ is an L^2 maps from \mathbb{R} into the space $so(m)$ of $m \times m$ antisymmetric matrices.

It is natural to extend the above mentioned results to $n/2$ harmonic maps in \mathbb{R}^n , with values in a k -dimensional sub-manifold $\mathcal{N} \subset \mathbb{R}^m$, where $m \geq 1$ and $n = 2p + 1$ is an odd integer. By analogy with the case $n = 1$, $n/2$ harmonic maps are functions u in the

space $\dot{H}^{n/2}(\mathbb{R}^n, \mathcal{N}) = \{u \in \dot{H}^{n/2}(\mathbb{R}^n, \mathbb{R}^m) : u(x) \in \mathcal{N}, \text{ a.e.}\}$, which are critical points for perturbation of the type $\Pi_{\mathcal{N}}^N(u + t\varphi)$, ($\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$) of the functional

$$\mathcal{L}_n(u) = \int_{\mathbb{R}^n} |(-\Delta)^{n/4}u(x)|^2 dx. \quad (5)$$

The Euler Lagrange equation associated to the non linear problem (5) can be written as follows :

$$(-\Delta)^{n/2}u \wedge \nu(u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \quad (6)$$

We first mention that the case of $1/2$ -harmonic maps into the circle S^1 might appear for instance in the asymptotic of equations in phase-field theory for fractional reaction-diffusion such as

$$\epsilon^2 (-\Delta)^{1/2}u + u(1 - |u|^2) = 0, \quad \text{in } \mathbb{R}$$

where u is a complex valued "wave function".

Moreover variational problems of the form (5) appear as simplified models for *renormalized energy* in general relativity, (see [1]).

There are also some strong geometric motivations in studying $n/2$ -harmonic maps in odd dimension $n \geq 1$ in relation with the so-called free boundary sub-manifolds and optimization problems of eigenvalues. This is the subject of a forthcoming paper [7].

As it has been already pointed out in [5, 6], the Euler Lagrange in the form (6) is hiding fundamental properties such as for instance its elliptic nature and it is difficult to use it directly for solving problems related to regularity and compactness. One of the first task is then to rewrite it in a form that will make some of its analysis features more apparent. This is the purpose of the next proposition. Before stating it we need some additional notations.

Denote by $P^T(z)$ and $P^N(z)$ the projections from \mathbb{R}^m to the tangent space $T_z\mathcal{N}$ and to the normal space $N_z\mathcal{N}$ to \mathcal{N} at $z \in \mathcal{N}$ respectively. For $u \in \dot{H}^{n/2}(\mathbb{R}^n, \mathcal{N})$ we denote simply by P^T and P^N the compositions $P^T \circ u$ and $P^N \circ u$. Under the assumption that \mathcal{N} is smooth, $P^T \circ u$ and $P^N \circ u$ are matrix valued maps in $\dot{H}^{n/2}(\mathbb{R}^n, M_m(\mathbb{R}^n))$. We will prove the following crucial formulation of the $n/2$ -harmonic map equation.

Proposition 1.1 *Let $u \in \dot{H}^{n/2}(\mathbb{R}^n, \mathcal{N})$ be a weak $n/2$ -harmonic map. Then the following equation holds*

$$(-\Delta)^{n/4}v = \Omega v + \tilde{\Omega}_1 v + \tilde{\Omega}_2 \quad (7)$$

where $v \in L^2(\mathbb{R}^n, \mathbb{R}^{2m})$ and $\Omega \in L^2(\mathbb{R}^n, so(2m))$ are given respectively by

$$v := \begin{pmatrix} P^T(-\Delta)^{n/4}u \\ P^N(-\Delta)^{n/4}u \end{pmatrix} \quad \text{and} \quad \Omega = 2 \begin{pmatrix} -\omega & \omega \\ \omega & -\omega \end{pmatrix},$$

the map ω is in $L^2(\mathbb{R}^n, so(m))$ and given by

$$\omega = \frac{(-\Delta)^{n/4} P^T P^T - P^T (-\Delta)^{n/4} P^T}{2}.$$

Finally $\tilde{\Omega}_1 = \tilde{\Omega}_1(P^T, P^N, (-\Delta)^{n/4} u)$ is in $L^{(2,1)}(\mathbb{R}^n, M_{2m}(\mathbb{R}^n))$, $\tilde{\Omega}_2 = \tilde{\Omega}_2(P^T, P^N)$ is in $\dot{W}^{-n/2, (2, \infty)}(\mathbb{R}^n, \mathbb{R}^{2m})$, and satisfy

$$\|\tilde{\Omega}_1\|_{L^{(2,1)}(\mathbb{R}^n)} \leq C(\|P^N\|_{\dot{H}^{n/2}(\mathbb{R}^n)}^2 + \|P^T\|_{\dot{H}^{n/2}(\mathbb{R}^n)}^2); \quad (8)$$

$$\|\tilde{\Omega}_2\|_{\dot{W}^{-n/2, (2, \infty)}(\mathbb{R}^n)} \leq C(\|P^N\|_{\dot{H}^{n/2}(\mathbb{R}^n)} + \|P^T\|_{\dot{H}^{n/2}(\mathbb{R}^n)}) \|(-\Delta)^{n/4} u\|_{L^{(2, \infty)}(\mathbb{R}^n)}. \quad \square \quad (9)$$

The explicit formulations of $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ in Proposition 1.1 are given in Section 3. The control on $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ is a consequence of *regularity by compensation* results on some operators that we now introduce.

Given $Q \in \mathcal{S}'(\mathbb{R}^n, \mathcal{M}_{\ell \times m}(\mathbb{R}^n))$ $\ell \geq 0^{(1)}$ and $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{R}^m)$, let us define the operator T_n as follows.

$$T_n(Q, u) = (-\Delta)^{n/4} [Q ((-\Delta)^{n/4} u)] - Q (-\Delta)^{n/2} u + (-\Delta)^{n/4} Q (-\Delta)^{n/4} u. \quad (10)$$

We prove the following commutator estimate.

Theorem 1.1 *Let $u \in \dot{W}^{n/2, (2, \infty)}(\mathbb{R}^n)$, $Q \in \dot{H}^{n/2}(\mathbb{R}^n)$. Then $T_n(Q, u) \in \dot{H}^{-n/2}(\mathbb{R}^n)$ and*

$$\|T_n(Q, u)\|_{\dot{H}^{-n/2}(\mathbb{R}^n)} \leq C \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|(-\Delta)^{n/4} u\|_{L^{(2, \infty)}(\mathbb{R}^n)}. \quad \square \quad (11)$$

Theorem 1.1 is a straightforward consequence of the following estimate for the dual operator of T_n defined by

$$T_n^*(Q, u) = (-\Delta)^{n/4} [((-\Delta)^{n/4} Q) u] - (-\Delta)^{n/2} [Q u] + (-\Delta)^{n/4} [Q ((-\Delta)^{n/4} u)]. \quad (12)$$

Theorem 1.2 *Let $u, Q \in \dot{H}^{n/2}(\mathbb{R}^n)$. Then $T_n^*(Q, u) \in W^{-n/2, (2, 1)}(\mathbb{R}^n)$, and*

$$\|T_n^*(Q, u)\|_{\dot{W}^{-n/2, (2, 1)}(\mathbb{R}^n)} \leq C \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)}. \quad \square \quad (13)$$

We recall that the spaces $W^{n/2, (2, \infty)}(\mathbb{R}^n)$ and $W^{-n/2, (2, 1)}(\mathbb{R}^n)$ are defined as

$$\dot{W}^{n/2, (2, \infty)}(\mathbb{R}^n) := \{f \in \mathcal{S}' : \mathcal{F}^{-1} [|\xi|^n \mathcal{F}[v]] \in L^{(2, \infty)}(\mathbb{R}^n)\};$$

$$\dot{W}^{-n/2, (2, 1)}(\mathbb{R}^n) := \{f \in \mathcal{S}' : \mathcal{F}^{-1} [|\xi|^{-n} \mathcal{F}[v]] \in L^{(2, 1)}(\mathbb{R}^n)\}.$$

Moreover $\dot{W}^{n/2, (2, \infty)}(\mathbb{R}^n)$ is the dual of $\dot{W}^{-n/2, (2, 1)}(\mathbb{R}^n)$. We refer the reader to Section 2 for the definition of Lorentz spaces $L^{(p, q)}$, $1 \leq p, q \leq +\infty$ and of the fractional Sobolev spaces.

⁽¹⁾ $\mathcal{M}_{\ell \times m}(\mathbb{R}^n)$ denotes, as usual, the space of $\ell \times m$ real matrices.

Theorems 1.1 and 1.2 correspond respectively to Theorem 1.2 and Theorem 1.4 in [5] for $n = 1$. Unlike the case $n = 1$, here we are not able to show that T_n^* is the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ but it is in the bigger space $\dot{W}^{-n/2,(2,1)}(\mathbb{R}^n)$, which is still enough in order to get our regularity results.

The main result of this paper is the following

Theorem 1.3 *Let \mathcal{N} be a smooth compact sub-manifold of \mathbb{R}^m without boundary. Let $u \in \dot{H}^{n/2}(\mathbb{R}^n, \mathcal{N})$ be a weak $n/2$ -harmonic map into \mathcal{N} , then $u \in C_{loc}^{0,\alpha}(\mathbb{R}^n, \mathcal{N})$, for every $0 < \alpha < 1$. \square*

Finally a classical "elliptic type" bootstrap argument leads to the following result (see [4] for the details of this argument).

Theorem 1.4 *Let \mathcal{N} be a smooth compact submanifold of \mathbb{R}^m without boundary. Let $u \in \dot{H}^{n/2}(\mathbb{R}^n, \mathcal{N})$ be a weak $n/2$ -harmonic map into \mathcal{N} , then u is $C^\infty(\mathbb{R}^n)$. \square*

We mention that Theorem 1.3 is deduced from Proposition 1.1 and a slight perturbation of the following result which concerns the sub-criticality of linear non-local Schrödinger systems. The proof of this result is given in [4] and it follows the same arguments with some suitable changes of the proof of Theorem 1.1 in [6] in dimension $n = 1$.

Theorem 1.5 *Let $\Omega \in L^2(\mathbb{R}^n, so(m))$ and $v \in L^2(\mathbb{R}^n)$ be a weak solution of*

$$(-\Delta)^{n/4}v = \Omega v. \quad (14)$$

Then $v \in L_{loc}^p(\mathbb{R}^n)$ for every $1 \leq p < +\infty$.

As it was already observed in [6], the fact that Ω is antisymmetric plays a crucial role in order to apply a suitable *gauge transformation* and rewrite the equation (14) with a *more regular* right hand side.

Next we would like to underline the difference and the novelty with respect to the case $n = 1$.

First of all the proof of Theorem 1.2 is not a mere extension of Theorem 1.4 in [6]. The fact that we are dealing with the dimension $n > 1$ requires a different analysis when we split the operator T_n^* in the so-called para-products. In particular we have to introduce in addition the operators (57) and (58) and to estimate them in a suitable way (see Proposition A.1 in the Appendix A). The fact that we need these two new operators will be evident in the proof of Theorem 1.2.

Moreover we observe that in the case of $n = 1$ the pseudo-differential operators ∇ and $(-\Delta)^{1/2}$ are of the same orders and this permits us to write the equations for $P^T(-\Delta)^{1/4}u$ and $P^N(-\Delta)^{1/4}u$ in a similar way (see Section 5 in [6]). More precisely $P^T(-\Delta)^{1/4}u$ and $P^N(-\Delta)^{1/4}u$ satisfy

$$(-\Delta)^{1/4}(P^T(-\Delta)^{1/4}u) = T_1(P^T, u) - [(-\Delta)^{1/4}P^T][(-\Delta)^{1/4}u], \quad (15)$$

and

$$(-\Delta)^{1/4}\mathcal{R}[P^N(-\Delta)^{1/4}u] = S_1(P^N, u) - [(-\Delta)^{1/4}P^N]\mathcal{R}[(-\Delta)^{1/4}u], \quad (16)$$

where \mathcal{R} is the Fourier multiplier of symbol $\sigma(\xi) = i\frac{\xi}{|\xi|}$ and for $Q, u \in \mathcal{S}'(\mathbb{R}^n)$

$$S_1(Q, u) := (-\Delta)^{1/4}[Q(-\Delta)^{1/4}u] - \mathcal{R}[Q\nabla u] + [(-\Delta)^{1/4}Q]\mathcal{R}[(-\Delta)^{1/4}u].$$

To write down (15) and (16) we use respectively the fact that $P^T(-\Delta)^{1/2}u = 0$ and $P^N\nabla u = 0$. In the case $n > 1$, ∇ and $(-\Delta)^{n/2}$ are pseudo-differential operators of order respectively 1 and n . The equation for $P^T(-\Delta)^{n/4}u$ is similar to equation (15), with T_1 replaced by T_n . On the contrary we cannot replace the equation (16) by an equation of the form

$$(-\Delta)^{n/4}\mathcal{R}[P^N(-\Delta)^{n/4}u] = S_n(P^N, u) - [(-\Delta)^{n/4}P^N]\mathcal{R}[(-\Delta)^{n/4}u], \quad (17)$$

where for $Q, u \in \mathcal{S}'(\mathbb{R}^n)$

$$S_n(Q, u) := (-\Delta)^{1/4}[Q(-\Delta)^{n/4}u] - \mathcal{R}(-\Delta)^{\frac{n-1}{2}}[Q\nabla u] + [(-\Delta)^{n/4}Q]\mathcal{R}[(-\Delta)^{n/4}u].$$

Actually even if S_n seems the natural extension of S_1 , it does not satisfy the same regularity estimates as S_1 , (see [5]).

Therefore we have to find a different formulation of the structure equation which still satisfies good estimates.

In the case of $n > 1$, the structure equation becomes

$$(-\Delta)^{n/4}(P^N(-\Delta)^{n/4}u) = ((-\Delta)^{n/4}\bar{\mathcal{R}})f(P^N, u). \quad (18)$$

where

$$f(P^N, u) := \mathcal{R}(P^N(-\Delta)^{n/4}u) - (-\Delta)^{\frac{n}{4}-\frac{1}{2}}(P^N\nabla u) \quad (19)$$

and $\bar{\mathcal{R}}$ is the Fourier multiplier of symbol $\sigma(\xi) = -i\frac{\xi}{|\xi|}$ (i.e. the conjugate of \mathcal{R}). We show that the right hand side of equation (18) is in $\dot{W}^{-n/2, (2, \infty)}(\mathbb{R}^n)$ and

$$\|((-\Delta)^{n/4}\bar{\mathcal{R}})f(P^N, u)\|_{\dot{W}^{-n/2, (2, \infty)}(\mathbb{R}^n)} \lesssim \|P^N\|_{\dot{H}^{n/2}} \|(-\Delta)^{n/4}u\|_{L^{(2, \infty)}}. \quad (20)$$

The estimate (20) is not straightforward, but we need to apply suitable interpolation arguments .

We conclude the present Section by recalling existing results in the literature on regularity of critical points of nonlocal Lagrangians and we refer the reader to [13] and [14] for a complete overview of analogous results in the local case.

The regularity of 1/2-harmonic maps with values into a sphere has been first investigated in [5] where new ‘‘three terms commutators’’ estimates have been obtained by using the technique of paraproducts. Analogous results have been extended in [6] to 1/2-harmonic maps with values into general submanifolds.

In [12] the author considers critical points to the functional that assigns to any $u \in \dot{H}^{1/2}(\mathbb{R}, \mathcal{N})$ the minimal Dirichlet energy among all possible extensions in \mathcal{N} , while in the papers [5, 6] the classical $\dot{H}^{1/2}$ Lagrangian corresponds to the minimal Dirichlet energy among all possible extensions in \mathbb{R}^m . Hence the approach in [12] consists in working with an intrinsic version of $H^{1/2}$ -energy instead of an extrinsic one. The drawback of considering the intrinsic energy is that the Euler Lagrange equation is almost impossible to write explicitly and is then implicit. However the intrinsic version of the 1/2-harmonic map is more closely related to the existing regularity theory of Dirichlet Energy minimizing maps into \mathcal{N} . Finally the regularity of $n/2$ harmonic maps in odd dimension $n > 1$ with values into a sphere has been recently investigated by Schikorra [16]. In this paper the author extends the results obtained in [5] by using an approach based on compensation arguments introduced by Tartar [17], moreover the fact that the map takes values on the sphere plays a crucial role in writing down the structure equation.

The paper is organized as follows.

- In Section 2 we recall some basic definitions and notations.
- In Section 3 we derive the Euler- Lagrangian equation (7) associated to the energy (5) and we prove Theorem 1.3.
- In Appendix A we prove the commutator estimates that are used in Section 3.

2 Preliminaries: function spaces and the fractional Laplacian

In this Section we introduce some notations and definitions that are used in the paper.

For $n \geq 1$, we denote respectively by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the spaces of Schwartz functions and tempered distributions. Moreover given a function v we will denote either by \hat{v} or by $\mathcal{F}[v]$ the Fourier Transform of v :

$$\hat{v}(\xi) = \mathcal{F}[v](\xi) = \int_{\mathbb{R}^n} v(x) e^{-i\langle \xi, x \rangle} dx.$$

Throughout the paper we use the convention that x, y denote variables in the space and ξ, ζ the variables in the phase.

We recall the definition of fractional Sobolev space (see for instance [18]).

Definition 2.1 For a real $s \geq 0$,

$$H^s(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : |\xi|^s \mathcal{F}[v] \in L^2(\mathbb{R}^n)\}, .$$

For a real $s < 0$,

$$H^s(\mathbb{R}^n) = \{v \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2} \mathcal{F}[v] \in L^2(\mathbb{R}^n)\} .$$

□

It is known that $H^{-s}(\mathbb{R}^n)$ is the dual of $H^s(\mathbb{R}^n)$.

For a submanifold \mathcal{N} of \mathbb{R}^m we can define

$$H^s(\mathbb{R}^n, \mathcal{N}) = \{u \in H^s(\mathbb{R}^n, \mathbb{R}^m) : u(x) \in \mathcal{N}, \text{ a.e.}\}.$$

Given $q > 1$ and $s \in \mathbb{R}$ we also set

$$W^{s,q}(\mathbb{R}^n) := \{v \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \mathcal{F}[v]] \in L^q(\mathbb{R}^n)\}$$

and

$$\dot{W}^{s,q}(\mathbb{R}^n) := \{v \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}[|\xi|^s \mathcal{F}[v]] \in L^q(\mathbb{R}^n)\}.$$

We shall make use of the Littlewood-Paley dyadic decomposition of unity that we recall here. Such a decomposition can be obtained as follows. Let $\phi(\xi)$ be a radial Schwartz function supported in $\{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$, which is equal to 1 in $\{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$. Let $\psi(\xi)$ be the function given by

$$\psi(\xi) := \phi(\xi) - \phi(2\xi).$$

ψ is then a ‘‘bump function’’ supported in the annulus $\{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$.

Let $\psi_0 = \phi$, $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j \neq 0$. The functions ψ_j , for $j \in \mathbb{Z}$, are supported in $\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ and realize a dyadic decomposition of the unity :

$$\sum_{j \in \mathbb{Z}} \psi_j(x) = 1.$$

We denote further

$$\phi_j(\xi) := \sum_{k=-\infty}^j \psi_k(\xi).$$

The function ϕ_j is supported on $\{\xi, |\xi| \leq 2^{j+1}\}$.

We recall the definition of the homogeneous Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and homogeneous Triebel-Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$ in terms of the above dyadic decomposition (see e.g [11, 15]).

Definition 2.2 Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we set

$$\begin{aligned} \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} &= \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} & \text{if } q < \infty \\ \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} &= \sup_{j \in \mathbb{Z}} 2^{js} \|\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty \end{aligned} \quad (21)$$

When $p, q < \infty$ we also set

$$\|u\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsq} |\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]|^q \right)^{1/q} \right\|_{L^p}.$$

□

The space of all tempered distributions u for which the quantity $\|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}$ is finite is called the homogeneous Besov space with indices s, p, q and it is denoted by $\dot{B}_{p,q}^s(\mathbb{R}^n)$. The space of all tempered distributions f for which the quantity $\|f\|_{\dot{F}_{p,q}^s(\mathbb{R}^n)}$ is finite is called the homogeneous Triebel-Lizorkin space with indices s, p, q and it is denoted by $\dot{F}_{p,q}^s(\mathbb{R}^n)$. A classical result says ⁽²⁾ that $\dot{W}^{s,q}(\mathbb{R}^n) = \dot{B}_{q,2}^s(\mathbb{R}^n) = \dot{F}_{q,2}^s(\mathbb{R}^n)$.

Finally we denote $\mathcal{H}^1(\mathbb{R}^n)$ the homogeneous Hardy Space in \mathbb{R}^n . A less classical results ⁽³⁾ asserts that $\mathcal{H}^1(\mathbb{R}^n) \simeq \dot{F}_{2,1}^0$ thus we have

$$\|u\|_{\mathcal{H}^1(\mathbb{R}^n)} \simeq \int_{\mathbb{R}^n} \left(\sum_j |\mathcal{F}^{-1}[\psi_j \mathcal{F}[u]]|^2 \right)^{1/2} dx.$$

We recall that

$$\dot{H}^{n/2}(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^0(\mathbb{R}^n), \quad (22)$$

where $BMO(\mathbb{R}^n)$ is the space of bounded mean oscillation dual to $\mathcal{H}^1(\mathbb{R}^n)$ (see for instance [15], page 31).

The s -fractional Laplacian of a function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as a pseudo differential operator of symbol $|\xi|^{2s}$:

$$(\widehat{-\Delta}^s u)(\xi) = |\xi|^{2s} \hat{u}(\xi). \quad (23)$$

Finally we introduce the definition of Lorentz spaces (see for instance [10] for a complete presentation of such spaces). For $1 \leq p < +\infty, 1 \leq q \leq +\infty$, the Lorentz space $L^{(p,q)}(\mathbb{R}^n)$ is the set of measurable functions satisfying

$$\begin{cases} \int_0^{+\infty} (t^{1/p} f^*(t))^q \frac{dt}{t} < +\infty, & \text{if } q < \infty, p < +\infty \\ \sup_{t>0} t^{1/p} f^*(t) < \infty & \text{if } q = \infty, p < \infty, \end{cases}$$

where f^* is the decreasing rearrangement of $|f|$.

We observe that $L^{p,\infty}(\mathbb{R}^n)$ corresponds to the weak L^p space. Moreover for $1 < p < +\infty, 1 \leq q \leq \infty$, the space $L^{(\frac{p}{p-1}, \frac{q}{q-1})}$ is the dual space of $L^{(p,q)}$.

Let us define

$$\dot{W}^{s,(p,q)}(\mathbb{R}^n) = \{f \in \mathcal{S}' : |\xi|^s \mathcal{F}[v] \in L^{(p,q)}(\mathbb{R}^n)\}.$$

⁽²⁾See for instance [10]

⁽³⁾See for instance [11].

In the sequel we will often use the Hölder inequality in the Lorentz spaces: if $f \in L^{p_1, q_1}$, $g \in L^{p_2, q_2}$, with $1 \leq p_1, p_2, q_1, q_2 \leq +\infty$. Then $fg \in L^{r, s}$, with $r^{-1} = p_1^{-1} + p_2^{-1}$ and $s^{-1} = q_1^{-1} + q_2^{-1}$, (see for instance [10]).

To conclude this section we introduce some basic notations.

$B_r(\bar{x})$ is the ball of radius r and centered at \bar{x} . If $\bar{x} = 0$ we simply write B_r . If $x, y \in \mathbb{R}^n$, $x \cdot y$ is the scalar product between x, y .

Given a *multindex* $\alpha = (\alpha_1, \dots, \alpha_n)$, where α_i is a nonnegative integer, we denote by $|\alpha| = \alpha_1 + \dots + \alpha_n$ the order of α .

For every function $u: \mathbb{R}^n \rightarrow \mathbb{R}$, $M(u)$ is the maximal function of u , namely

$$M(u) = \sup_{r>0, x \in \mathbb{R}^n} |B(x, r)|^{-1} \int_{B(x, r)} |u(y)| dy. \quad (24)$$

Given $q > 1$ we denote by q' the conjugate of q : $q^{-1} + q'^{-1} = 1$.

In the sequel we will often use the symbols \lesssim and \simeq instead of \leq and $=$, if the constants appearing in the estimates are not relevant and therefore they are omitted.

3 Euler Equation for $n/2$ -Harmonic Maps into Manifolds

We consider a compact k dimensional smooth manifold without boundary $\mathcal{N} \subset \mathbb{R}^m$. Let $\Pi_{\mathcal{N}}$ be the orthogonal projection on \mathcal{N} . We also consider the Dirichlet energy (5).

The weak $n/2$ -harmonic maps are defined as critical points of the functional (5) with respect to perturbation of the form $\Pi_{\mathcal{N}}(u + t\phi)$, where ϕ is an arbitrary compacted supported smooth map from \mathbb{R}^n into \mathbb{R}^m .

Definition 3.1 *We say that $u \in H^{n/2}(\mathbb{R}^n, \mathcal{N})$ is a weak $n/2$ -harmonic map if and only if, for every maps $\phi \in H^{n/2}(\mathbb{R}^n, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ we have*

$$\frac{d}{dt} \mathcal{L}_n(\Pi_{\mathcal{N}}(u + t\phi))|_{t=0} = 0. \quad (25)$$

We introduce some notations. We denote by $\bigwedge(\mathbb{R}^m)$ the exterior algebra (or Grassmann Algebra) of \mathbb{R}^m and by the symbol \wedge the *exterior or wedge product*. For every $p = 1, \dots, m$, $\bigwedge_p(\mathbb{R}^m)$ is the vector space of p -vectors.

If $(\epsilon_i)_{i=1, \dots, m}$ is the canonical orthonormal basis of \mathbb{R}^m , then every element $v \in \bigwedge_p(\mathbb{R}^m)$ is written as $v = \sum_I v_I \epsilon_I$ where $I = \{i_1, \dots, i_p\}$ with $1 \leq i_1 \leq \dots \leq i_p \leq m$, $v_I := v_{i_1, \dots, i_p}$ and $\epsilon_I := \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_p}$.

By the symbol \llcorner we denote the interior multiplication $\llcorner: \bigwedge_p(\mathbb{R}^m) \times \bigwedge_q(\mathbb{R}^m) \rightarrow \bigwedge_{q-p}(\mathbb{R}^m)$ defined as follows.

Let $\epsilon_I = \epsilon_{i_1} \wedge \dots \wedge \epsilon_{i_p}$, $\epsilon_J = \epsilon_{j_1} \wedge \dots \wedge \epsilon_{j_q}$, with $q \geq p$. Then $\epsilon_J \llcorner \epsilon_I = 0$ if $I \not\subset J$, otherwise $\epsilon_J \llcorner \epsilon_I = (-1)^M \epsilon_K$ where ϵ_K is a $q-p$ vector (with $K \cup I = J$) and M is the number of pairs $(i, j) \in I \times J$ with $j > i$.

Finally by the symbol $*$ we denote the Hodge-star operator, $*$: $\bigwedge_p(\mathbb{R}^m) \rightarrow \bigwedge_{m-p}(\mathbb{R}^m)$, defined by $*\beta = (\epsilon_1 \wedge \dots \wedge \epsilon_n) \llcorner \beta$. For an introduction of the Grassmann Algebra we refer the reader to the first Chapter of the book by Federer [9].

In the sequel we denote by P^T and P^N respectively the tangent and the normal projection from \mathbb{R}^m to the manifold \mathcal{N} .

They verify the following properties: $(P^T)^t = P^T$, $(P^N)^t = P^N$ (namely they are symmetric operators), $(P^T)^2 = P^T$, $(P^N)^2 = P^N$, $P^T + P^N = Id$, $P^N P^T = P^T P^N = 0$.

We set $e = \epsilon_1 \wedge \dots \wedge \epsilon_k$ and $\nu = \epsilon_{k+1} \wedge \dots \wedge \epsilon_m$. For every $z \in \mathcal{N}$, $e(z)$ and $\nu(z)$ give the orientation respectively of the tangent k -plane and the normal $m-k$ -plane to $T_z \mathcal{N}$.

We observe that for every $v \in \mathbb{R}^m$ we have

$$P^T v = (-1)^{k-1} * ((e \llcorner v) \wedge \nu). \quad (26)$$

$$P^N v = (-1)^{m-1} * (e \wedge (\nu \llcorner v)). \quad (27)$$

Hence P^N and P^T can be seen as matrices in $\dot{H}^{n/2}(\mathbb{R}^n, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

Next we write the Euler equation associated to the functional (5).

Proposition 3.1 *All weak $n/2$ -harmonic maps $u \in H^{n/2}(\mathbb{R}^n, \mathcal{N})$ satisfy in a weak sense i) the equation*

$$\int_{\mathbb{R}^n} ((-\Delta)^{n/2} u) \cdot v \, dx = 0, \quad (28)$$

for every $v \in \dot{H}^{n/2}(\mathbb{R}^n, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and $v \in T_{u(x)} \mathcal{N}$ almost everywhere, or in a equivalent way

ii) the equation

$$P^T (-\Delta)^{n/2} u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \quad (29)$$

or

iii) the equation

$$(-\Delta)^{n/4} (P^T (-\Delta)^{n/4} u) = T_n(P^T, u) - ((-\Delta)^{n/4} P^T) (-\Delta)^{n/4} u, \quad (30)$$

where T_n is the operator defined in (10).

Together with the Euler Lagrange equation (30) we consider the following "structure equation":

Proposition 3.2 *All maps in $\dot{H}^{n/2}(\mathbb{R}^n, \mathcal{N})$ satisfy the following identity*

$$(-\Delta)^{n/4} (P^N (-\Delta)^{n/4} u) = ((-\Delta)^{n/4} \bar{\mathcal{R}}) f(P^N, u), \quad (31)$$

where

$$f(P^N, u) := \mathcal{R}(P^N(-\Delta)^{n/4}u) - (-\Delta)^{\frac{n}{4}-\frac{1}{2}}(P^N \nabla u) \quad (32)$$

is in $L^{(2,\infty)}(\mathbb{R}^n)$ and

$$\|((-\Delta)^{n/4}\bar{\mathcal{R}})f(P^N, u)\|_{\dot{W}^{-n/2,(2,\infty)}(\mathbb{R}^n)} \lesssim \|P^N\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|(-\Delta)^{n/4}u\|_{L^{(2,\infty)}(\mathbb{R}^n)}. \quad (33)$$

We give the proof of Proposition 3.2. For the proof of Proposition 3.1 we refer the reader to [5].

Proof of Proposition 3.2. We first observe that $P^N \nabla u = 0$ (see Proposition 1.2 in [5]). Thus we can write:

$$\begin{aligned} (-\Delta)^{n/4}(P^N(-\Delta)^{n/4}u) &= ((-\Delta)^{n/4}\bar{\mathcal{R}}) [\mathcal{R}(P^N(-\Delta)^{n/4}u)] \\ &= \underbrace{((-\Delta)^{n/4}\bar{\mathcal{R}}) [\mathcal{R}(P^N(-\Delta)^{n/4}u) - P^N((-\Delta)^{n/4}\mathcal{R}u)]}_{(1)} \\ &\quad + \underbrace{((-\Delta)^{n/4}\bar{\mathcal{R}}) [P^N((-\Delta)^{n/4}\mathcal{R}u) - (-\Delta)^{\frac{n}{4}-\frac{1}{2}}(P^N \nabla u)]}_{(2)} \\ &= ((-\Delta)^{n/4}\bar{\mathcal{R}})f(P^N, u). \end{aligned} \quad (34)$$

Corollary A.2 and Theorem A.2 imply respectively that (1) and (2) $\in \dot{W}^{-n/2,(2,\infty)}(\mathbb{R}^n)$ and

$$\begin{aligned} \|(1)\|_{\dot{W}^{-n/2,(2,\infty)}(\mathbb{R}^n)} &\lesssim \|P^N\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|(-\Delta)^{n/4}u\|_{L^{(2,\infty)}(\mathbb{R}^n)}; \\ \|(2)\|_{\dot{W}^{-n/2,(2,\infty)}(\mathbb{R}^n)} &\lesssim \|P^N\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|(-\Delta)^{n/4}u\|_{L^{(2,\infty)}(\mathbb{R}^n)}. \end{aligned}$$

Hence $((-\Delta)^{n/4}\bar{\mathcal{R}})f(P^N, u) \in \dot{W}^{-n/2,(2,\infty)}(\mathbb{R}^n)$ and (33) holds. \square

Next we see that by combining (30) and (31) we can obtain the new equation (7) for the vector field $v = (P^T(-\Delta)^{n/4}u, P^N(-\Delta)^{n/4}u)$ where an antisymmetric potential appears.

We introduce the following matrices

$$\omega_1 = \frac{((-\Delta)^{n/4}P^T)P^T + P^T(-\Delta)^{n/4}P^T - (-\Delta)^{n/4}(P^T P^T)}{2} \quad (35)$$

$$\omega_2 = ((-\Delta)^{n/4}P^T)P^N + P^T(-\Delta)^{n/4}P^N - (-\Delta)^{n/4}(P^T P^N), \quad (36)$$

$$\omega = \frac{((-\Delta)^{n/4}P^T)P^T - P^T(-\Delta)^{n/4}P^T}{2}. \quad (37)$$

We observe that Theorem 1.2 implies that $\omega_1, \omega_2 \in L^{(2,1)}(\mathbb{R}^n)$. Moreover it holds

$$\|\omega_1\|_{L^{(2,1)}(\mathbb{R}^n)} + \|\omega_2\|_{L^{(2,1)}(\mathbb{R}^n)} \lesssim \|P^T\|_{\dot{H}^{n/2}(\mathbb{R}^n)}^2.$$

The matrix ω is **antisymmetric**.

Proof of Proposition 1.1.

From Propositions 3.1 and 3.2 it follows that u satisfies in a weak sense the equations (30) and (31).

The key point is to rewrite the the terms $((-\Delta)^{n/4}P^T)(-\Delta)^{n/4}u$ and equation (31) in a different way.

- **Re-writing of $((-\Delta)^{n/4}P^T)(-\Delta)^{n/4}u$.**

$$\begin{aligned} ((-\Delta)^{n/4}P^T)(-\Delta)^{n/4}u &= ((-\Delta)^{n/4}P^T)(P^T(-\Delta)^{n/4}u + P^N(-\Delta)^{n/4}u) \\ &= [((-\Delta)^{n/4}P^T)P^T][P^T(-\Delta)^{n/4}u] \\ &\quad + [((-\Delta)^{n/4}P^T)P^N][P^N(-\Delta)^{n/4}u]. \end{aligned}$$

Now we have

$$((-\Delta)^{n/4}P^T)P^T = \omega_1 + \omega + \frac{(-\Delta)^{n/4}P^T}{2}; \quad (38)$$

and

$$\begin{aligned} ((-\Delta)^{n/4}P^T)P^N &= ((-\Delta)^{n/4}P^T)P^N + P^T(-\Delta)^{n/4}P^N - P^T(-\Delta)^{n/4}P^N \\ &\quad - (-\Delta)^{n/4}(P^T P^N) \\ &= \omega_2 + P^T(-\Delta)^{n/4}P^T \\ &= \omega_2 + \omega_1 - \omega + \frac{(-\Delta)^{n/4}P^T}{2}. \end{aligned} \quad (39)$$

Thus

$$\frac{((-\Delta)^{n/4}P^T)(P^T(-\Delta)^{n/4}u)}{2} = \omega_1(P^T(-\Delta)^{n/4}u) + \omega(P^T(-\Delta)^{n/4}u) \quad (40)$$

$$\frac{((-\Delta)^{n/4}P^T)(P^N(-\Delta)^{n/4}u)}{2} = (\omega_1 + \omega_2)(P^N(-\Delta)^{n/4}u) - \omega(P^N(-\Delta)^{n/4}u). \quad (41)$$

• **Re-writing of equation (31).** Equation (31) can be rewritten as follows:

$$\begin{aligned}
& ((-\Delta)^{n/4}(P^N(-\Delta)^{n/4}u) = ((-\Delta)^{n/4}\bar{\mathcal{R}})f(P^N, u) \\
& + \underbrace{(-\Delta)^{n/4}[P^T(-\Delta)^{n/4}u] - (-\Delta)^{\frac{n-1}{2}}[P^T(-\Delta)^{1/2}u]}_{(3)} \\
& + (-\Delta)^{\frac{n-1}{2}}[P^T(-\Delta)^{1/2}u] - (-\Delta)^{n/4}[P^T(-\Delta)^{n/4}u] + P^T(-\Delta)^{\frac{n}{2}}u \\
& + ((-\Delta)^{n/4}P^T)((-\Delta)^{n/4}u) - ((-\Delta)^{n/4}P^T)((-\Delta)^{n/4}u) \tag{42} \\
& = \underbrace{((-\Delta)^{\frac{n-1}{2}}(P^T(-\Delta)^{1/2}u) - T_n(P^T, u))}_{(4)} - \underbrace{((-\Delta)^{n/4}P^N)((-\Delta)^{n/4}u)}_{(5)}.
\end{aligned}$$

The term (3) in (42) is in $W^{-n/2, (2, \infty)}(\mathbb{R}^n)$ by Corollary A.1 . The term (4) is in $\dot{W}^{-n/2, (2, \infty)}(\mathbb{R}^n)$ by Theorem 1.1 and Corollary A.1. We finally observe that in (4) we use the fact that $P^T(-\Delta)^{\frac{n}{2}}u = 0$ and in (5) the fact that $(-\Delta)^{n/4}P^T = -(-\Delta)^{n/4}P^N$.

Given u, Q we set

$$\begin{aligned}
R(Q, u) &= ((-\Delta)^{n/4})(Q(-\Delta)^{n/4}u) - ((-\Delta)^{\frac{n-1}{2}})(Q(-\Delta)^{1/2}u) \\
&+ ((-\Delta)^{\frac{n-1}{2}})(Q(-\Delta)^{1/2}u) - T_n(Q, u).
\end{aligned}$$

We remark that $R(P^T, u)$ is the sum of (3), (4) in (42) .

• **Re-writing of $((-\Delta)^{n/4}P^N)(-\Delta)^{n/4}u$.**

We have

$$((-\Delta)^{n/4}P^N)(-\Delta)^{n/4}u = ((-\Delta)^{n/4}P^N)(P^T((-\Delta)^{n/4}u) + P^N((-\Delta)^{n/4}u)).$$

We estimate $((-\Delta)^{n/4}P^N)P^T((-\Delta)^{n/4}u)$ and $((-\Delta)^{n/4}P^N)P^N((-\Delta)^{n/4}u)$. We have

$$\begin{aligned}
((-\Delta)^{n/4}P^N)P^T &= -((-\Delta)^{n/4}P^T)P^T \tag{43} \\
&= -\omega_1 - \omega - \frac{((-\Delta)^{n/4}P^T)}{2} \\
&= -\omega_1 - \omega + \frac{((-\Delta)^{n/4}P^N)}{2},
\end{aligned}$$

and

$$((-\Delta)^{n/4}P^N)P^N = -((-\Delta)^{n/4}P^T)P^N + P^T((-\Delta)^{n/4}P^N) - P^T((-\Delta)^{n/4}P^N) \quad (44)$$

$$= -[((-\Delta)^{n/4}P^T)P^N + P^T((-\Delta)^{n/4}P^N) - (-\Delta)^{n/4}(P^N P^T)] \quad (45)$$

$$+ P^T((-\Delta)^{n/4}P^N)$$

$$= -\omega_2 - P^T((-\Delta)^{n/4}P^T) \quad (46)$$

$$= -\omega_2 - \omega_1 + \omega + \frac{((-\Delta)^{n/4}P^N)}{2}.$$

In (44) we use (38) and the fact that $P^T = -P^N$. Thus

$$\frac{((-\Delta)^{n/4}P^N)P^T(-\Delta)^{n/4}u}{2} = -\omega_1(P^T(-\Delta)^{n/4}u) - \omega(P^T(-\Delta)^{n/4}u) \quad (47)$$

$$\begin{aligned} \frac{((-\Delta)^{n/4}P^N)P^N(-\Delta)^{n/4}u}{2} &= -\omega_2(P^N(-\Delta)^{n/4}u) - \omega_1(P^N(-\Delta)^{n/4}u) \quad (48) \\ &+ \omega(P^N(-\Delta)^{n/4}u). \end{aligned}$$

By combining (40), (41), (47), (48) we obtain

$$\begin{aligned} (-\Delta)^{n/4} \begin{pmatrix} P^T(-\Delta)^{n/4}u \\ P^N(-\Delta)^{n/4}u \end{pmatrix} &= 2\tilde{\Omega}_1 \begin{pmatrix} P^T(-\Delta)^{n/4}u \\ P^N(-\Delta)^{n/4}u \end{pmatrix} + \tilde{\Omega}_2 \quad (49) \\ &+ 2 \begin{pmatrix} -\omega & \omega \\ \omega & -\omega \end{pmatrix} \begin{pmatrix} P^T(-\Delta)^{n/4}u \\ P^N(-\Delta)^{n/4}u \end{pmatrix}, \end{aligned}$$

where $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given by

$$\tilde{\Omega}_1 = \begin{pmatrix} -\omega_1 & -(\omega_1 + \omega_2) \\ \omega_1 & (\omega_1 + \omega_2) \end{pmatrix};$$

$$\tilde{\Omega}_2 = \begin{pmatrix} T_n(P^T, u) \\ R(P^T, u) + \mathcal{R}(-\Delta)^{n/4}f(P^N, u) \end{pmatrix}.$$

The matrix

$$\Omega = 2 \begin{pmatrix} -\omega & \omega \\ \omega & -\omega \end{pmatrix}$$

is antisymmetric.

We observe that from the estimates on the operators T_n , R and f it follows that $\tilde{\Omega}_2 \in \dot{W}^{-n/2, (2, \infty)}(\mathbb{R}^n, \mathbb{R}^{2m})$ and

$$\|\tilde{\Omega}_2\|_{\dot{W}^{-n/2, (2, \infty)}(\mathbb{R}^n, \mathbb{R}^{2m})} \lesssim \left(\|P^N\|_{\dot{H}^{n/2}(\mathbb{R}^n)} + \|P^T\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \right) \|(-\Delta)^{n/4}u\|_{L^{(2, \infty)}(\mathbb{R}^n)}. \quad (50)$$

On the other hand $\tilde{\Omega}_1 \in L^{(2, 1)}(\mathbb{R}^n, \mathcal{M}_{2m \times 2m})$ and

$$\|\tilde{\Omega}_1\|_{L^{(2, 1)}(\mathbb{R}^n, \mathcal{M}_{2m \times 2m})} \lesssim (\|P^N\|_{\dot{H}^{n/2}(\mathbb{R}^n)}^2 + \|P^T\|_{\dot{H}^{n/2}(\mathbb{R}^n)}^2). \quad (51)$$

□

Now we prove Theorem 1.3.

Proof of Theorem 1.3. We give just a sketch of proof, since the arguments are similar to those of Theorem 1.1 and Theorem 1.7. in [6]. From Proposition 1.1 it follows that

$$v = (P^T((-\Delta)^{n/4}u), P^N((-\Delta)^{n/4}u))$$

solves equation (49) which of the type (14) up to the terms $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$.

We aim at obtaining that $(-\Delta)^{n/4}u \in L_{loc}^p(\mathbb{R}^n)$, for all $p \geq 1$. To this purpose we take $\rho > 0$ such that

$$\|\Omega\|_{L^2(B(0, \rho))}, \|P^T\|_{\dot{H}^{n/2}(B(0, \rho))}, \|P^N\|_{\dot{H}^{n/2}(B(0, \rho))} \leq \varepsilon_0,$$

with $\varepsilon_0 > 0$ small enough. Let $x_0 \in B(0, \rho/4)$ and $r \in (0, \rho/8)$. We argue by duality and multiply (49) by ϕ which is given as follows. Let $g \in L^{(2, 1)}(\mathbb{R}^n)$, with $\|g\|_{L^{(2, 1)}} \leq 1$ and set $g_{r\alpha} = \mathbb{1}_{B(x_0, r\alpha)}g$, with $0 < \alpha < 1/4$ and $\phi = (-\Delta)^{-n/4}(g_{r\alpha}) \in L^\infty(\mathbb{R}^n) \cap \dot{W}^{n/2, (2, 1)}(\mathbb{R}^n)$. We multiply both sides of equation (49) by ϕ and we integrate.

By using the same ‘‘localization arguments’’ in the proof of Theorem 1.7 in [6] one can show that v satisfies for all $x_0 \in B(0, \rho/4)$ and $0 < r < \rho/8$, $\|v\|_{L^{(2, \infty)}(B(x_0, r))} \leq Cr^\beta$, for some $\beta \in (0, 1/2)$. Then by bootstrapping into the equation one can deduce that $v \in L_{loc}^p(\mathbb{R}^n)$, for all $p \geq 1$. Therefore $(-\Delta)^{n/4}u \in L_{loc}^p(\mathbb{R}^n)$, for all $p \geq 1$ as well.

This implies that $u \in C_{loc}^{0, \alpha}$ for all $0 < \alpha < 1$, since $W_{loc}^{n/2, p}(\mathbb{R}^n) \hookrightarrow C_{loc}^{0, \alpha}(\mathbb{R}^n)$ if $p > 2$ (see for instance [2]). This concludes the proof of Theorem 1.3. □

A Commutator Estimates

In this appendix we present a series of commutator estimates which have been used in the previous sections. We consider the Littlewood-Paley decomposition of unity introduced in Section 2. For every $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the Littlewood-Paley projection operators P_j and $P_{\leq j}$ by

$$\widehat{P_j f} = \psi_j \hat{f} \quad \widehat{P_{\leq j} f} = \phi_j \hat{f}.$$

Informally P_j is a frequency projection to the annulus $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, while $P_{\leq j}$ is a frequency projection to the ball $\{|\xi| \leq 2^{j+1}\}$. We will set $f_j = P_j f$ and $f^j = P_{\leq j} f$.

We observe that $f^j = \sum_{k=-\infty}^j f_k$ and $f = \sum_{k=-\infty}^{+\infty} f_k$ (where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$).

Given $f, g \in \mathcal{S}'(\mathbb{R}^n)$ we can split the product in the following way

$$fg = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g), \quad (52)$$

where

$$\begin{aligned} \Pi_1(f, g) &= \sum_{-\infty}^{+\infty} f_j \sum_{k \leq j-4} g_k = \sum_{-\infty}^{+\infty} f_j g^{j-4}; \\ \Pi_2(f, g) &= \sum_{-\infty}^{+\infty} f_j \sum_{k \geq j+4} g_k = \sum_{-\infty}^{+\infty} g_j f^{j-4}; \\ \Pi_3(f, g) &= \sum_{-\infty}^{+\infty} f_j \sum_{|k-j| < 4} g_k. \end{aligned}$$

We observe that for every j we have

$$\begin{aligned} \text{supp} \mathcal{F}[f^{j-4} g_j] &\subset \{2^{j-2} \leq |\xi| \leq 2^{j+2}\}; \\ \text{supp} \mathcal{F}[\sum_{k=j-3}^{j+3} f_j g_k] &\subset \{|\xi| \leq 2^{j+5}\}. \end{aligned}$$

The three pieces of the decomposition (52) are examples of paraproducts. Informally the first paraproduct Π_1 is an operator which allows high frequencies of f ($\sim 2^{nj}$) multiplied by low frequencies of g ($\ll 2^{nj}$) to produce high frequencies in the output. The second paraproduct Π_2 multiplies low frequencies of f with high frequencies of g to produce high frequencies in the output. The third paraproduct Π_3 multiply high frequencies of f with high frequencies of g to produce comparable or lower frequencies in the output. For a presentation of these paraproducts we refer to the reader for instance to the book [11]. The following Lemma will be often used in the sequel. For the proof of the first one we refer the reader to [5].

Lemma A.1 *For every $f \in \mathcal{S}'$ we have*

$$\sup_{j \in \mathbb{Z}} |f^j| \leq M(f).$$

Lemma A.2 *Let ψ be a Schwartz radial function such that $\text{supp}(\psi) \subset B(0, 4)$. Then for every $s \geq \lceil \frac{n}{2} \rceil + 1$ we have*

$$\|(-\Delta)^s \mathcal{F}^{-1} \psi\|_{L^1} \leq C_{\psi, n} (1 + s^{n+1}) 4^{2s},$$

where $C_{\psi, n}$ is a positive constant depending on the C^{n+1} norm of ψ and the dimension.

Proof of Lemma A.2. We assume that $n \geq 1$ is odd, so that $\lceil \frac{n}{2} \rceil + 1 = \frac{n+1}{2}$, (the case n even is similar). We recall that

$$(-\Delta)^s \mathcal{F}^{-1}\psi(\xi) = \mathcal{F}^{-1}[|\xi|^{2s}\psi](\xi).$$

We write

$$\int_{\mathbb{R}^n} |(-\Delta)^s \mathcal{F}^{-1}\psi(\xi)| d\xi = \int_{|\xi| \leq 1} |(-\Delta)^s \mathcal{F}^{-1}\psi(\xi)| d\xi + \int_{|\xi| \geq 1} |(-\Delta)^s \mathcal{F}^{-1}\psi(\xi)| d\xi.$$

The following estimates hold.

$$\begin{aligned} \int_{|\xi| \leq 1} |(-\Delta)^s \mathcal{F}^{-1}\psi(\xi)| d\xi &\leq \omega_n \|(-\Delta)^s \mathcal{F}^{-1}\psi(\xi)\|_{L^\infty} \\ &\leq \omega_n \| |x|^{2s} \psi \|_{L^1} \leq \omega_n 4^{2s} \|\psi\|_{L^1}, \end{aligned} \quad (53)$$

where $\omega_n = |B_1(0)|$.

$$\begin{aligned} \int_{|\xi| \geq 1} |(-\Delta)^s \mathcal{F}^{-1}\psi(\xi)| d\xi &= \int_{|\xi| \geq 1} \left(\frac{1}{|\xi|^{n+1}} \right) \left[\int_{\mathbb{R}^n} (-\Delta)_x^{\frac{n+1}{2}} e^{i\langle x, \xi \rangle} \psi(x) |x|^{2s} dx \right] d\xi \\ &= \int_{|\xi| \geq 1} \left(\frac{1}{|\xi|^{n+1}} \right) \left[\int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} (-\Delta)_x^{\frac{n+1}{2}} (\psi(x) |x|^{2s}) dx \right] d\xi \\ &\leq \int_{|\xi| \geq 1} \frac{1}{|\xi|^{n+1}} d\xi \left[\sum_{k=0}^{n+1} (2s)^k 4^{2s-k} \|\psi\|_{C^{n+1}} \right]. \end{aligned} \quad (54)$$

By combining (53) and (54) we obtain

$$\begin{aligned} \|(-\Delta)^s \mathcal{F}^{-1}\psi(\xi)\|_{L^1} &\leq \omega_n 4^{2s} \|\psi\|_{L^1} \\ &\quad + (4^{2s} (1 + s^{n+1}) \|\psi\|_{C^{n+1}}) \int_{|\xi| \geq 1} \frac{1}{|\xi|^{n+1}} d\xi \\ &\leq C_{\psi, n} (1 + s^{n+1}) 4^{2s}. \end{aligned} \quad (55)$$

This concludes the proof of Lemma A.2. \square

Lemma A.3 *Let $f \in B_{\infty, \infty}^0(\mathbb{R}^n)$. Then for all $s \geq \lceil \frac{n}{2} \rceil + 1$ and for all $j \in Z$ we have*

$$2^{-2sj} \|(-\Delta)^s f_j\|_{L^\infty} \leq C_{\psi, n} (1 + s^{n+1}) 4^{2s} \|f\|_{B_{\infty, \infty}^0(\mathbb{R}^n)}.$$

Proof of Lemma A.3. Let Ψ be a Schwartz radial function such that $\Psi = 1$ in B_2 and $\Psi = 0$ in $B^c(0, 4)$.

Since $\text{supp } \mathcal{F}[f_j] \subseteq B_{2^{j+1}} \setminus B_{2^j}$ we have

$$\mathcal{F}[(-\Delta)^s f_j] \simeq |\xi|^{2s} \mathcal{F}[f_j] = 2^{2sj} \psi(2^{-j}\xi) \frac{|\xi|^{2s}}{2^{2sj}} \mathcal{F}[f_j]. \quad (56)$$

Observe that

$$\begin{aligned} & \|\mathcal{F}^{-1}[\psi(2^{-j}\xi) \frac{|\xi|^{2s}}{2^{2sj}}]\|_{L^1} \\ &= \left\| \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \psi(2^{-j}\xi) \frac{|\xi|^{2s}}{2^{2sj}} d\xi \right\|_{L^1} \\ &= 2^{nj} \left\| \int_{\mathbb{R}^n} e^{i2^j \langle x, \xi \rangle} \psi(\xi) |\xi|^{2s} d\xi \right\|_{L^1} \\ &= 2^{nj} \|(-\Delta)^s \mathcal{F}^{-1}[\psi](2^j \cdot)\|_{L^1}. \end{aligned}$$

Thus

$$\begin{aligned} 2^{-2sj} \|(-\Delta)^s f_j\|_{L^\infty} &\leq \|\mathcal{F}^{-1}[\psi(2^{-j}\xi) \frac{\xi^{2s}}{2^{2sj}}] * f_j\|_{L^\infty} \\ &\leq \|\mathcal{F}^{-1}[\psi(2^{-j}\xi) \frac{|\xi|^{2s}}{2^{2sj}}]\|_{L^1} \|f_j\|_{L^\infty} \\ &= \|(-\Delta)^s (\mathcal{F}^{-1}[\psi])\|_{L^1} \|f_j\|_{L^\infty} \leq C_{\psi, n} (1 + s^{n+1}) 4^{2s} \|f_j\|_{L^\infty} \\ &\leq C_{\psi, n} (1 + s^{n+1}) 4^{2s} \|f\|_{B_{\infty, \infty}^0(\mathbb{R}^n)}. \quad \square \end{aligned}$$

In the sequel we suppose that $n > 1$ is an odd integer.

Given $Q \in \dot{H}^{n/2}(\mathbb{R}^n, \mathcal{M}_{\ell \times m}(\mathbb{R}^n))$ $\ell \geq 0$ and $u \in \dot{H}^{n/2}(\mathbb{R}^n, \mathbb{R}^m)$, we introduce the following operators

$$\begin{aligned} M_1(Q, u) &= \sum_{\substack{1 \leq |\alpha| \leq [n/2] \\ |\alpha| \text{ odd}}} \frac{c_\alpha}{\alpha!} (-\Delta)^{n/4} [(-\Delta)^{n/4 - |\alpha|} (\nabla^\alpha Q)] \nabla^\alpha u \\ &+ \sum_{\substack{1 \leq |\alpha| \leq [n/2] \\ |\alpha| \text{ even}}} \frac{c_\alpha}{\alpha!} (-\Delta)^{n/4} [(-\Delta)^{n/4 - |\alpha|/2} Q], \nabla^\alpha u; \end{aligned} \quad (57)$$

$$\begin{aligned}
M_2(Q, u) &= \sum_{\substack{1 \leq |\alpha| \leq [n/2] \\ |\alpha| \text{ odd}}} \frac{c_\alpha}{\alpha!} (-\Delta)^{n/4} (\nabla^\alpha Q [(-\Delta)^{n/4-|\alpha|} (\nabla^\alpha u)]) \\
&+ \sum_{\substack{1 \leq |\alpha| \leq [n/2] \\ |\alpha| \text{ even}}} \frac{c_\alpha}{\alpha!} (-\Delta)^{n/4} (\nabla^\alpha Q [(-\Delta)^{n/4-|\alpha|/2} u])
\end{aligned} \tag{58}$$

where $c_\alpha = \partial^{|\alpha|} |x|_{|x|=1}^{n/2}$.

Proposition A.1 *Let $u, Q \in \dot{H}^{n/2}(\mathbb{R}^n)$. Then $M_1(Q, u), M_2(Q, u) \in \dot{W}^{-n/2, (2,1)}(\mathbb{R}^n)$ and*

$$\|M_1(Q, u)\|_{\dot{W}^{-n/2, (2,1)}(\mathbb{R}^n)} \lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|(-\Delta)^{n/4} u\|_{L^2(\mathbb{R}^n)}; \tag{59}$$

$$\|M_2(Q, u)\|_{\dot{W}^{-n/2, (2,1)}(\mathbb{R}^n)} \lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|(-\Delta)^{n/4} u\|_{L^2(\mathbb{R}^n)}. \tag{60}$$

Proof of Proposition A.1. We prove only (59), since the estimate of (60) similar.

We recall that for $0 < s < n/2$ we have

$$\dot{H}^{n/2}(\mathbb{R}^n) \hookrightarrow \dot{W}^{s, (\frac{n}{s}, 2)}(\mathbb{R}^n),$$

(see for instance [19]).

Thus if $n = 2p + 1 > 1$, ($p \geq 1$), is an odd integer number and $0 < |\alpha| \leq [n/2]$ then $\nabla^\alpha u \in L^{(\frac{n}{|\alpha|}, 2)}$ and $((-\Delta)^{n/4-|\alpha|/2} (\nabla^\alpha Q)); ((-\Delta)^{n/4-|\alpha|/2} Q)$ belong to $L^{(\frac{n}{n/2-|\alpha|}, 2)}$. Thus by Hölder Inequality the following products

$$[(-\Delta)^{n/4-|\alpha|} (\nabla^\alpha Q)] \nabla^\alpha u, \quad [(-\Delta)^{n/4-|\alpha|/2} Q] \nabla^\alpha u$$

are in $L^{(2,1)}(\mathbb{R}^n)$ and

$$\begin{aligned}
\|[(-\Delta)^{n/4-|\alpha|} (\nabla^\alpha Q)] \nabla^\alpha u\|_{L^{(2,1)}} &\lesssim \|[(-\Delta)^{n/4-|\alpha|} (\nabla^\alpha Q)]\|_{L^{(\frac{n}{n/2-|\alpha|}, 2)}} \|\nabla^\alpha u\|_{L^{(\frac{n}{|\alpha|}, 2)}} \\
&\lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)},
\end{aligned}$$

$$\begin{aligned}
\|[(-\Delta)^{n/4-|\alpha|/2} Q] \nabla^\alpha u\|_{L^{(2,1)}} &\lesssim \|[(-\Delta)^{n/4-|\alpha|/2} Q]\|_{L^{(\frac{n}{n/2-|\alpha|}, 2)}} \|\nabla^\alpha u\|_{L^{(\frac{n}{|\alpha|}, 2)}} \\
&\lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)}.
\end{aligned}$$

It follows that $M_1(Q, u) \in \dot{W}^{-n/2, (2,1)}(\mathbb{R}^n)$ and (59) holds. This concludes the proof of Proposition A.1. \square

Next we prove Theorem 1.2.

Proof of Theorem 1.2. We group as follows :

$$\begin{aligned}\Pi_1[T_n^*(Q, u)] &= \underbrace{\Pi_1[(-\Delta)^{n/4}(((-\Delta)^{n/4}Q) u) - (-\Delta)^{n/2}(Qu)]}_{\text{Term 1}} + \underbrace{\Pi_1[(-\Delta)^{n/4}(Q((-\Delta)^{n/4}u))]}_{\text{Term 2}} \\ \Pi_2[T_n^*(Q, u)] &= \underbrace{\Pi_2[(-\Delta)^{n/4}(((-\Delta)^{n/4}Q) u)]}_{\text{Term 3}} + \underbrace{\Pi_2[-(-\Delta)^{n/2}(Qu) + (-\Delta)^{n/4}(Q((-\Delta)^{n/4}u))]}_{\text{Term 4}} \\ \Pi_3[T_n^*(Q, u)] &= \underbrace{\Pi_3[(-\Delta)^{n/4}(((-\Delta)^{n/4}Q) u)]}_{\text{Term 5}} - \underbrace{\Pi_3[(-\Delta)^{n/2}(Qu)]}_{\text{Term 6}} + \underbrace{\Pi_3[(-\Delta)^{n/4}(Q((-\Delta)^{n/4}u))]}_{\text{Term 7}}.\end{aligned}$$

Some terms appearing in $T_n^*(Q, u)$ satisfy a better estimate in the sense that they belong in \mathcal{H}^1 or in $\dot{B}_{1,1}^0$. We recall that $B_{1,1}^0 \hookrightarrow \mathcal{H}^1 \hookrightarrow \dot{W}^{-n/2, (2,1)}$.

- Estimate of $\|\Pi_1[(-\Delta)^{n/4}(Q(-\Delta)^{n/4}u)]\|_{\mathcal{H}^1(\mathbb{R}^n)}$.

$$\|\Pi_1[(-\Delta)^{n/4}(Q(-\Delta)^{n/4}u)]\|_{\mathcal{H}^1(\mathbb{R}^n)} \simeq \int_{\mathbb{R}^n} \left(\sum_j (2^{\frac{n}{2}j} Q_j(-\Delta)^{n/4}u^j)^2 \right)^{1/2} dx \quad (61)$$

$$\begin{aligned}&\lesssim \int_{\mathbb{R}^n} \sup_j [(-\Delta)^{n/4}u^j] \left(\sum_j 2^{nj} Q_j^2 \right)^{1/2} dx \\ &\lesssim \left(\int_{\mathbb{R}^n} (\sup_j (-\Delta)^{n/4}u^j)^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \sum_j 2^{nj} Q_j^2 dx \right)^{1/2} \\ &\lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)}.\end{aligned}$$

- Estimate of $\|\Pi_3[(-\Delta)^{n/4}(Q(-\Delta)^{n/4}u)]\|_{\dot{B}_{1,1}^0(\mathbb{R}^n)}$.

$$\|\Pi_3[(-\Delta)^{n/4}(Q(-\Delta)^{n/4}u)]\|_{\dot{B}_{1,1}^0(\mathbb{R}^n)} \quad (62)$$

$$\simeq \sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \int_{\mathbb{R}^n} \sum_j Q_j(-\Delta)^{n/4}u_j [(-\Delta)^{n/4}h^{j-6} + \sum_{t=j-5}^{j+6} (-\Delta)^{n/4}h_t]$$

by Lemma A.3 (63)

$$\begin{aligned}&\lesssim \sup_{\|h\|_{B_{\infty,\infty}^0} \leq 1} \|h\|_{B_{\infty,\infty}^0} \int_{\mathbb{R}^n} \sum_j 2^{\frac{n}{2}j} |Q_j| |(-\Delta)^{n/4}u_j| dx \\ &\lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)}.\end{aligned}$$

The estimate of $\Pi_3[(-\Delta)^{n/2}(Qu)]$, $\Pi_3[(-\Delta)^{n/4}(Q((-\Delta)^{n/4}u))]$, $\Pi_2[(-\Delta)^{n/4}(((-\Delta)^{n/4}Q) u)]$ are similar to (61) and (62) and we omit them.

- Estimate of $\|\Pi_1[(-\Delta)^{n/4}((-\Delta)^{n/4}Q)u - (-\Delta)^{n/2}(Qu)]\|_{\dot{W}^{-n/2,(2,1)}(\mathbb{R}^n)}$.

$$\|\Pi_1[(-\Delta)^{n/4}((-\Delta)^{n/4}Q)u - (-\Delta)^{n/2}(Qu)]\|_{\dot{W}^{-n/2,(2,1)}(\mathbb{R}^n)} \quad (64)$$

$$\begin{aligned} &= \sup_{\|h\|_{\dot{W}^{n/2,(2,\infty)}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j| \leq 3} [(-\Delta)^{n/4}((-\Delta)^{n/4}Q_j u^{j-4}) - (-\Delta)^{n/2}(Q_j u^{j-4})] h_t dx \\ &= \sup_{\|h\|_{\dot{W}^{n/2,(2,\infty)}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j| \leq 3} \mathcal{F}[u^{j-4}] \mathcal{F}[\Delta^{n/4}Q_j(-\Delta)^{n/4}h_t - Q_j(-\Delta)^{n/2}h_t] d\xi \\ &= \sup_{\|h\|_{\dot{W}^{n/2,(2,\infty)}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_j \sum_{|t-j| \leq 3} \mathcal{F}[u^{j-4}](\xi) \\ &\quad \left(\int_{\mathbb{R}^n} \mathcal{F}[Q_j](\zeta) \mathcal{F}[(-\Delta)^{n/4}h_t](\xi - \zeta) (|\zeta|^{n/2} - |\xi - \zeta|^{n/2}) d\zeta \right) d\xi. \end{aligned}$$

Now we observe that in (64) we have $|\xi| \leq 2^{j-3}$ and $2^{j-2} \leq |\eta| \leq 2^{j+2}$. Thus $|\frac{\xi}{\zeta}| \leq \frac{1}{2}$.

Hence

$$\begin{aligned} &|\zeta|^{n/2} - |\xi - \zeta|^{n/2} = |\xi|^{n/2} \left[1 - \left| \frac{\xi}{|\zeta|} - \frac{\zeta}{|\zeta|} \right|^{n/2} \right] \\ &= |\zeta|^{n/2} \left[\sum_{\substack{|\alpha| \geq 1 \\ |\alpha| \text{ odd}}} \frac{c_\alpha}{\alpha!} \left(\frac{\xi}{|\zeta|} \right)^\alpha \left(\frac{\zeta}{|\zeta|} \right)^\alpha + \sum_{\substack{|\alpha| \geq 2 \\ |\alpha| \text{ even}}} \frac{c_\alpha}{\alpha!} \left(\frac{\xi}{|\zeta|} \right)^\alpha \right]. \end{aligned}$$

We may suppose the series in (65) is convergent if $|\frac{\xi}{|\zeta|}| \leq \frac{1}{2}$, otherwise one may consider a different Littlewood-Paley decomposition by replacing the exponent $j - 4$ with $j - s$, $s > 4$ large enough.

Unlike the case $n = 1$ (see the proof of estimate (35) in [5]) we need to separate two cases: $|\alpha| \geq [n/2] + 1$ and $1 \leq |\alpha| \leq [n/2]$.

Case 1: $|\alpha| \geq [n/2] + 1$. Here we use the fact that $\dot{W}^{n/2,(2,\infty)}(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^0(\mathbb{R}^n)$ and

the crucial property that for every vector field $X \in \dot{H}^{n/2}(\mathbb{R}^n)$ we have

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} 2^{-jn} (X^j)^2 dx &= \int_{\mathbb{R}^n} \sum_{k,\ell} X_k X_\ell \sum_{j-4 \geq k, j-4 \geq \ell} 2^{-jn} dx \\ &\simeq \int_{\mathbb{R}^n} \sum_k X_k \left(\sum_{|k-\ell| \leq 2} X_\ell \right) 2^{-(k-2)n} dx \end{aligned}$$

by Cauchy-Schwarz Inequality

$$\begin{aligned} &\lesssim \int_{\mathbb{R}^n} \left(\sum_k 2^{-kn} X_k^2 \right)^{1/2} \left(\sum_k 2^{-kn} X_k^2 \right)^{1/2} dx \\ &= \int_{\mathbb{R}^n} \sum_{j=-\infty}^{+\infty} 2^{-kn} (X_k)^2 dx, \end{aligned} \tag{65}$$

(see also Section 4.4.2 in [15], page 165).

We are going to estimate

$$\begin{aligned} &\sup_{\|h\|_{\dot{W}^{n/2, (2, \infty)}} \leq 1} \left[\sum_{\substack{|\alpha| \geq [n/2]+1 \\ |\alpha| \text{ odd}}} \frac{c_\alpha}{\alpha!} \int_{\mathbb{R}^n} \sum_j \nabla^\alpha u^{j-4} (-\Delta)^{n/4-|\alpha|} (\nabla^\alpha Q_j) (-\Delta)^{n/4} h_j dx \right. \\ &\quad \left. + \sum_{\substack{|\alpha| \geq [n/2]+1 \\ |\alpha| \text{ even}}} \frac{c_\alpha}{\alpha!} \int_{\mathbb{R}^n} \sum_j |\nabla^\alpha u^{j-4} (-\Delta)^{n/4-|\alpha|/2} (Q_j) (-\Delta)^{n/4} h_j dx \right]. \end{aligned} \tag{66}$$

By applying Lemma A.3 ($\|(-\Delta)^{n/4} h_j\|_{\dot{B}_{\infty, \infty}^0} \lesssim 2^{\frac{nj}{2}} 4^{n/2} \|h\|_{\dot{B}_{\infty, \infty}^0}$) we get

$$\begin{aligned}
(66) &\lesssim \sup_{\|h\|_{\dot{W}^{n/2, (2, \infty)}} \leq 1} \|h\|_{\dot{B}_{\infty, \infty}^0} \\
& \left[\sum_{\substack{|\alpha| \geq [n/2] + 1 \\ |\alpha| \text{ odd}}} \frac{C_\alpha}{\alpha!} 2^n \int_{\mathbb{R}^n} \sum_j 2^{\frac{nj}{2}} |\nabla^\alpha u^{j-4}| |(-\Delta)^{n/4 - |\alpha|} (\nabla^\alpha Q_j)| dx \right. \\
& + \sum_{\substack{|\alpha| \geq [n/2] + 1 \\ |\alpha| \text{ even}}} \frac{C_\alpha}{\alpha!} 2^n \int_{\mathbb{R}^n} \sum_j 2^{\frac{nj}{2}} |\nabla^\alpha u^{j-4}| |(-\Delta)^{n/4 - |\alpha|/2} (Q_j)| dx \left. \right] \\
&\lesssim \sup_{\|h\|_{\dot{W}^{n/2, (2, \infty)}} \leq 1} \|h\|_{\dot{B}_{\infty, \infty}^0} \\
& \left[\sum_{\substack{|\alpha| \geq [n/2] + 1 \\ |\alpha| \text{ odd}}} \frac{C_\alpha}{\alpha!} 2^{3n-4|\alpha|} \right. \tag{67} \\
& \left(\int_{\mathbb{R}^n} \sum_j 2^{(n-2|\alpha|)(j-4)} |\nabla^\alpha u^{j-4}|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \sum_j 2^{2|\alpha|j} |(-\Delta)^{n/4 - |\alpha|} (\nabla^\alpha Q_j)|^2 dx \right)^{1/2} \left. \right] \\
& + \left[\sum_{\substack{|\alpha| \geq [n/2] + 1 \\ |\alpha| \text{ even}}} \frac{C_\alpha}{\alpha!} 2^{3n-4|\alpha|} \right. \\
& \left(\int_{\mathbb{R}^n} \sum_j 2^{(n-2|\alpha|)(j-4)} |\nabla^\alpha u^{j-4}|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \sum_j 2^{2|\alpha|j} |(-\Delta)^{n/4 - |\alpha|/2} (Q_j)|^2 dx \right)^{1/2} \left. \right] \\
&\lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)}.
\end{aligned}$$

Case 2: $1 \leq |\alpha| \leq [n/2]$. In this case we apply Proposition A.1.

We have:

$$\begin{aligned}
& \sup_{\|h\|_{\dot{W}^{n/2, (2, \infty)}} \leq 1} \\
& \left[\sum_{\substack{1 \leq |\alpha| \leq [n/2] \\ |\alpha| \text{ odd}}} \frac{c_\alpha}{\alpha!} \int_{\mathbb{R}^n} \sum_j \nabla^\alpha u^{j-4} (-\Delta)^{n/4-|\alpha|} (\nabla^\alpha Q_j) (-\Delta)^{n/4} h_j dx \right. \\
& \left. + \sum_{\substack{1 \leq |\alpha| \leq [n/2] \\ |\alpha| \text{ even}}} \frac{c_\alpha}{\alpha!} \int_{\mathbb{R}^n} \sum_j |\nabla^\alpha u^{j-4} (-\Delta)^{n/4-|\alpha|/2} (Q_j) (-\Delta)^{n/4} h_j dx \right] \\
& \lesssim \sup_{\|h\|_{\dot{W}^{n/2, (2, \infty)}} \leq 1} \|(-\Delta)^{n/4} h\|_{L^{(2, \infty)}} \\
& \left[\sum_{\substack{1 \leq |\alpha| \leq [n/2] \\ |\alpha| \text{ odd}}} \frac{c_\alpha}{\alpha!} \|(-\Delta)^{n/4-|\alpha|} (\nabla^\alpha Q)\|_{L^{(\frac{n}{n/2-|\alpha|}, 2)}} \|\nabla^\alpha u\|_{L^{(\frac{n}{|\alpha|}, 2)}} \right. \\
& \left. + \left[\sum_{\substack{1 \leq |\alpha| \leq [n/2] \\ |\alpha| \text{ even}}} \frac{c_\alpha}{\alpha!} \|(-\Delta)^{n/4-|\alpha|/2} Q\|_{L^{(\frac{n}{n/2-|\alpha|}, 2)}} \|\nabla^\alpha u\|_{L^{(\frac{n}{|\alpha|}, 2)}} \right] \right] \\
& \lesssim \sup_{\|h\|_{\dot{W}^{n/2, (2, \infty)}} \leq 1} \|(-\Delta)^{n/4} h\|_{L^{(2, \infty)}} \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \\
& \lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|u\|_{\dot{H}^{n/2}(\mathbb{R}^n)}.
\end{aligned}$$

The estimate of $\Pi_2[(-\Delta)^{n/4}(Q((-\Delta)^{n/4}u)) - (-\Delta)^{n/2}(Qu)]$ is analogous to (64) and we omit it. This concludes the proof of Theorem 1.2. \square

The next result permits us to estimate the right hand side of equation (31).

We denote by r' the conjugate of $1 < r < +\infty$.

Theorem A.1 *Let $n > 2$, $1 < r < \frac{2n}{n-2}$, $h \in L^{r'}(\mathbb{R}^n)$, $Q \in \dot{H}^{n/2}(\mathbb{R}^n)$. Then*

$$(-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qh) - Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h \in \dot{W}^{-(\frac{n}{2}-1), r'}(\mathbb{R}^n), \quad (68)$$

and

$$\|(-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qh) - Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h\|_{\dot{W}^{-(\frac{n}{2}-1), r'}(\mathbb{R}^n)} \lesssim \|h\|_{L^r} \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)}. \quad (69)$$

Theorem A.1 implies “by duality” the following result.

Theorem A.2 *Let $n > 2$, $1 < r < \frac{2n}{n-2}$, $Q \in \dot{H}^{n/2}(\mathbb{R}^n)$, $f \in \dot{W}^{\frac{n}{2}-1,r}(\mathbb{R}^n)$ then*

$$Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}f - (-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qf) \in L^r(\mathbb{R}^n), \quad (70)$$

and

$$\|Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}f - (-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qf)\|_{L^r(\mathbb{R}^n)} \lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|f\|_{\dot{W}^{\frac{n}{2}-1,r}}. \quad \square \quad (71)$$

Proof of Theorem A.1. Throughout the proof we use the following embeddings:

$$\dot{W}^{\frac{n}{2}-1,r}(\mathbb{R}^n) \hookrightarrow L^s(\mathbb{R}^n), \quad \frac{1}{s} = \frac{1}{r} - \frac{\frac{n}{2}-1}{n}; \quad (72)$$

$$\dot{H}^{n/2}(\mathbb{R}^n) \hookrightarrow \dot{W}^{\frac{n}{2}-1,(q,2)}(\mathbb{R}^n) \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{n} = \frac{\frac{n}{2}-1}{n}. \quad (73)$$

We also use the fact that

$$\frac{1}{r'} + \frac{1}{s} + \frac{1}{q} = 1. \quad (74)$$

• **Estimate of $\|\Pi_1[(-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qh)]\|_{\dot{W}^{-(\frac{n}{2}-1),r'}(\mathbb{R}^n)}$.**

$$\begin{aligned} & \|\Pi_1[(-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qh)]\|_{\dot{W}^{-(\frac{n}{2}-1),r'}(\mathbb{R}^n)} \\ & \simeq \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_j Q_j h^j (-\Delta)^{\frac{n}{4}-\frac{1}{2}} g_j dx \\ & \lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \underbrace{\sup_j h^j}_{\in L^{r'}} \underbrace{\left(\sum_j 2^{2(\frac{n}{2}-1)j} Q_j^2\right)^{1/2}}_{\in L^q} \underbrace{\left(\sum_j 2^{-2(\frac{n}{2}-1)j} ((-\Delta)^{\frac{n}{4}-\frac{1}{2}} g_j)^2\right)^{1/2}}_{\in L^s} dx \end{aligned}$$

by (74)

$$\begin{aligned} & \lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \|h\|_{L^{r'}} \|(-\Delta)^{\frac{n}{4}-\frac{1}{2}} Q\|_{L^q} \|g\|_{L^s} \\ & \lesssim \|h\|_{L^{r'}} \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)}. \end{aligned} \quad (75)$$

• **Estimate of $\|\Pi_1[Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h]\|_{\dot{W}^{-(\frac{n}{2}-1),r'}(\mathbb{R}^n)}$.**

$$\|\Pi_1[Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h]\|_{\dot{W}^{-(\frac{n}{2}-1),r'}(\mathbb{R}^n)} \simeq \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_j Q_j(-\Delta)^{\frac{n}{4}-\frac{1}{2}} h^j g_j dx \quad (76)$$

$$\lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_j \underbrace{\sup_j 2^{(\frac{1}{2}-\frac{n}{2})j} (-\Delta)^{\frac{n}{4}-\frac{1}{2}} h^j}_{\in L^{r'}} \underbrace{\left(\sum_j 2^{2(\frac{n}{2}-1)j} Q_j^2\right)^{1/2}}_{\in L^q} \underbrace{\left(\sum_j g_j^2\right)^{1/2}}_{\in L^s} dx$$

by (74)

$$\lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \|h\|_{L^{r'}} \|(-\Delta)^{\frac{n}{4}-\frac{1}{2}} Q\|_{L^q} \|g\|_{L^s}$$

$$\lesssim \|h\|_{L^{r'}} \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)}.$$

• **Estimate of** $\|\Pi_3[(-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qh)]\|_{\dot{W}^{-(\frac{n}{2}-1),r'}(\mathbb{R}^n)}$.

$$\|\Pi_3[(-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qh)]\|_{\dot{W}^{-(\frac{n}{2}-1),r'}(\mathbb{R}^n)}$$

$$\simeq \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_j Q_j h_j (-\Delta)^{\frac{n}{4}-\frac{1}{2}} g^j dx$$

$$\lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sup_j 2^{-(\frac{n}{2}-1)j} ((-\Delta)^{\frac{n}{4}-\frac{1}{2}} g^j) \sum_j 2^{(\frac{n}{2}-1)j} Q_j h_j dx \quad (77)$$

by (74)

$$\lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \|h\|_{L^{r'}} \|(-\Delta)^{\frac{n}{4}-\frac{1}{2}} Q\|_{L^q} \|g\|_{L^s}$$

$$\lesssim \|h\|_{L^{r'}} \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)}.$$

The estimates of $\Pi_1[Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h]$ and $\Pi_3[Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h]$ are similar to (75) and (77) and we omit them.

• **Estimate of** $\|\Pi_2[(-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qh) - Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h]\|_{\dot{W}^{-(\frac{n}{2}-1),r'}(\mathbb{R}^n)}$.

We denote by \tilde{c}_α the coefficients of the Taylor expansion of $|x|^{\frac{n}{2}-1}$ at $x = 1$.

$$\begin{aligned}
& \|\Pi_2[(-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qh) - Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h]\|_{\dot{W}^{-(\frac{n}{2}-1),r'}(\mathbb{R}^n)} \\
& \simeq \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \left((-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Q^j h_j) - Q^j(-\Delta)^{\frac{n}{4}-\frac{1}{2}}h_j \right) g_j dx. \quad (78)
\end{aligned}$$

Now we argue as in (64) and we get

$$\begin{aligned}
(78) & \lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \left[\sum_{\substack{1 \leq |\alpha| \\ |\alpha| \text{ odd}}} \frac{\tilde{C}_\alpha}{\alpha!} \int_{\mathbb{R}^n} \sum_j \nabla^\alpha Q^{j-4}(-\Delta)^{\frac{n}{4}-\frac{1}{2}-|\alpha|}(\nabla^\alpha h_j) g_j dx \right. \\
& \left. + \sum_{\substack{1 \leq |\alpha| \\ |\alpha| \text{ even}}} \frac{\tilde{C}_\alpha}{\alpha!} \int_{\mathbb{R}^n} \sum_j |\nabla^\alpha Q^{j-4}(-\Delta)^{\frac{n}{4}-\frac{1}{2}-|\alpha|/2}(h_j) g_j dx \right]
\end{aligned}$$

by Lemma A.3

$$\begin{aligned}
& \lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \|Q\|_{B_{\infty,\infty}^0} \\
& \left[\sum_{1 \leq |\alpha|} \frac{\tilde{C}_\alpha}{\alpha!} 2^{-2|\alpha|} \int_{\mathbb{R}^n} 2^{|\alpha|j} 2^{-(\frac{n}{2}-1)j} |(-\Delta)^{\frac{n}{4}-\frac{1}{2}-|\alpha|/2}(h_j)| 2^{(\frac{n}{2}-1)j} |g_j| dx \right] \\
& \lesssim \sup_{\|g\|_{\dot{W}^{(\frac{n}{2}-1),r}(\mathbb{R}^n)} \leq 1} \|Q\|_{B_{\infty,\infty}^0} \|h\|_{L^{r'}} \|(-\Delta)^{\frac{n}{4}-\frac{1}{2}}g\|_{L^r} \\
& \lesssim \|Q\|_{\dot{H}^{n/2}} \|h\|_{L^{r'}}. \quad \square
\end{aligned}$$

Since $\frac{2n}{n-2} > 2$ we can now apply the interpolation Theorem 3.3.3 in [8] and obtain the following:

Corollary A.1 *Let $n > 2$, $Q \in \dot{H}^{n/2}(\mathbb{R}^n)$, $f \in \dot{H}^{\frac{n}{2}-1}(\mathbb{R}^n)$ then*

$$Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}f - (-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qf) \in L^{(2,\infty)}(\mathbb{R}^n), \quad (79)$$

and

$$\|Q(-\Delta)^{\frac{n}{4}-\frac{1}{2}}f - (-\Delta)^{\frac{n}{4}-\frac{1}{2}}(Qf)\|_{L^{(2,\infty)}(\mathbb{R}^n)} \lesssim \|Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)} \|f\|_{\dot{W}^{\frac{n}{2}-1,(2,\infty)}}. \quad (80)$$

We finally recall a commutator estimate obtained in [3] .

Lemma A.4 *Let $p > 1$, $Q \in BMO(\mathbb{R}^n)$, $u \in L^p(\mathbb{R}^n)$ and let $\mathcal{P}^{(4)}$ a pseudo-differential operator of order zero. Then $\mathcal{P}(Qu) - Q\mathcal{P}u \in L^p(\mathbb{R}^n)$ and*

$$\|\mathcal{P}(Qu) - Q\mathcal{P}u\|_{L^p(\mathbb{R}^n)} \lesssim \|Q\|_{BMO(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)}. \quad \square$$

The interpolation Theorem 3.3.3 in [8], and Lemma A.4 imply the following result.

Corollary A.2 *Let $Q \in BMO(\mathbb{R}^n)$, $u \in L^{(2,\infty)}(\mathbb{R}^n)$ and let \mathcal{P} a pseudo-differential operator of order zero. Then $\mathcal{P}(Qu) - Q\mathcal{P}u \in L^{2,\infty}(\mathbb{R}^n)$ and*

$$\|\mathcal{P}(Qu) - Q\mathcal{P}u\|_{L^{(2,\infty)}(\mathbb{R}^n)} \lesssim \|Q\|_{BMO(\mathbb{R}^n)} \|u\|_{L^{(2,\infty)}(\mathbb{R}^n)} \quad \square$$

We observe that Corollary A.2 implies that for every $h \in L^{(2,1)}(\mathbb{R}^n)$, $u \in L^{(2,\infty)}(\mathbb{R}^n)$ the operator $u\mathcal{P}h - (\mathcal{P}u)h \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|u\mathcal{P}h - (\mathcal{P}u)h\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim \|u\|_{L^{(2,\infty)}(\mathbb{R}^n)} \|h\|_{L^{(2,1)}(\mathbb{R}^n)}. \quad (81)$$

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⁽⁴⁾We recall that a pseudo-differential operator \mathcal{P} can be formally defined as

$$\mathcal{F}[\mathcal{P}f(x)] = \sigma(x, \xi)\mathcal{F}[f],$$

where σ , the symbol of \mathcal{P} , is a complex-valued function defined $\mathbb{R}^n \times \mathbb{R}^n$. If $\sigma(x, \xi) = m(\xi)$ is independent of x , then \mathcal{P} is the Fourier multiplier associated with m . Given $k \in \mathbb{Z}$ we say that σ is of order k if for every multi-indexes $\beta, \alpha \in \mathbb{N}^n$

$$|D^\beta D_\xi^\alpha \sigma(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{k-\alpha}.$$

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