Lecture n°3 — Bessel functions and spherical harmonics

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1 Solving the Helmholtz equation in R^d

Bessel functions arise in the solution of the Helmholtz equation in a spherical domain.

Proposition 1. *The Laplacian in spherical coordinates in $\mathbb{R}^d$ is given by*

$$\Delta\phi = \frac{1}{r^{d-1}}\frac{\partial(r^{d-1}\phi)}{\partial r} + \frac{1}{r^2}\Delta_\Gamma\phi = \partial_{rr}\phi + \frac{d-1}{r}\partial_r\phi + \frac{1}{r^2}\Delta_\Gamma\phi,$$

where $\Delta_\Gamma\phi$ is the Laplace-Beltrami operator.

Proof. Consider functions $\phi, \psi \in \mathcal{C}_c^\infty$. An integration by parts yields

$$\begin{aligned}
\int_{\mathbb{R}^d} \Delta\phi\psi \mathrm{d}x &= -\int_{\mathbb{R}^d} \nabla\phi\cdot\nabla\psi \mathrm{d}x \\
&= -\int_0^{+\infty}\int_{S^{d-1}} \left(\partial_r\phi\boldsymbol{n} + \frac{1}{r}\nabla_\Gamma\phi\right)\cdot\left(\partial_r\psi\boldsymbol{n} + \frac{1}{r}\nabla_\Gamma\psi\right) r^{d-1}\mathrm{d}\sigma\mathrm{d}r \\
&= -\int_0^{+\infty}\int_{S^{d-1}} \left(\partial_r\phi\partial_r\psi r^{d-1} + \frac{r^{d-1}}{r^2}\nabla_\Gamma\phi\cdot\nabla_\Gamma\psi\right)\mathrm{d}\sigma\mathrm{d}r \\
&= \int_0^{+\infty}\int_{S^{d-1}} \left(\frac{\partial_r(r^{d-1}\partial_r\phi)}{r^{d-1}} + \frac{1}{r^2}\Delta_\Gamma\phi\right) r^{d-1}\mathrm{d}\sigma\mathrm{d}r.
\end{aligned}$$

This implies the result. □

We seek solutions to the Helmholtz equation in $\mathbb{R}^d$:

$$(\Delta + k^2)u = 0 \text{ in } \mathbb{R}^d. \tag{1}$$

We follow the derivation of [4]. Equation (1) reads in shperical coordinates

$$\partial_{rr}u + \frac{d-1}{r}\partial_r u + \frac{1}{r^2}\Delta_\Gamma u + k^2 u = 0. \tag{2}$$

We use the method of separation of variables: we seek a solution of the form

$$u(x) = f(kr)\psi(\omega) \text{ with } r := |x| \text{ and } \omega = x/|x|.$$

Inserting into (2) and denoting by $t := kr$, we obtain

$$\begin{aligned}
&\left(k^2 f'' + k\frac{d-1}{r}f' + k^2 f\right)\psi + \frac{1}{r^2}f\Delta_\Gamma\psi = 0 \\
\Leftrightarrow &\left(f'' + \frac{d-1}{t}f' + f\right)\psi + \frac{1}{t^2}f\Delta_\Gamma\psi = 0.
\end{aligned}$$

Let us introduce the basis of eigenfunctions $(\mathcal{Y}_i)_{i\geqslant 0}$ of $L^2(S^{d-1})$ for the Laplace-Beltrami operator $-\Delta_\Gamma$:

$$-\Delta_\Gamma\mathcal{Y}_i = \lambda_i\mathcal{Y}_i, \tag{3}$$

where λ_i is the eigenvalue associated to $\mathcal{Y}_i$. Decomposing

$$\psi(\omega) = \sum_{i=0}^{+\infty} \langle \psi, \mathcal{Y}_i \rangle \mathcal{Y}_i,$$

this leads us to consider rather the ansatz

$$u(x) = \sum_{i=0}^{+\infty} f_i(kr)\psi(\omega) \tag{4}$$

and we obtain the following ordinary differential equation for f_i:

$$f_i'' + \frac{d-1}{t} f_i' + \left(1 - \frac{\lambda_i}{t^2}\right) f_i = 0 \tag{5}$$

Equation (5) is a Bessel equation, while (3) is the definition of spherical harmonics. The object of this lecture is to review properties associated to these two notions, which comes into play when providing explicit formulas.

2 Spherical harmonics

We consider the eigenvalue problem

$$-\Delta_\Gamma Y = \lambda Y \tag{6}$$

where Δ_Γ is the Laplace-Beltrami operator. In this section, we follow [5]. We start with the following lemma.

Lemma 1. *For any $\mu \in \mathbb{R}$,*

$$\Delta(r^\mu g(\omega)) = 0 \Leftrightarrow \Delta_\Gamma g = \mu(\mu + d - 2)g. \tag{7}$$

Since $\mu \mapsto \mu(\mu+d-2)$ maps $(0, +\infty)$ to $(0, +\infty)$ surjectively, all eigenvalues of (6) can be written of the form

$$\lambda = \mu(\mu + d - 2) \text{ with } \mu \geqslant 0 \text{ and } \Delta(r^\mu g(\omega)) = 0.$$

Proof.

$$\Delta(r^\mu g(\omega)) = \mu(\mu-2)r^{\mu-2}g + \frac{2-d}{r}\mu r^{\mu-1}g + r^{\mu-2}\Delta_\Gamma g = r^{\mu-2}(\mu(\mu-2) + (2-d)\mu)g = 0.$$

□

In the next proposition, we denote by Γ the fundamental solution to the Laplace operator:

$$\Gamma(x) := \begin{cases} \dfrac{1}{2\pi} \log|x| \text{ if } d = 2, \\ \dfrac{2-d}{|S^{d-1}|}\dfrac{1}{|x|^{d-2}} \text{ if } d \geqslant 3. \end{cases}$$

It turns out that a function of the form $r^\mu g(\omega)$ with $\mu \geqslant 0$ satisfying $\Delta(r^\mu g(\omega)) = 0$ in $\mathbb{R}^d \setminus \{0\}$ must be a polynomial. This is a consequence of the two following results.

Proposition 2 (Removable singularity theorem). *If h satisfies $\Delta h = 0$ in $\mathbb{R}^d \setminus \{0\}$ and $h(x) = o(\Gamma(x))$ as $|x| \to 0$, then h can be extended into a harmonic function on $\mathbb{R}^d$.*

Proof. See [1, Theorem 2.69]. □

Proposition 3 (Liouville theorem). *If g is a harmonic function in $\mathbb{R}^d$ satisfying*

$$g(x) = O(|x|^p) \text{ as } |x| \to +\infty$$

for some exponent $p \geqslant 0$, then g must be a polynomial of degree lower or equal to p.

Proof. The asymptotic implies that g is a tempered distribution. Therefore it has a Fourier transform $\hat{g}$ which satisfies

$$|\xi|^2 \hat{g} = 0.$$

This implies that $\hat{g}$ must be of the form

$$\hat{g} = \sum_{|\alpha| \leqslant m} a_\alpha \delta_0^\alpha$$

for some number m and multi-indices coefficients a_α. Taking the inverse Fourier transform, we obtain that g is a polynomial. □

Corollary 1. *The spectral decomposition of the Laplace-Beltrami operator Δ_Γ on the sphere has the following characterization:*

1. *The spectrum of Δ_Γ is given by*

$$\sigma(\Delta_\Gamma) = \{m(m + d - 2) \,|\, m \in \mathbb{N}\backslash\{0\}\}.$$

2. *For every $m \in \mathbb{N}\backslash\{0\}$, the eigenspace associated to the eigenvalue $m(m + d - 2)$ is the space*

$$\mathcal{H}^m := \{\mathcal{Y}_m(x) \,|\, \mathcal{Y}_m \text{ is a harmonic polynomial of degree } m\}.$$

The space $\mathcal{H}^m$ is finite-dimensional and its dimension can be explicitly characterized [5].

To conclude with spherical harmonics, let us provide the following decomposition theorem.

Proposition 4. *Any function $g(x)$ can be decomposed as*

$$g(r\omega) := \sum_{i=0}^{+\infty} g_i(r) \mathcal{Y}_i(\omega)$$

for a fixed r where $g_i(t)$ is a radial function, where $\mathcal{Y}_i$ is a harmonic polynomial of degree i, and where the above series is convergent in $L^2(\mathcal{S}^{d-1})$.

Proof. This is the decomposition of $\omega \mapsto g(r\omega)$ onto the eigenvectors of Δ_Γ on the sphere. □

Remark 1. *The above proposition justifies the validity of the ansatz* (4) *which was suggested by the method of separation of variables.*

3 Bessel functions

Definition 1. *We call "Bessel equation of order μ" the ODE*

$$x'' + \frac{1}{t} x' + \left(1 - \frac{\mu^2}{t^2}\right) x = 0. \tag{8}$$

Equation (5) can be reformulated as a Bessel equation of order μ. For this, consider

$$g_i(t) := t^{-\alpha} f_i(t),$$

which is equivalent to $f_i(t) := t^\alpha g_i(t)$. Inserting this into (5) yields

$$\alpha(\alpha - 1) t^\alpha \frac{1}{t^2} g_i + 2\alpha t^\alpha \frac{1}{t} g_i' + t^\alpha g_i'' + \alpha \frac{d-1}{t^2} t^\alpha g_i + t^\alpha \frac{d-1}{t} g_i' + \left(1 - \frac{\lambda_i}{t^2}\right) t^\alpha g_i = 0$$

Setting

$$2\alpha + d - 1 = 1 \Leftrightarrow \alpha = 1 - \frac{d}{2},$$

we obtain that g_i is solution to the following Bessel equation of order μ:

$$g_i'' + \frac{1}{t}g_i' + \left(1 - \frac{\mu^2}{t^2}\right) = 0 \text{ with } \mu^2 = \left(1 - \frac{d}{2}\right)^2 + \lambda_i. \tag{9}$$

Finally, we know that $\lambda_i = i(i + d - 2)$ so that

$$\begin{aligned}\mu^2 &= \left(1 - \frac{d}{2}\right)^2 - 2i\left(1 - \frac{d}{2} - \frac{i}{2}\right) \\ &= \left(1 - \frac{d}{2}\right)^2 - 2i\left(1 - \frac{d}{2}\right) + i^2 \\ &= \left(i - 1 + \frac{d}{2}\right)^2.\end{aligned}$$

Therefore, g_i is solution to the Bessel equation of order $\mu = \left(i + \frac{d}{2} - 1\right)$.

Remark 2. *The Bessel equation of order μ of (8) is a purely conventional choice of canonical form. We use it to keep the conventions of the widely used literature, however it would certainly be possible to write Bessel functions for the solution to (5).*

Definition 2. *We call J_μ "Bessel function of the first kind" of order μ the function defined by*

$$J_\mu(t) := \sum_{p=0}^{+\infty} \frac{(-1)^p (t/2)^{\mu+2p}}{p!\Gamma(\mu + p + 1)}, \tag{10}$$

where Γ is the Gamma function (generalizing the factorial). The function J_μ is a solution to the Bessel equation (5) of order μ.

Proof. The function J_μ is obtained by seeking a solution of the form $x(t) = t^\mu \sum_{p=0}^{+\infty} c_p x^p$ where the exponent μ is motivated by the fact that (8) leads to expect $x(t) \sim t^\mu$ as $t \to 0$. This leads to a recurrence which allows to identify the functions c_p as those given by (10). See [6, 3] for more details. □

The function J_μ is smooth at the origin for $\mu > 0$ and singular otherwise, with a singularity of order $t^{-\mu}$.

Definition 3. *The Bessel function of the second kind Y_μ is defined as*

$$Y_\mu := \frac{J_\mu \cos(\pi\mu) - J_{-\mu}}{\sin(\pi\mu)}$$

if $\mu \notin \mathbb{Z}$, and

$$Y_\mu := \lim_{\mu' \to \mu} \frac{J_\mu \cos(\pi\mu) - J_{-\mu}}{\sin(\pi\mu)}$$

otherwise. The functions J_μ and Y_μ form a basis of solutions to (5).

Proposition 5. *For $\mu \geqslant 0$:*

1. *J_μ is smooth at the origin.*
2. *Y_μ is singular at the origin and the singularity is of order $O(t^{-\mu})$ if $\mu \geqslant 0$, and $O(\log \mu)$ if $\mu = 0$.*
3. *The behavior of J_μ and Y_μ at infinity are given by*

$$J_\mu(t) \sim \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{1}{2}\pi\mu - \frac{\pi}{4}\right)$$

$$Y_\mu(t) \sim \sqrt{\frac{2}{\pi t}} \sin\left(t - \frac{1}{2}\pi\mu - \frac{\pi}{4}\right)$$

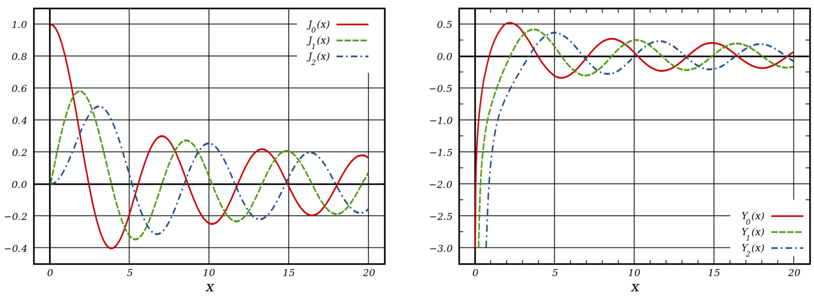

Figure 1: Bessel functions of the first and second kinds (source: wikipedia).

We provide plots of the bessel functions J_μ and Y_μ on fig. 1. They are other types of Bessel functions: Hankel functions, spherical Bessel functions, and spherical Hankel functions.

Definition 4. *We call:*

- *spherical Bessel functions of the first and second kind the functions*

$$\mathrm{j}_m(t,d) := \sqrt{\frac{\pi}{2}}\frac{J_{m+d/2-1}(t)}{t^{d/2-1}}, \qquad \mathrm{y}_m(t,d) := \sqrt{\frac{\pi}{2}}\frac{Y_{m+d/2-1}(t)}{t^{d/2-1}}$$

- *Hankel functions of first and second kind the functions*

$$H^{(1)}_\mu := J_\mu + \mathrm{i}Y_\mu, \qquad H^{(2)}_\mu = J_\mu - \mathrm{i}Y_\mu.$$

- *spherical Hankel functions of first and second kind the functions*

$$\mathrm{h}^{(1)}_m(\cdot,d) := \mathrm{j}_m(\cdot,d) + \mathrm{i}\mathrm{y}_m(\cdot,d), \qquad \mathrm{h}^{(2)}_m = \mathrm{j}_m(\cdot,d) - \mathrm{i}\mathrm{y}_m(\cdot,d).$$

Remark 3. *The scaling constant $\sqrt{\pi/2}$ ensures that*

$$\mathrm{h}^{(1)}_m(k|x|,d) \sim \mathrm{j}_m(k|x|,d) + \mathrm{i}\mathrm{y}_m(k|x|,d) \sim \frac{1}{(k|x|)^{(d-1)/2}} e^{\mathrm{i}k|x|} e^{-\mathrm{i}\frac{1}{2}\pi\left(m+\frac{d-1}{2}\right)} \text{ as } k|x| \to +\infty.$$

Going back to the Helmholtz equation, we obtain that solutions to the homogeneous Helmholtz equation in $\mathbb{R}^d \setminus \{0\}$ are linear combinations of spherical Hankel functions of the first and second kind.

Corollary 2. *Any function u solving the Helmholtz equation in $\mathbb{R}^d \setminus \{0\}$ can be decomposed as*

$$\begin{aligned} u(x) &:= \sum_{m=0}^{+\infty} \sum_{l=0}^{\dim(\mathcal{H}^m)} (\alpha_{m,l}\mathrm{j}_m(k|x|,d) + \beta_{m,l}\mathrm{y}_m(k|x|,d))\mathcal{Y}_{m,l}(x/|x|) \\ &= \sum_{m=0}^{+\infty} \sum_{l=0}^{\dim(\mathcal{H}^m)} (\alpha'_{m,l}\mathrm{h}^{(1)}_m(k|x|,d) + \beta'_{m,l}\mathrm{h}^{(2)}_m(k|x|,d))\mathcal{Y}_{m,l}(x/|x|) \end{aligned}$$

where for any $m \geqslant 0$, $(\mathcal{Y}_{m,l})_{1\leqslant l\leqslant \dim(\mathcal{H}^m)}$ is a orthonormal basis of harmonic polynomials of degree m on the sphere. As a particular case the fundamental solution of the Helmholtz operator is given by

$$\Gamma^k(x) := -\frac{k^{d-2}}{2(2\pi)^{(d-1)/2}}\mathrm{i}\mathrm{h}^{(1)}_0(d,k|x|).$$

Let us mention a few additional useful properties.

Proposition 6. *For any* $(z,t) \in \mathbb{C} \times \mathbb{C}^*$, *let us define*

$$g(z,t) = e^{\frac{z}{2}\left(t-\frac{1}{t}\right)}.$$

The function g *is holomorphic on* $\mathbb{C} \times \mathbb{C}^*$ *and admits the following Laurent series expansion:*

$$g(z,t) = \sum_{n\in\mathbb{Z}} t^n J_n(z).$$

Corollary 3. *For any* $\theta \in \mathbb{R}$,

$$e^{\mathrm{i}z\sin(\theta)} = \sum_{n\in\mathbb{Z}} J_n(z) e^{\mathrm{i}n\theta}$$

and

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin(\theta)} e^{-\mathrm{i}n\theta}\mathrm{d}\theta.$$

Corollary 2 can be physically interpreted as the decomposition of plane waves into cylindrical waves. Indeed

$$e^{\mathrm{i}z\cos(\theta)} = e^{\mathrm{i}z\sin\left(\frac{\pi}{2}-\theta\right)} = \sum_{n\in\mathbb{Z}} J_n(z)e^{-\mathrm{i}n\theta}e^{\mathrm{i}n\frac{\pi}{2}}$$

so that if k is a unit vector,

$$e^{ik\cdot x} = e^{i|x|\cos(\theta_x)} = \sum_{n\in\mathbb{Z}} J_n(|x|)e^{-\mathrm{i}n\theta_x}e^{\mathrm{i}n\frac{\pi}{2}}$$

where θ_x is the angle between x and k.

4 Far field expansions

In this last part, we show how spherical harmonics yield far-field expansions of the solutions to the Helmholtz equation. The method is rather general, let us illustrate it first on the Laplace operator Δ. We seek a far field expansion of

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi}\log|x| \text{ if } d=2 \\ \dfrac{2-d}{|S^{d-1}|}\dfrac{1}{|x|^{d-2}} \text{ if } d \geqslant 3. \end{cases}$$

We seek to expand $\Gamma(x-y)$ as $|x| > |y|$. For this, let us start by decomposing $(\rho,\omega) \mapsto \Gamma(x-\rho\omega)$ on spherical harmonics:

$$\Gamma(x-y) = \sum_{m=0}^{+\infty}\sum_{l=0}^{\deg(\mathcal{H}^m)} \int_{\mathcal{S}^{d-1}} \Gamma(x-\rho\eta)\overline{\mathcal{Y}^{m,l}(\eta)}\mathrm{d}\eta\,\mathcal{Y}^{m,l}(\omega).$$

We compute the integral by making use of the two following identities which hold for any $m > 0$:

$$\Delta(\rho^m\mathcal{Y}^{m,l}(\omega)) = 0 \text{ in } \mathbb{R}^d, \qquad \Delta(\rho^{2-d-m}\mathcal{Y}^{m,l}(\omega)) = 0 \text{ in } \mathbb{R}^d\setminus\{0\}.$$

Let us denote by $u(x) := |x|^m\mathcal{Y}^{m,l}(x/|x|)$ and $v(x) := |x|^{2-d-m}\mathcal{Y}^{m,l}(x/|x|)$. Since v decays "sufficiently" at infinity[1], the following integration by parts holds true for $x \in \mathbb{R}^d \setminus B(0,\rho)$:

$$0 = \int_{\mathbb{R}^d\setminus B(0,\rho)} \Delta_t v\Gamma(x-t)\mathrm{d}t = v(x) + \int_{\partial B(0,\rho)} \left(\frac{\partial\Gamma}{\partial n}(x-t)v(t) - \Gamma(x-t)\frac{\partial v}{\partial n}(t)\right)\mathrm{d}\sigma(t).$$

Furthermore, since u is smooth inside $\mathcal{B}(0,\rho)$ and $u(t) = \rho^{d-2+2m}v(t)$ on $\partial B(0,\rho)$, we have

$$\int_{B(0,\rho)} \Delta_y u\Gamma(x-y)\mathrm{d}y = \int_{\partial B(0,\rho)} \left(-\rho^{d-2+2m}\frac{\partial\Gamma}{\partial n}(x-y)v(y) + \Gamma(x-y)\frac{\partial u}{\partial n}(y)\right)\mathrm{d}\sigma(y)$$

[1] It is a function of the Deny-Lions space $\mathcal{D}^{1,2}(\mathbb{R}^d\setminus\{0\})$

Eliminating $\frac{\partial \Gamma}{\partial n}(x-y)$, we obtain thus

$$\rho^{d-2+2m} v(x) + \int_{\partial B(0,\rho)} \Gamma(x-y)\left(\frac{\partial u}{\partial n} - \rho^{d-2+2m}\frac{\partial v}{\partial n}\right) \mathrm{d}\sigma(y) = 0$$

Since

$$\frac{\partial u}{\partial n} = l\rho^{m-1}\mathcal{Y}^{m,l}, \qquad \frac{\partial v}{\partial n} = (2-d-m)\rho^{1-d-m}\mathcal{Y}^{m,l} \text{ on } \partial B(0,\rho),$$

we find

$$\rho^{d-2+2m}|x|^{2-d-m}\mathcal{Y}^{m,l}\left(x/|x|\right) + (2m+d-2)\rho^{m+d-2}\int_{\partial\mathcal{B}(0,\rho)} \Gamma(x-\rho\eta)\mathcal{Y}^{m,l}(\eta)\mathrm{d}\sigma(\eta) = 0.$$

Whence for $m > 0$:

$$\int_{\mathcal{S}^{d-1}} \Gamma(x-\rho\eta)\overline{\mathcal{Y}^{m,l}(\eta)}\mathrm{d}\eta = -\frac{\rho^m}{|x|^{d-2+m}}\frac{1}{2m+d-2}\overline{\mathcal{Y}^{m,l}(x/|x|)}. \tag{11}$$

For $l = 0$ we have instead from the mean value theorem:

$$\frac{1}{|\mathcal{S}^{d-1}|}\int_{\mathcal{S}^{d-1}} \Gamma(x-\rho\eta)\mathrm{d}\sigma(\eta) = \frac{1}{\partial B(0,\rho)}\int_{\partial B(0,\rho)} \Gamma(x-y)\mathrm{d}\sigma(y) = \Gamma(x).$$

We deduce:

Proposition 7. *The following addition theorem holds for the fundamental solution to the Laplace equation for $d \geqslant 2$:*

$$\Gamma(x-y) = \Gamma(x) - \sum_{m=1}^{+\infty}\sum_{l=0}^{\deg(\mathcal{H}^m)} \frac{1}{2m+d-2}|x|^{2-d-m}|y|^m\overline{\mathcal{Y}^{m,l}}(x/|x|)\mathcal{Y}^{m,l}(y/|y|). \tag{12}$$

In fact, we have also

$$\Gamma(x-y) = \sum_{m=0}^{+\infty}(-1)^m\frac{1}{m!}\nabla^m\Gamma(x)\cdot y^m \text{ for } |x| > |y|$$

which shows that, for $m > 0$:

$$\nabla^m\Gamma\left(\omega\right)\cdot\eta^m = (-1)^{m+1}\frac{m!}{2m+d-2}\sum_{l=0}^{\deg(\mathcal{H}^m)}\overline{\mathcal{Y}^{m,l}(\omega)}\mathcal{Y}^{m,l}(\eta)$$

Remark 4. *The addition formula* (12) *enables to decompose a wave centered at y in terms of higher order waves centered at 0. This property is at the origin of the multipole expansion method of Greengard and Roeklin [2].*

Remark 5. *For $d = 2$, the space $\mathcal{H}^m$ is a one dimensional complex vector space, whose basis is the polynomial*

$$P(x_1,x_2) := \frac{1}{\sqrt{2\pi}}(x_1+\mathrm{i}x_2)^n = \frac{1}{\sqrt{2\pi}}|x|^n e^{\mathrm{i}n\theta_x} \textit{ for } n \geqslant 0.$$

Consequently (12) *reads*

$$\Gamma(x-y) = \Gamma(x) - \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{2n}|x|^{-|n|}|y|^{|n|}e^{-\mathrm{i}n\theta_x}e^{\mathrm{i}n\theta_y}.$$

For the fundamental solution to the Helmholtz equation, a similar reasoning provides the following addition theorem.

Proposition 8. *The following addition formula holds for the fundamental solution to the Helmholtz equation:*

$$\Gamma^k(x-y) = -\mathrm{i}k^{d-2}\sum_{m=0}^{+\infty}\sum_{l=0}^{\dim(\mathcal{H}^m)} \mathrm{h}_m^{(1)}(d,k|x|)\mathrm{j}_m(k|y|)\overline{\mathcal{Y}}^{m,l}(y/|y|)\mathcal{Y}^{m,l}(y/|y|).$$

In dimension 2:

$$\Gamma^k(x-y) = -\frac{\mathrm{i}}{4}\sum_{n\in\mathbb{Z}} H_n^{(1)}(k|x|)e^{\mathrm{i}n\theta_x}J_n(k|y|)e^{-\mathrm{i}n\theta_y} \text{ for } |x| > |y|.$$

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