Lecture n°1 — Motivations

Florian Feppon

October 5, 2021

The story starts with the wave equation in the heterogeneous domain. Let D be a smooth bounded domain with density ρ_b and bulk modulus κ_b. The complement of D is an acoustic material of density ρ and bulk modulus κ. The wave equation in such material reads in the whole space

$$\frac{1}{\kappa(x)}\frac{\partial^2 u}{\partial^2 t} - \operatorname{div}\left(\frac{1}{\rho(x)}\nabla u\right) = 0 \text{ in } \mathbb{R}^d, \tag{1}$$

where

$$\kappa(x) = \begin{cases} \kappa_b \text{ if } x \in D, \\ \kappa \text{ if } x \in \mathbb{R}^d \setminus D, \end{cases} \qquad \rho(x) = \begin{cases} \rho_b \text{ if } x \in D, \\ \rho \text{ if } x \in \mathbb{R}^d \setminus D. \end{cases}$$

One of the objectives of the lecture is to understand the resonances of the system (1) in the high contrast regime

$$\delta := \frac{\rho_b}{\rho} \to 0. \tag{2}$$

Minnaert was the first to demonstrate physically that air bubbles [4] have the property to interact with waves whose wavelength is several order of magnitudes of the size of the bubble. Such result can be obtained by a thorough asymptotic analysis of (1) in the low frequency regime. See

`https://www.college-de-france.fr/site/en-pierre-louis-lions/seminar-2017-06-02-11h15.htm`

for a presentation on Minnaert resonnaces (in French) including a demonstrating experiment.

1 Derivation of the Helmholtz equation

In order to study the behavior of (1) at a given frequency ω, one considers an incident wave $u_{\text{in}}(t,x) = e^{\mathrm{i}\omega t}u_{\text{in}}(x)$, oscillating at the frequency t. Since u_{in} comes from the far field, we can consider it solves

$$\frac{1}{\kappa}\frac{\partial^2 u_{\text{in}}}{\partial^2 t} - \frac{1}{\rho}\Delta u = 0 \text{ in } \mathbb{R}^d.$$

Inserting $u_{\text{in}}(t,x) = e^{\mathrm{i}\omega t}u_{\text{in}}(x)$ implies that u_{in} solves the Helmholtz equation in $\mathbb{R}^d$:

$$\Delta u_{\text{in}} + k^2 u_{\text{in}} = 0 \text{ in } \mathbb{R}^d$$

with

$$k := \omega\sqrt{\frac{\rho}{\kappa}}.$$

The quantity

$$v := \sqrt{\frac{\kappa}{\rho}}$$

is homogeneous to a velocity and is the speed of the sound in the medium $\mathbb{R}^d \setminus D$. When the wave u_{in} encounters the heterogeneity D, it generates a scattered field $u_s(t,x) := u(t,x) - u_{\text{in}}(t,x)$ which oscillates at the same frequency ω. Inserting $u(t,x) = e^{\mathrm{i}\omega t}u(x)$ in (1), we find that u satisfies

$$\operatorname{div}\left(\frac{1}{\rho(x)}\nabla u\right) + \frac{\omega^2}{\kappa(x)}u = 0 \text{ in } \mathbb{R}^d.$$

This equation holds in a distributional sense, which means more precisely

$$\begin{cases} \Delta u + k_b^2 u = 0 \text{ in } D, \\ \Delta u + k^2 u = 0 \text{ in } \mathbb{R}^d \setminus D, \\ \left.\dfrac{1}{\rho}\dfrac{\partial u}{\partial n}\right|_+ = \left.\dfrac{1}{\rho_b}\dfrac{\partial u}{\partial n}\right|_- \text{ on } \partial D, \\ \quad u|_+ = u|_- \text{ on } \partial D, \end{cases} \tag{3}$$

where we have denoted

$$k_b := \omega\sqrt{\frac{\rho}{\kappa}}.$$

Furthermore, the scattered field $u_s := u - u_{\text{in}}$ must be *outgoing*, which can be mathematically formulated as the *Sommerfeld radiation condition*:

$$\left(\frac{\partial}{\partial|x|} - \mathrm{i}k\right)(u - u_{\text{in}}) = O\left(|x|^{-(d+1)/2}\right) \text{ as } |x| \to +\infty. \tag{4}$$

We can prove that the formulation (3) and (4) admits a unique solution. It is useful to reformulate the third line of (3) in terms of the contrast parameter δ of (2):

$$\left.\frac{\partial u}{\partial n}\right|_- = \delta \left.\frac{\partial u}{\partial n}\right|_+ \text{ on } \partial D. \tag{5}$$

2 Layer potentials

As we will see generically in this lecture, the asymptotic analysis of (3) in terms of the parameter δ is possible once he have an (rather) explicit representation of the solution u in terms of the parameter δ. Such is possible by the use of *layer potentials*.

Another method to obtain asymptotic expansions in parameter dependent partial differential equations is the method of matched asymptotic expansions, see e.g. [2].

Layer potentials exploit the knowledge of the fundamental solution of a differential operator. In fact, there may be infinitely many fundamental solutions for a given differental operator. For the Helmholtz equation, a fundamental solution Γ^k satisfies

$$(\Delta + k^2)\Gamma^k = \delta_0$$

in the sense of distributions. The physically relevant one is the outgoing fundamental solution given by

$$\Gamma^k(x) := \frac{k^{\frac{d}{2}-1}}{4(2\pi)^{d/2-1}} \frac{Y_{d/2-1}(k|x|) - \mathrm{i}J_{d/2-1}(k|x|)}{|x|^{d/2-1}}$$

where $J_{d/2-1}$ and $Y_{d/2-1}$ are the Bessel functions of the first and second kind of order $(d/2-1)$, see [3] for a derivation. In dimensions 2 and 3, Γ^k is given by

$$\Gamma^k(x) = \begin{cases} -\dfrac{\mathrm{i}}{4} H_0^{(1)}(k|x|), & \text{if } d = 2, \\ -\dfrac{e^{\mathrm{i}k|x|}}{4\pi|x|}, & \text{if } d = 3. \end{cases}$$

where

$$H_0^{(1)}(k|x|) = J_0 + \mathrm{i}Y_0.$$

is the Hankel function of the first kind. The definition of the single and double layer potentials are motivated by the following result.

Proposition 1. *Let u a function satisfying*

$$\begin{cases} (\Delta + k^2)u = 0 \ \text{in}\ \mathbb{R}^d \setminus \partial D \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right) u = O\left(|x|^{-(d+1)/2}\right) \ \text{as}\ |x| \to +\infty. \end{cases} \tag{6}$$

and $D \subset \mathbb{R}^d$ a smooth domain of $\mathbb{R}^d$. Then for any $x \in \mathbb{R}^d \setminus \partial D$, $u(x)$ can be expressed in terms of its trace and its normal derivative on ∂D:

$$\forall x \in \mathbb{R}^d \setminus \partial D, \quad u(x) = \int_{\partial D} \Gamma^k(x-y) \left[\!\!\left[\frac{\partial u}{\partial n}(y) \right]\!\!\right] \mathrm{d}\sigma(y) - \int_{\partial D} \nabla_y \Gamma^k(x-y) \cdot \boldsymbol{n}(y) [\![u(y)]\!] \, \mathrm{d}\sigma(y), \tag{7}$$

where $[\![\cdot]\!]$ denotes the jump accross the surface ∂D:

$$[\![u]\!] := u|_+ - u|_-, \quad \left[\!\!\left[\frac{\partial u}{\partial n} \right]\!\!\right] = \left.\frac{\partial u}{\partial n}\right|_+ - \left.\frac{\partial u}{\partial n}\right|_- = \nabla u|_+ \cdot \boldsymbol{n} - \nabla u|_- \cdot \boldsymbol{n} \ \text{on}\ \partial D,$$

and where u_+ and u_- denote the outer and inner traces of a function u:

$$u|_+(x_0) = \lim_{\substack{s\to 0 \\ s>0}} u(x_0 + s\boldsymbol{n}(x_0)), \quad u|_-(x_0) = \lim_{\substack{s\to 0 \\ s>0}} u(x_0 - s\boldsymbol{n}(x_0)), \qquad x_0 \in \partial D.$$

Proof. Multiply (4) by the fundamental solution and integrate formally by parts, by neglecting the behavior at infinity (this can be justified thanks to the Sommerfeld radiation condition):

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} ((\Delta + k^2)u(y))\Gamma^k(x-y)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} u(y)((\Delta + k^2)\Gamma^k(x-y))\mathrm{d}y - \int_{\partial D} \left[\!\!\left[\frac{\partial u}{\partial n} \right]\!\!\right] \Gamma^k(x-y)\mathrm{d}\sigma(y) \\ &\quad + \int_{\partial D} [\![u]\!] \nabla_y \Gamma^k(x-y) \cdot \boldsymbol{n}(y)\mathrm{d}\sigma(y). \end{aligned}$$

The result follows because

$$\int_{\mathbb{R}^d} u(y)(\Delta + k^2)\Gamma^k(x-y)\mathrm{d}y = u(x).$$

□

Remark 1. *It is useful to remember the identity*

$$\int_D \Delta u v \mathrm{d}x = \int_D u \Delta v \mathrm{d}x + \int_{\partial D} \left(\frac{\partial u}{\partial n} v - u \frac{\partial v}{\partial n} \right) \mathrm{d}\sigma.$$

Remark 2. *In (7), the notation $\nabla_y \Gamma^k(x-y)$ means $\nabla_y(\Gamma^k(x-y))$ and not $-(\nabla_y \Gamma^k)(x-y)$.*

Remark 3. *In many text books, one consider rather the fundamental solution of $-\Delta - k^2$ which induces a minus sign on Γ^k. Then the formula (7) is obtained with an opposite sign. This convention induces some variations in the definitions and properties of the layer potentials. I recommend that these can be retrieved quickly from formal arguments to avoid mistakes.*

Equation (7) shows that one can retrieve the values of $u(x)$ everywhere in the space $\mathbb{R}^d \setminus \partial D$ as soon as one knows the values of the jumps of u and $\frac{\partial u}{\partial n}$ on the interface ∂D. This is at the basis of the method of integral equations, where we seek to reduce the problem (4) in the infinite space on a problem set on the bounded surface ∂D, where the jumps become the unknown of the problem. To achieve this, one needs to take the limit $x \to x_0$ for a given $x_0 \in \partial D$ in (7), which motivates the definition of the following surface operators.

Definition 1. *Assume D to be a smooth bounded domain. The single layer potential is the operator $\mathcal{S}_D^k$ defined for a function $\phi \in L^2(\partial D)$ on the boundary ∂D by:*

$$\mathcal{S}_D^k[\phi](t) := \int_{\partial D} \Gamma^k(t-t')\phi(t')\mathrm{d}\sigma(t'), \qquad t \in \partial D.$$

The Neumann-Poincaré operator is the operator $\mathcal{K}_D$ defined for a function ϕ defined on the boundary ∂D by:

$$\mathcal{K}_D^k[\phi](t) := \int_{\partial D} \boldsymbol{n}(t') \cdot \nabla_{t'}\Gamma^k(t-t')\phi(t')\mathrm{d}\sigma(t'), \qquad t \in \partial D,$$

where the integral is a convergent because the smoothness of D implies the existence of a constant $c > 0$ such that

$$\boldsymbol{n}(t') \cdot \frac{t-t'}{|t-t'|} \leqslant c|t-t'| \text{ uniformly in } t, t' \in \partial D.$$

Remark 4. *When D is a Lipschitz domain, it is still possible to define the Neumann-Poincaré operator as a Cauchy principal value integral:*

$$\mathcal{K}_D^k[\phi](t) := \mathrm{p.v.}\int_{\partial D} \boldsymbol{n}(t') \cdot \nabla_{t'}\Gamma^k(t-t')\phi(t')\mathrm{d}\sigma(t') = \lim_{\epsilon \to 0}\int_{\partial D \setminus B(t,\epsilon)} \boldsymbol{n}(t') \cdot \nabla_{t'}\Gamma^k(t-t')\phi(t')\mathrm{d}\sigma(t'),$$

where $B(t,\epsilon)$ is the ball of center t and radius ϵ. It turns out that the above limit exists, although $\boldsymbol{n} \cdot \nabla_y \Gamma^k(t-\cdot)$ may not be integrable on the surface ∂D.

Remark 5. *In view of the preceding remark, the discretization of the Neumann Poincaré operator proposed by [1, section 2.4.5.1] relies on the smoothness of the boundary, and not on the computation of a Cauchy principal value integral.*

These operators are not to be confused with the single and double layer potentials which are defined in the exterior domain $\mathbb{R}^d \setminus D$.

Definition 2. *We also call "single layer potential" the operator $\mathcal{S}_D^k$ mapping a function $\phi \in L^2(\partial D)$ to a function of $\mathbb{R}^d \setminus D$ and defined by*

$$\mathcal{S}_D^k[\phi](x) := \int_{\partial D} \Gamma^k(x-t')\phi(t')\mathrm{d}\sigma(t'), \qquad x \in \mathbb{R}^d \setminus \partial D.$$

The double-layer potential is the operator denoted by $\mathcal{D}_D^k$ which maps a function $\phi \in L^2(\partial D)$ to a function of $\mathbb{R}^d \setminus D$ given by

$$\mathcal{D}_D^k[\phi](x) := \int_{\partial D} \boldsymbol{n}(y) \cdot \nabla_y \Gamma^k(x-y)\phi(y)\mathrm{d}\sigma(y), \qquad x \in \mathbb{R}^d \setminus \partial D.$$

In order to reduce a partial differential equation posed on $\mathbb{R}^d \setminus D$ to an integral equation for a function defined only on the boundary ∂D, we need to take the limit as $x \to x_0$ with $x_0 \in \partial D$ and $x \in \mathbb{R}^3 \setminus \partial D$ of $\mathcal{S}_D^k[\phi]$, $\mathcal{D}_D^k[\phi]$ or their normal derivatives. The most useful jump identities are summarized in the following proposition.

Proposition 2 (Jump relations)**.**

1. *The single layer potential is continuous accross the interface ∂D:*

$$\mathcal{S}_D^k[\phi]\big|_\pm = \mathcal{S}_D^k[\phi].$$

2. *The double layer potential is discontinuous accross the interface ∂D, and we have*

$$\mathcal{D}_D^k[\phi]\big|_\pm = \mp\frac{1}{2}\mathrm{I} + \mathcal{K}_D^k \tag{8}$$

3. *The normal derivative of the single layer potential is discontinuous accross the interface ∂D, and it holds:*

$$\frac{\partial}{\partial n}\mathcal{S}_D^k\bigg|_\pm = \pm\frac{1}{2}\mathrm{I} + \mathcal{K}_D^{k*}, \tag{9}$$

where $\mathcal{K}_D^{k*}$ is the adjoint of the Neumann-Poincaré operator and is given (when D is a smooth domain) by

$$\mathcal{K}_D^{k*}[\phi](t) = \int_{\partial D} \boldsymbol{n}(t) \cdot \nabla_t \Gamma^k(t-t')\phi(t')\mathrm{d}\sigma(t').$$

Proof. 1. The continuity is due to the fact that the singularity of Γ^k is Lebesgue integrable on ∂D. For instance, for a small h and if $d = 3$, there exists a uniform constant $C > 0$ such that

$$\left| \frac{1}{4\pi|x_0 + h - t|} - \frac{1}{4\pi|x_0 - t|} \right| \leqslant C \frac{1}{4\pi|x_0 - t|}$$

for any $h \in \mathbb{R}^3$ small enough. By the Lebesgue dominated convergence theorem, this shows that $h \to 1/(4\pi|x_0 + h - t|)$ is continuous and the result.

2. We need to handle the singularity. The "trick" is to decompose $\mathcal{D}_D^k[\phi]_\pm$ to bring cancellations of the singularities:

$$\begin{aligned}
\mathcal{D}_D^k[\phi](x) &= \int_{\partial D} \boldsymbol{n}(y) \cdot \nabla_y (\Gamma^k(x-y) - \Gamma^k(x_0 - y))(\phi(y) - \phi(x_0)) \mathrm{d}\sigma(y) \\
&+ \int_{\partial D} \boldsymbol{n}(y) \cdot \nabla_y \Gamma^k(x-y) \mathrm{d}\sigma(y) \phi(x_0) + \int_{\partial D} \boldsymbol{n}(y) \cdot \Gamma^k(x_0 - y)(\phi(y) - \phi(x_0)) \mathrm{d}\sigma(y) \\
&= \int_{\partial D} \boldsymbol{n}(y) \cdot \nabla_y (\Gamma^k(x-y) - \Gamma^k(x_0 - y))(\phi(y) - \phi(x_0)) \mathrm{d}\sigma(y) \\
&+ (g(x) - g(x_0))\phi(x_0) + \mathcal{K}_D^k[\phi](x_0),
\end{aligned}$$

where g is the function

$$g(x) := \begin{cases} \displaystyle\int_{\partial D} \boldsymbol{n}(y) \cdot \nabla_y \Gamma^k(x-y) \mathrm{d}\sigma(y) \text{ if } x \in \mathbb{R}^d \setminus \partial D, \\ \displaystyle\quad \int_{\partial D} \boldsymbol{n}(y) \cdot \nabla_y \Gamma^k(x-y) \mathrm{d}\sigma(y) \text{ if } x \in \partial D, \end{cases}$$

and where

$$\int_{\partial D} \boldsymbol{n}(y) \cdot \nabla_y (\Gamma^k(x-y) - \Gamma^k(x_0 - y))(\phi(y) - \phi(x_0)) \mathrm{d}\sigma(y)$$

is a continuous function at $x = x_0$ "due to the removal of the singularity".

A little reasoning using an integration by parts and a drawing shows that

$$g(x) = \begin{cases} -k^2 \displaystyle\int_D \Gamma^k(x-y) \mathrm{d}y \text{ if } x \in \mathbb{R}^d \setminus D, \\ -k^2 \displaystyle\int_D \Gamma^k(x-y) \mathrm{d}y + 1 \text{ if } x \in D, \\ -k^2 \displaystyle\int_D \Gamma^k(x-y) \mathrm{d}y + \frac{1}{2} \text{ if } x \in \partial D. \end{cases}$$

We obtain as such

$$g|_\pm - g_{\partial D} = \mp \frac{1}{2},$$

from where (8) follows.

The proof is identical to the point 2., where one needs to study the limit of

$$\boldsymbol{n}(x_0) \cdot \nabla_x \mathcal{S}_D^k[\phi](x).$$

□

References

[1] H. Ammari, B. Fitzpatrick, H. Kang, M. Ruiz, S. Yu, and H. Zhang, *Mathematical and Computational Methods in Photonics and Phononics*, American Mathematical Society, oct 2018.

[2] V. Kozlov, V. Maz'ya, and A. Movchan, *Asymptotic analysis of fields in multi-structures*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1999. Oxford Science Publications.

[3] W. C. H. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge university press, 2000.

[4] M. Minnaert, *Xvi. on musical air-bubbles and the sounds of running water*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 16 (1933), pp. 235–248.