Lecture n°5 — Deny-Lions space and exterior Laplace problems

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In this lecture, we consider the exterior Laplace problem

$$\begin{cases} -\Delta v = f \text{ in } \mathbb{R}^d \setminus D \\ v = g \text{ on } \partial D \end{cases} \tag{1}$$

with a suitable decay condition at infinity, where $d \geqslant 2$ and f is a function with compact support in $\mathbb{R}^d$. It is clear that (1) can be solved by using layer-potential. Here, we show that the well-posedness of the exterior boundary problem (1) can be inferred by the Lax-Milgram theorem up to the introduction of suitable Hilbert spaces. We then relate these properties to the invertibility of the single layer potential and the capacity. This material is strongly inspired from discussions with G. Allaire, his work [1], and the appendix A.1 of the thesis of Chetboun [5].

1 Deny-Lions spaces

Let us start by introducing the so-called *Deny-Lions* (also called *Beppo-Levi* or homogeneous Sobolev space) $\mathcal{D}^{1,2}(\mathbb{R}^d)$ (see [7, 9]) defined by the completion of the space of compactly supported functions with respect to the L^2 norm of the gradient:

$$\mathcal{D}^{1,2}(\mathbb{R}^d) := \overline{\mathcal{C}_c^\infty(\mathbb{R}^d)}^{||\nabla \cdot||_{L^2(\mathbb{R}^d)}}$$

It is clear from the definition that

$$\mathcal{D}^{1,2}(\mathbb{R}^d) \subset \{v \,|\, \nabla v \in L^2(\mathbb{R}^d)\}.$$

The reverse inclusion is not true at least for $d \geqslant 3$. More precisely, we have the following characterization, which shows that constant functions do not belong to $\mathcal{D}^{1,2}(\mathbb{R}^d)$ if $d \geqslant 3$.

Proposition 1. *1. Assume $d \geqslant 3$. Then the following Poincaré inequality holds:*

$$\forall v \in \mathcal{D}^{1,2}(\mathbb{R}^d) \quad \left|\left|\frac{v}{1+|x|}\right|\right|_{L^2(\mathbb{R}^d)} \leqslant C||\nabla v||_{L^2(\mathbb{R}^d)}, \tag{2}$$

for some constant $C > 0$. Reciprocally, the space $\mathcal{D}^{1,2}(\mathbb{R}^d)$ has the following characterization:

$$\mathcal{D}^{1,2}(\mathbb{R}^d) = \left\{ v \,\middle|\, \frac{v}{1+|x|} \in L^2(\mathbb{R}^d) \text{ and } \nabla v \in L^2(\mathbb{R}^d) \right\}, \tag{3}$$

and

$$||v||_{\mathcal{D}^{1,2}(\mathbb{R}^d)} := ||\nabla v||_{L^2(\mathbb{R}^d)}$$

defines a norm on $\mathcal{D}^{1,2}(\mathbb{R}^d)$.

2. Assume $d = 2$. Then the following Poincaré inequality holds:

$$\forall v \in \mathcal{D}^{1,2}(\mathbb{R}^2), \quad \left|\left|\frac{v}{(|x|+1)\log(|x|+2)}\right|\right|_{L^2(\mathbb{R}^2)} \leqslant C(||\nabla v||_{L^2(\mathbb{R}^2)} + ||v||_{L^2(B(0,1))}) \tag{4}$$

where $B(0,1)$ is the unit ball of $\mathbb{R}^2$ and for some constant $C > 0$. Reciprocally, $\mathcal{D}^{1,2}(\mathbb{R}^2)$ has the following characterization:

$$\mathcal{D}^{1,2}(\mathbb{R}^2) = \left\{ v \mid \frac{v}{(|x|+1)\log(|x|+2)} \in L^2(\mathbb{R}^2) \text{ and } \nabla v \in L^2(\mathbb{R}^2) \right\} \tag{5}$$

and

$$||v||_{\mathcal{D}^{1,2}(\mathbb{R}^2)} := \left(\left|\left| \frac{v}{(|x|+1)\log(|x|+2)} \right|\right|^2_{L^2(\mathbb{R}^2)} + ||\nabla v||^2_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \tag{6}$$

or

$$||v||_{\mathcal{D}^{1,2}(\mathbb{R}^2)} := \left(||v||^2_{L^2(B(0,1))} + ||\nabla v||^2_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \tag{7}$$

define two equivalent norms on $\mathcal{D}^{1,2}(\mathbb{R}^2)$.

Remark 1. *The weights occurring in (2) and (5) states that v is controlled at infinity. The condition $v/(1+|x|) \in L^2(\mathbb{R}^d)$ can be interpreted formally as $v(x) = o(|x|^{1-d/2})$ at infinity, while $v/((|x|+1)\log(|x|+2))$ can be interpreted as $v(x) = o(\log(|x|))$.*

Remark 2. *The main difference between the 2D case and $d \geqslant 3$ lies in the fact that $\mathcal{D}^{1,2}(\mathbb{R}^2)$ contains constant functions, but not $\mathcal{D}^{1,2}(\mathbb{R}^d)$ with $d \geqslant 3$.*

Remark 3. *The constant $+1$ and $+2$ (2) and (5) are used to avoid singularity of the weighting functions. However since $1/|x|^2$ has an integrable singularity at the origin when $d \geqslant 3$, one could remove the constant 1 in the definitions (2) and (3).*

Proof. We sketch the proof of the inequalities (2) and (4). The characterizations (3) and (5) are obtained by standard mollification arguments and we refer the reader to [1, 2] for a proof.
1. *Proof of* (2). We prove the inequality for a compactly supported function $v \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, the result following by density. By using spherical coordinates, it is sufficient to assume that v is a radial function: $v \equiv v(r)$ with $r > 0$, and (2) reduces to

$$\int_0^{+\infty} \frac{|v(r)|^2}{(1+r)^2} r^{d-1} \mathrm{d}r \leqslant C \int_0^{+\infty} |v'(r)|^2 r^{d-1} \mathrm{d}r. \tag{8}$$

This inequality is called a *Hardy inequality* (see for instance [9]). It is obtained by writing that for such v vanishing for r sufficiently large and $\alpha > 0$:

$$\begin{aligned} \left| -\alpha \int_0^{+\infty} v(r)^2 r^{\alpha-1} \mathrm{d}r \right| &= \left| -\int_0^{+\infty} v(r)^2 \frac{\mathrm{d}r^\alpha}{\mathrm{d}r} \mathrm{d}r \right| = \left| \int_0^{+\infty} \frac{\mathrm{d}|v(r)|^2}{\mathrm{d}r} r^\alpha \mathrm{d}r \right| \\ &= 2 \left| \int_0^{+\infty} v(r) v'(r) r^\alpha \mathrm{d}r \right| \leqslant 2 \left(\int_0^{+\infty} v(r)^2 r^{2\alpha-(d-1)} \mathrm{d}r \right)^{\frac{1}{2}} \left(\int_0^{+\infty} |v'(r)|^2 r^{d-1} \mathrm{d}r \right)^{\frac{1}{2}}. \end{aligned}$$

Setting $\alpha = d-2$, we obtain

$$\left(\int_0^{+\infty} \frac{v(r)^2}{r^2} r^{d-1} \mathrm{d}r \right)^{\frac{1}{2}} \leqslant \frac{2}{d-2} \left(\int_0^{+\infty} |v'(r)|^2 r^{d-1} \mathrm{d}r \right)^{\frac{1}{2}},$$

from where (8) follows.
2. *Proof of* (4). Using spherical coordinates, we need to prove

$$\int_0^{+\infty} \frac{|v(r)|^2}{(r+1)^2 \log(r+2)^2} r \mathrm{d}r \leqslant C \left(\int_0^{+\infty} |v'(r)|^2 r \mathrm{d}r + \int_0^1 |v(r)|^2 r \mathrm{d}r \right).$$

This inequality is obtained by a variant of the Hardy inequality. Considering this time with $\beta \neq 0$ and $v(r) = 0$ for $r < R$ with $R > 0$ and r large enough,

$$\begin{aligned} \left| \beta \int_R^{+\infty} v(r)^2 \frac{1}{r} \log(r)^{\beta-1} \mathrm{d}r \right| &= \left| \int_R^{+\infty} \frac{\mathrm{d}v(r)^2}{\mathrm{d}r} \log(r)^\beta \mathrm{d}r \right| = \left| \int_R^{+\infty} 2v'(r)v(r) \log(r)^\beta \mathrm{d}r \right| \\ &\leqslant 2 \left(\int_R^{+\infty} v'(r)^2 r \mathrm{d}r \right)^{\frac{1}{2}} \left(\int_R^{+\infty} v(r)^2 \log(r)^{2\beta} \frac{1}{r} \mathrm{d}r \right)^{\frac{1}{2}}. \end{aligned}$$

Choosing $\beta = -1$, we obtain thus for such v:

$$\left(\int_R^{+\infty} \frac{v(r)^2}{r^2 \log(r)^2} r\mathrm{d}r\right)^{\frac{1}{2}} \leqslant 2 \left(\int_R^{+\infty} v'(r)^2 r\mathrm{d}r\right)^{\frac{1}{2}}.$$

For a general v, we consider the decomposition $v = hv + v(1-h)$ where $h \in \mathcal{C}^\infty(\mathbb{R})$ satisfies $h(r) = 0$ for $r \leqslant R$ and $h(r) = 1$ for $r \geqslant 1$. Then the above inequality applies to hv instead of v and we get

$$\begin{aligned}
\int_0^{+\infty} \frac{v(r)^2}{(r+1)^2 \log(r+2)^2} r\mathrm{d}r &\leqslant C \int_0^1 v(r)^2 r\mathrm{d}r + \int_1^{+\infty} \frac{v(r)^2}{(r+1)^2 \log(r+2)^2} r\mathrm{d}r \\
&\leqslant C \left(\int_0^1 v(r)^2 r\mathrm{d}r + \int_1^{+\infty} \frac{h(r)v(r)^2}{r^2 \log(r)^2} r\mathrm{d}r\right) \leqslant C \left(\int_0^1 v(r)^2 r\mathrm{d}r + \int_R^{+\infty} \frac{h(r)^2 v(r)^2}{r^2 \log(r)^2} r\mathrm{d}r\right) \\
&\leqslant C \left(\int_0^1 v(r)^2 r\mathrm{d}r + \int_R^{+\infty} (h'(r)v(r) + h(r)v'(r))^2 r\mathrm{d}r\right) \\
&\leqslant C \left(\int_0^1 v(r)^2 r\mathrm{d}r + \int_R^1 h'(r)^2 v(r)^2 r\mathrm{d}r + \int_R^{+\infty} h(r)^2 v'(r)^2 r\mathrm{d}r\right).
\end{aligned}$$

The result follows easily. □

When $d = 2$, it is not coercive due to constants. However, it is coercive when considering Dirichlet conditions on some boundary or zero average value in some bounded set. Before proceeding further, let us add a few remarks about the space $\mathcal{D}^{1,2}(\mathbb{R}^d)$.

Proposition 2 (Integration by parts in $\mathcal{D}^{1,2}(\mathbb{R}^d)$)**.** *Let $v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ and $\boldsymbol{u}$ a vector field satisfying* $\mathrm{div}(\boldsymbol{u}) = 0$ *in $\mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d \setminus D} \boldsymbol{u} \cdot \nabla v \mathrm{d}x = -\int_{\partial D} v \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}\sigma, \tag{9}$$

where $\boldsymbol{n}$ is the outward normal to D.

Proof. The result is obvious by density. □

Remark 4. *Note the negative sign in* (9) *because the normal outward to $\mathbb{R}^d \setminus D$ points inward D.*

Let us recall the definition of the fundamental solution $\Gamma(x)$ of the Laplace operator (which decays at infinity):

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \log|x| \text{ if } d = 2 \\ \dfrac{1}{(2-d)|S^{d-1}|} \dfrac{1}{|x|^{d-2}} \text{ if } d \geqslant 3. \end{cases}$$

which satisfies

$$\Delta \Gamma = \delta_0$$

in the sense of distributions.

Remark 5. *Using the characterizations* (3) *and* (5)*, we see that $\Gamma \in \mathcal{D}^{1,2}(\mathbb{R}^d \setminus D)$ if $d \geqslant 3$, but $\Gamma \notin \mathcal{D}^{1,2}(\mathbb{R}^2 \setminus D)$ if $d = 2$.*

2 The exterior Dirichlet problem and the capacity in dimension $d \geqslant 3$

As a consequence of the Poincaré inequality (2), the bilinear form

$$(u, v) \mapsto \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \mathrm{d}x$$

is continuous on $\mathcal{D}^{1,2}(\mathbb{R}^d)$ and coercive when $d \geqslant 3$. Using the Lax-Milgram theorem [8], we have the following result when no boundary condition is applied on some obstacle D.

Proposition 3. *Let $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ a smooth compactly supported function.*

1. *There exists a unique solution $v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ to the problem*

$$\begin{cases} -\Delta v = f \text{ in } \mathbb{R}^d \\ \quad v \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^d). \end{cases} \tag{10}$$

2. *The solution v admits the explicit representation*

$$v(x) = -\int_{\mathbb{R}^d} \Gamma(x-y) f(y) \mathrm{d}y, \tag{11}$$

and hence satisfies the following asymptotic at infinity:

$$v(x) = -\left(\int_{\mathbb{R}^d} f(y)\mathrm{d}y\right)\Gamma(x) + O\left(\frac{1}{|x|^{d-1}}\right). \tag{12}$$

Proof. The solution to (10) is given by the unique Lax-Milgram solution to the variational problem

$$\text{find } v \in \mathcal{D}^{1,2}(\mathbb{R}^d), \text{ such that } \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^d), \int_{\mathbb{R}^d} \nabla v \cdot \nabla v \mathrm{d}x = \int_{\mathbb{R}^d} f v \mathrm{d}x.$$

It is obvious from the property of the fundamental solution that the explicit representation (11) satisfies $\Delta v = f$ in $\mathbb{R}^d$. Furthermore, it belongs to $\mathcal{D}^{1,2}(\mathbb{R}^d)$ due to the decay of Γ at infinity. The asymptotic (12) follows. □

Remark 6. *Due to the asymptotic behavior at infinity, we often prefer to write the problem* (10) *as*

$$\begin{cases} -\Delta v = f \text{ in } \mathbb{R}^d, \\ v(x) = O(|x|^{2-d}) \text{ as } |x| \to +\infty. \end{cases}$$

In order to treat a Dirichlet boundary condition on the obstacle D, we consider the space

$$\mathcal{D}_0^{1,2}(\mathbb{R}^d \setminus \partial D) := \{v \in \mathcal{D}^{1,2}(\mathbb{R}^d) \,|\, v = 0 \text{ on } \Gamma_D\},$$

where the trace makes sense due to the inclusion $\mathcal{D}^{1,2}(\mathbb{R}^d) \subset H^1_{loc}(\mathbb{R}^d)$. We recall the definition of the single layer potental:

$$\mathcal{S}_D[\psi](x) := \int_{\partial D} \Gamma(x-y)\psi(y)\mathrm{d}y, \qquad x \in \mathbb{R}^d, \psi \in L^2(\partial D).$$

Proposition 4. *Let $g \in H^{\frac{1}{2}}(\partial D)$.*

1. *There exists a unique solution to the problem*

$$\begin{cases} -\Delta v = 0 \text{ in } \mathbb{R}^d \setminus \partial D \\ \quad v = g \text{ on } \partial D \\ \quad v \in \mathcal{D}^{1,2}(\mathbb{R}^d). \end{cases} \tag{13}$$

2. *The solution v can be represented as a single layer potential:*

$$v = \mathcal{S}_D\left[\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right] \text{ in } \mathbb{R}^d. \tag{14}$$

where $[\![\frac{\partial v}{\partial n}]\!] = \frac{\partial v}{\partial n}\big|_+ - \frac{\partial v}{\partial n}\big|_-$ is the jump of the normal derivative accross ∂D. Consequently, v has the following asymptotic expansion at infinity:

$$v(x) = \left(\int_{\partial D}\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\mathrm{d}\sigma\right)\Gamma(x) + O\left(\frac{1}{|x|^{d-1}}\right)$$

Proof. The solution to (13) is given by $v := \tilde{v} + \tilde{f}$ where $\tilde{f}$ is a lifting of the boundary condition (i.e. $\tilde{f} \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ and $\tilde{f} = g$ on ∂D) and $\tilde{v}$ is the unique Lax-Milgram solution to the variational problem

$$\text{find } \tilde{v} \in \mathcal{D}_0^{1,2}(\mathbb{R}^d \setminus \partial D), \text{ such that } \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^d \setminus D), \int_{\mathbb{R}^d} \nabla v \cdot \nabla v \mathrm{d}x = \int_{\mathbb{R}^d} \Delta \tilde{f} v \mathrm{d}x.$$

Using an integration by parts on (13), we can then show that for any compactly supported function v,

$$\int_{\mathbb{R}^d} v \Delta v \mathrm{d}x = \int_{\partial D} \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] v \mathrm{d}\sigma. \tag{15}$$

i.e. $\Delta v = [\![\frac{\partial v}{\partial n}]\!] \mathrm{d}\sigma$ in the sense of distributions, where $\mathrm{d}\sigma$ is the surface measure of ∂D. Consider the function

$$\hat{v}(x) := \left(\Gamma * [\![\frac{\partial v}{\partial n}]\!] \right)(x) = \int_{\partial D} \Gamma(x-y) \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] (y) \mathrm{d}\sigma(y) = \mathcal{S}_D \left[\left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \right].$$

Due to (15), $\Delta(v - \hat{v}) = 0$ in the sense of distributions, which implies that $v - \hat{v}$ is a harmonic function in $\mathbb{R}^d$ (see e.g. [6]). Furthermore, $v - \hat{v} \in \mathcal{D}^{1,2}(\mathbb{R}^2)$ from the decay property of Γ. Since $\Delta(v - \hat{v}) = 0$, the uniqueness result of proposition 3 implies $v - \hat{v} = 0$ and the representation formula (14) is proved. □

Remark 7. *Similarly, we often prefer to write the problem* (13) *as*

$$\begin{cases} -\Delta v = 0 \text{ in } \mathbb{R}^d \setminus \partial D \\ v = g \text{ on } \partial D \\ v(x) = O(|x|^{2-d}) \text{ as } |x| \to +\infty. \end{cases} \tag{16}$$

where the second condition can be interpreted as a boundary condition at infinity.

Corollary 1. *The single-layer potential* $\mathcal{S}_D : H^{-1/2}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ *defined by*

$$\mathcal{S}_D[v](x) := \int_{\partial D} \Gamma(x-y) v(y) \mathrm{d}y$$

is an invertible operator when $d = 3$. *In fact,* (14) *implies that*

$$(\mathcal{S}_D)^{-1}[g] = \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right],$$

where v *is the unique solution to* (13).

We conclude with the definition of the capacity.

Definition 1. *Consider the exterior problem*

$$\begin{cases} -\Delta \Phi = 0 \text{ in } \mathbb{R}^d \setminus D \\ \Phi = 1 \text{ on } \partial D \\ \Phi(x) = O(|x|^{2-d}) \text{ as } |x| \to +\infty. \end{cases}$$

The capacity of the set D *is defined as the positive quantity*

$$\mathrm{cap}\,(D) := \int_{\mathbb{R}^d \setminus D} |\nabla \Phi|^2 \mathrm{d}x.$$

By integration by part, it is also given by

$$\mathrm{cap}\,(D) = -\int_{\partial D} \left. \frac{\partial \Phi}{\partial n} \right|_+ \mathrm{d}\sigma = -\int_{\partial D} (\mathcal{S}_D)^{-1}[1_{\partial D}] \mathrm{d}\sigma.$$

3 The exterior Dirichlet problem and the capacity in dimension $d = 2$

When $d = 2$, the bilinear form $(u, v) \mapsto \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \mathrm{d}x$ is not coercive anymore on $\mathcal{D}^{1,2}(\mathbb{R}^2)$. This implies some important variations in the statements of propositions 3 and 4.

Proposition 5. *Let $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ a smooth compactly supported function.*

1. *There exists a unique solution $v \in \mathcal{D}^{1,2}(\mathbb{R}^2)$ defined up to a constant to the problem*

$$\begin{cases} -\Delta v = f \ \text{in } \mathbb{R}^2, \\ \quad v \ \text{in } \mathcal{D}^{1,2}(\mathbb{R}^2), \end{cases} \tag{17}$$

if and only if the compatibility condition

$$\int_{\mathbb{R}^2} f \mathrm{d}x = 0 \tag{18}$$

is satisfied.

2. *When it is the case, the solution v admits, up to the choice of an additive constant $v_\infty \in \mathbb{R}$, the explicit representation*

$$v(x) = v_\infty - \int_{\mathbb{R}^2} \Gamma(x - y) f(y) \mathrm{d}y. \tag{19}$$

3. *The solution v has the following asymptotic behavior at infinity:*

$$v(x) = v_\infty + \nabla\Gamma(x) \cdot \int_{\mathbb{R}^2} y f(y) \mathrm{d}y + O(|x|^{-2}) \ \text{as } |x| \to +\infty.$$

Proof. 1. The problem (17) reads in variational form: find $v \in \mathcal{D}^{1,2}(\mathbb{R}^2)$ such that for any $\psi \in \mathcal{D}^{1,2}(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla \psi \mathrm{d}x = \int_{\mathbb{R}^2} f \psi \mathrm{d}x.$$

Since $\mathcal{D}^{1,2}(\mathbb{R}^2)$ contains constant functions, we can set $\psi = 1$ in the above equation and we thus obtain the compatibility condition (18). Reciprocally, let us assume that (18) is satisfied and consider the space

$$V := \left\{ v \in \mathcal{D}^{1,2}(\mathbb{R}^2) \mid \int_{B(0,1)} v \mathrm{d}x = 0 \right\}.$$

Due to the Poincaré-Wirtinger inequality,

$$\left|\left| v - \frac{1}{|B(0,1)|} \int_{B(0,1)} v \mathrm{d}x \right|\right|_{L^2(B(0,1))} \leqslant C ||\nabla v||_{L^2(B(0,1))}$$

it is clear that $||\nabla \cdot ||_{L^2(\mathbb{R}^2)}$ defines an equivalent norm on V and that the bilinear form $(u, v) \mapsto \int_{\mathbb{R}^2} \nabla u \cdot \nabla v \mathrm{d}x$ is coercive on V. Therefore, there exists a unique solution $v \in V$ to the problem

$$\forall \psi \in V, \int_{\mathbb{R}^2} \nabla v \cdot \nabla \psi \mathrm{d}x = \int_{\mathbb{R}^2} f \psi \mathrm{d}x.$$

Classically, this implies the existence of a constant $\lambda \in \mathbb{R}$ such that

$$\forall \psi \in \mathcal{D}^{1,2}(\mathbb{R}^2), \int_{\mathbb{R}^2} \nabla v \cdot \nabla \psi \mathrm{d}x = \int_{\mathbb{R}^2} f \psi \mathrm{d}x + \lambda \int_{B(0,1)} \psi \mathrm{d}x.$$

Setting $\psi = 1$, and using the compatibility condition, we find $\lambda = 0$ and the first point of the proposition is proved.

2 and 3. The function

$$\tilde{v}(x) := -\int_{\mathbb{R}^2} \Gamma(x-y) f(y) \mathrm{d}y$$

satisfies $-\Delta\tilde{v} = f$ in $\mathbb{R}^2$. Furthermore, $\tilde{v} = O(|x|^{-1})$ at infinity, which is enough to ensure that $\tilde{v} \in \mathcal{D}^{1,2}(\mathbb{R}^2)$. Hence v and $\tilde{v}$ must differ by a constant. □

Remark 8. *Due to the explicit representation* (19) *and the possible choice $v_\infty = 0$ for the constant, we often prefer to write the problem* (17) *as*

$$\begin{cases} -\Delta v = f \ \text{ in } \mathbb{R}^2, \\ \quad\ v = O(|x|^{-1}) \ \text{ as } |x| \to +\infty, \end{cases}$$

which admits a unique solution if the compatibility condition (18) *is satisfied.*

The Dirichlet problem has a slightly different conclusion because the constant v_∞ is determined by the boundary condition on ∂D and can possibly be different from 0.

Proposition 6. *Let $g \in H^{\frac{1}{2}}(\partial D)$.*

1. *There exists a unique solution $v \in \mathcal{D}^{1,2}(\mathbb{R}^2)$ to the problem*

$$\begin{cases} -\Delta v = 0 \ \text{ in } \mathbb{R}^2 \backslash \partial D, \\ \quad\ v = g \ \text{ on } \partial D, \\ \quad\ v \in \mathcal{D}^{1,2}(\mathbb{R}^2). \end{cases} \tag{20}$$

2. *The outer normal flux (as well as the inner normal flux) of v vanishes:*

$$\int_{\partial D} \left.\frac{\partial v}{\partial n}\right|_+ \mathrm{d}\sigma = 0. \tag{21}$$

3. *There exists a constant v_∞ such that the function v admits the following single layer potential representation:*

$$v(x) = v_\infty + \int_{\partial D} \Gamma(x-y) \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right](y) \mathrm{d}\sigma(y). \tag{22}$$

Consequently, the asymptotic behavior of v at infinity reads

$$v(x) = v_\infty - \nabla\Gamma(x) \cdot \int_{\partial D} y \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right](y) \mathrm{d}\sigma(y) + O(|x|^{-2}) \ \text{ as } |x| \to +\infty. \tag{23}$$

Proof. 1. Existence and uniqueness of a solution to (20) is obtained as in the case $d \geqslant 3$: we use a lifting of the Dirichlet condition and the fact that the bilinear form

$$(v, v') \mapsto \int_{\mathbb{R}^2 \backslash D} \nabla v \cdot \nabla v' \mathrm{d}x$$

is coercive on $\mathcal{D}_0^{1,2}(\mathbb{R}^2 \setminus \partial D)$. This results from the Poincaré inequality in the space

$$H_0^1(B(0,R) \setminus D) := \{v \in H^1(B(0,R) \setminus D) \,|\, v = 0 \text{ on } \partial D\},$$

which enables to write $||v||_{\mathcal{D}_0^{1,2}(\mathbb{R}^2 \setminus \partial D)} \leqslant C||\nabla v||_{L^2(\mathbb{R}^2)}$ using the definition (7) of the norm with $B(0,1)$ replaced with $B(0,R)$ for $R > 0$ large enough to contain D.
2. Integrating $-\Delta v = 0$ against the constant test function 1 in the whole set $\mathbb{R}^2 \setminus D$ implies (21) (this is possible because $1 \in \mathcal{D}^{1,2}(\mathbb{R}^2 \setminus D)$).
3. Consider the function $\tilde{v}$ defined by the single layer potential

$$\tilde{v}(x) := \mathcal{S}_D\left[\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right](x) = \int_{\partial D} \Gamma(x-y) \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right](y) \mathrm{d}\sigma(y).$$

The function $\tilde{v}$ satisfies $\Delta\tilde{v} = \Delta v = \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] \mathrm{d}\sigma$ in the distributional sense. This implies that $v - \tilde{v}$ is a harmonic function in $\mathbb{R}^2$. Since $\tilde{v} = O(|x|^{-1})$ because of (21) $v(x) - \tilde{v}(x) \in \mathcal{D}^{1,2}\mathbb{R}^2$ with $\Delta(v - \tilde{v}) = 0$. From the result of proposition 5, this function must be a constant, which we denote by v_∞. The asymptotic behavior follows from (21). □

Remark 9. *We prefer to write* (20) *as*

$$\begin{cases} -\Delta v = 0 \text{ in } \mathbb{R}^2 \setminus \partial D, \\ \quad v = g \text{ on } \partial D, \\ \quad v = v_\infty + O(|x|^{-1}) \text{ as } |x| \to +\infty, \end{cases} \tag{24}$$

where the unknown of the problem is $(v, v_\infty) \in \mathcal{D}^{1,2}(\mathbb{R}^2 \setminus \partial D) \times \mathbb{R}$.

We are now able to give the definition of the capacity in dimension 2.

Proposition 7. *1. There exists a unique solution* Φ *to the problem*

$$\begin{cases} -\Delta \Phi = 0 \text{ in } \mathbb{R}^2 \setminus D, \\ \quad \Phi = 0 \text{ on } \partial D, \\ \Phi(x) \sim \dfrac{1}{2\pi} \log |x| \text{ as } |x| \to +\infty. \end{cases} \tag{25}$$

satisfying $\Phi - \frac{1}{2\pi} \log |x| \in \mathcal{D}^{1,2}(\mathbb{R}^2 \setminus D)$.

2. *There exists a constant* Φ_∞ *such that* Φ *admits the following single layer potential representation:*

$$\Phi(x) = \mathcal{S}_D \left[\left. \frac{\partial \Phi}{\partial n} \right|_+ \right] (x) + \Phi_\infty. \tag{26}$$

Consequently, we have the asymptotic expansion

$$\Phi(x) = \frac{1}{2\pi} \log |x| + \Phi_\infty + O(|x|^{-1}). \tag{27}$$

3. *Independently of the shape of the obstacle* D*, the normal flux of* Φ *is equal to minus one:*

$$- \int_{\partial D} \left. \frac{\partial \Phi}{\partial n} \right|_+ \mathrm{d}\sigma = -1. \tag{28}$$

Proof. 1. The solution Φ is given by $\Phi(x) = \frac{1}{2\pi} \log |x| + \Psi(x)$ where Ψ is the unique solution in $\mathcal{D}^{1,2}(\mathbb{R}^2 \setminus D)$ to the difference problem

$$\begin{cases} -\Delta \Psi = 0 \text{ in } \mathbb{R}^2 \setminus D, \\ \quad \Psi = -\dfrac{1}{2\pi} \log |x| \text{ on } \partial D, \\ \quad \Psi = \Psi_\infty + O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases}$$

We use crucially the fact that $\Delta \log |x| = 0$ in $\mathbb{R}^2 \setminus \{0\}$.
2. The reasonning is the same as in point 2. of proposition 6, noticing that $\frac{\partial \Phi}{\partial n}\big|_- = 0$.
3. Due to (21), it holds, using $\Delta \log |x| = 0$ in the domain $D \setminus B(0, \epsilon)$, where $B(0, \epsilon)$ is a small ball of radius ϵ centered at 0 and contained in D:

$$\begin{aligned} - \int_{\partial D} \left. \frac{\partial \Phi}{\partial n} \right|_+ \mathrm{d}\sigma &= -\frac{1}{2\pi} \int_{\partial D} \frac{\partial \log |y|}{\partial n} \mathrm{d}\sigma(y) = -\frac{1}{2\pi} \int_{\partial B(0,\epsilon)} \frac{\partial \log |y|}{\partial n} \mathrm{d}\sigma(y) \\ &= -\frac{1}{2\pi} \int_{\partial B(0,\epsilon)} \frac{1}{\epsilon} \mathrm{d}\sigma(y) = -1. \end{aligned}$$

□

Definition 2. *The capacity of the obstacle* D *in dimension 2 is defined to be the positive constant*

$$\mathrm{cap}\,(D) := e^{2\pi \Phi_\infty} \tag{29}$$

where Φ_∞ *is the constant arising in the asymptotic* (27).

Remark 10. *The existence of a solution* Φ *growing logarithmically is peculiar to the dimension 2. Note that* $\Phi \notin \mathcal{D}^{1,2}(\mathbb{R}^2)$.

Remark 11. *The property* (28) *occurs only if* $d = 2$ *and is remarkable; it is related to the so-called "Stokes paradox" [1]. The Stokes paradox states that in dimension 2, there is no solution* $(\boldsymbol{v}, p)$ *to the problem*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla p = 0 \text{ in } \mathbb{R}^2 \setminus D, \\ \boldsymbol{v} = 0 \text{ on } \partial D, \\ \boldsymbol{v}(x) \sim \boldsymbol{e}_i \text{ as } |x| \to +\infty \end{cases}$$

where $\boldsymbol{e}_i$ *the unit direction of the flow (such a solution exists if* $d \geqslant 3$*) (or in other word, there does not exist an infinit flow creeping around a cylinder bounded at infinity). In fact, there exists a unique solution*

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla p = 0 \text{ in } \mathbb{R}^2 \setminus D, \\ \boldsymbol{v} = 0 \text{ on } \partial D, \\ \boldsymbol{v}(x) \sim \Gamma(x)\boldsymbol{e}_i \text{ as } |x| \to +\infty \end{cases}$$

where $\Gamma(x) \in \mathbb{R}^{2\times 2}$ *is the Kelvin matrix of the Stokes problem (see [4]) which grows logarithmically. Worse, such flow generates a drag force*

$$\boldsymbol{F} = -\int_{\partial D} (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T - pI)\boldsymbol{n} \mathrm{d}\sigma$$

which is independent of the shape of the obstacle.

Let us conclude with some remarks about the invertibility of the single layer potential in dimension 2, leading to a numerical way for solving (24).

Proposition 8. *The map*

$$\begin{array}{cccc} \pi & : & H_0^{-1/2}(\partial D) \times \mathbb{R} & \to & H^{\frac{1}{2}}(\partial D) \\ & & (\varphi, a) & \mapsto & a + \mathcal{S}_D[\phi] \end{array}$$

is invertible, where $H_0^{-\frac{1}{2}}(\partial D)$ *is the space*

$$H_0^{-\frac{1}{2}}(\partial D) := \left\{ \phi \in H^{-\frac{1}{2}}(\partial D) \,\Big|\, \int_{\partial D} \phi \mathrm{d}\sigma = 0 \right\},$$

where the integral is taken in the sense of the $H^{\frac{1}{2}}$*—*$H^{-\frac{1}{2}}$ *duality product. In fact,*

$$\pi^{-1}(g) = \left(\left[\!\left[\frac{\partial v}{\partial n} \right]\!\right], v_\infty \right)$$

where (v, v_∞) *is solution to* (24).

Corollary 2. *The single layer potential* $\mathcal{S}_D : H^{-\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ *is invertible in dimension 2 if and only if the condition*

$$\Phi_\infty \neq 0$$

is satisfied, where Φ_∞ *is the constant of* (27)*. In that case, the inverse reads explicitly*

$$\mathcal{S}_D^{-1}[g] = \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] - \frac{v_\infty}{\Phi_\infty} \left. \frac{\partial \Phi}{\partial n} \right|_+, \tag{30}$$

where (v, v_∞) *is the unique solution to* (24).

Proof. In view of (22), we can write

$$g = v_\infty + \mathcal{S}_D \left[\left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \right].$$

Furthermore, (26) reads on the boundary

$$0 = \Phi_\infty + \mathcal{S}_D \left[\frac{\partial \Phi}{\partial n}\bigg|_+ \right] \tag{31}$$

so that

$$v_\infty = -\frac{v_\infty}{\Phi_\infty} \mathcal{S}_D \left[\frac{\partial \Phi}{\partial n}\bigg|_+ \right],$$

and (30) is obtained. Reciprocally, if $\Phi_\infty = 0$, then (31) implies that $\mathcal{S}_D$ is not injective, hence not invertible. □

In computational practice, the result of proposition 8 can be used to solve the exterior problem (24). Indeed, it states that the system

$$\begin{cases} \mathcal{S}_D[\varphi] + a = g, \\ \int_{\partial D} \varphi \mathrm{d}y = 0, \end{cases} \tag{32}$$

can be inverted. Then the solution to (24) is given by

$$u(x) = \mathcal{S}_D[\phi](x) + a.$$

Remark 12. *In [3], the capacity is defined as* $\operatorname{cap}(D) = e^{2\pi a}$ *where* a *is the unique number such that there exists a function* ϕ *satisfying*

$$\mathcal{S}_D[\phi] = -a \text{ and } \int_{\partial D} \phi \mathrm{d}y = 1. \tag{33}$$

In fact, it is clear that $\phi = \frac{\partial \Phi}{\partial n}\big|_+$ *and* $a = \Phi_\infty$ *provides the unique solution to* (33)*, which shows the equivalence with the definition* (29).

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