Lecture n°9 — Topological asymptotics of the perforated poisson problem

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The object of this lecture is to show how layer potentials and Deny-Lions spaces enable to determine full asymptotic expansions of singularly perturbed PDE problems.

We consider the problem of determining asymptotic expansions of the solution u_ϵ to the problem

$$\begin{cases} -\Delta u_\epsilon = f \text{ in } \Omega, \\ \quad u_\epsilon = 0 \text{ on } \partial\omega_\epsilon, \\ \quad u_\epsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

as $\epsilon \to 0$. The problem (1) (illustrated on fig. 1) is known as the perforated Poisson problem; it

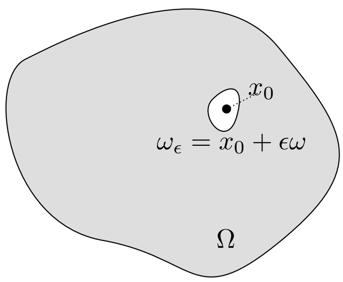

Figure 1: Schematic for the problem (1). Dirichlet boundary conditions are applied on the background domain Ω and on a small inclusion ω_ϵ.

finds applications e.g. in structural design [3, 4]. Ω is a smooth bounded open subdomain of $\mathbb{R}^d$ with $d \geqslant 2$, $f \in \mathcal{C}^\infty(\Omega)$ is a smooth right-hand side, and $\omega_\epsilon := x_0 + \epsilon\omega$ is a small inclusion obtained by centering a smooth domain ω around $x_0 \in \Omega$ and rescaling it by a size factor ϵ. In the limit where ϵ converges to zero, it can easily be shown that u_ϵ converges to the solution u to the same Dirichlet problem with no hole,

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

and the problem of finding the asymptotic of u_ϵ as $\epsilon \to 0$ is to understand how the solution to (2) is perturbed by a perforation of the domain Ω with the small hole ω_ϵ.

In what follows, we outline the derivation of higher order asymptotic expansions of u_ϵ. This material is based from [2] and considers the following method:

- first, an explicit representation of the physical solution u_ϵ in terms of layer potentials is sought. Owing to a change of variable in these integral operators, the explicit dependence with respect to the small parameter enables to determine the correct form of the ansatz satisfied by the full asymptotic expansion of the solution;

- second, the method of compound asymptotic expansions is applied to characterize the terms of this ansatz as the solutions to a cascade of equations. It is then possible to prove error bounds *a posteriori* with standard variational estimates and the structure of exterior Laplace solutions in $\mathbb{R}^d$.

This procedure is systematic and could be applied as well for many other types of perforated Dirichlet problems. It yields correct derivations in the delicate case $d = 2$.

1 Integral representation of the perforated solution

The first step is to determine an integral representation of the solution u_ϵ.

In what follows, we recall the expression of the fundamental solution Γ of the Laplace operator (i.e. $\Delta\Gamma = \delta_0$ in $\mathbb{R}^d$):

$$\Gamma(x) := \begin{cases} \dfrac{1}{2\pi}\log|x| \text{ if } d = 2, \\ -\dfrac{1}{(d-2)|\partial B(0,1)|}\dfrac{1}{|x|^{d-2}} \text{ if } d \geqslant 3, \end{cases} \tag{3}$$

where $|\partial B(0,1)|$ is the measure of the unit sphere of $\mathbb{R}^d$. For any Lipschitz open set $D \subset \mathbb{R}^d$, we denote by $\mathcal{S}_D$ the single layer potential on ∂D:

$$\forall x \in \mathbb{R}^d, \forall \phi \in H^{-\frac{1}{2}}(\partial D), \quad \mathcal{S}_D[\phi](x) := \int_{\partial D} \Gamma(x-y)\phi(y)\mathrm{d}\sigma(y). \tag{4}$$

From the fundamental solution Γ of (3), we construct the Laplace Green kernel $G_\Omega(x,y)$ with Dirichlet boundary conditions on Ω, defined as the unique solution to

$$\begin{cases} \Delta_y G_\Omega(x,\cdot) = \delta_x \text{ in } \Omega, \\ G_\Omega(x,\cdot) = 0 \text{ on } \partial\Omega, \end{cases} \qquad \text{for any } x \in \Omega. \tag{5}$$

Classically, the function $G_\Omega(x,\cdot)$ is constructed by using a difference problem [1].

Proposition 1. *The Green kernel G_Ω is given by*

$$G_\Omega(x,y) := \Gamma(x-y) + R_\Omega(x,y),$$

where for any $x \in \Omega$, $R_\Omega(x,\cdot)$ is the unique solution to the difference problem

$$\begin{cases} \Delta_y R_\Omega(x,\cdot) = 0 \textit{ in } \Omega, \\ R_\Omega(x,\cdot) = -\Gamma(x-\cdot) \textit{ on } \partial\Omega. \end{cases} \tag{6}$$

The function R_Ω satisfies $R_\Omega(x,y) = R_\Omega(y,x)$ for any $(x,y) \in \Omega \times \Omega$. Furthermore, for any $x \in \Omega$, R_Ω is a smooth function of Ω.

This allows to define the single layer potential $\mathcal{S}_{\Omega,\omega_\epsilon}$ with a Dirichlet boundary condition on $\partial\Omega$ from the formula

$$\forall \phi \in H^{-\frac{1}{2}}(\partial\omega_\epsilon), \quad \forall x \in \Omega, \quad \mathcal{S}_{\Omega,\omega_\epsilon}[\phi](x) := \int_{\partial\omega_\epsilon} G_\Omega(x,y)\phi(y)\mathrm{d}\sigma(y). \tag{7}$$

We note that the operator $\mathcal{S}_{\Omega,\omega_\epsilon}$ is a compact perturbation of the "classical" single layer potential $\mathcal{S}_{\omega_\epsilon}$:

$$\mathcal{S}_{\Omega,\omega_\epsilon}[\phi] = \mathcal{S}_{\omega_\epsilon}[\phi] + \int_{\partial\omega_\epsilon} R_\Omega(\cdot,y)\phi(y)\mathrm{d}\sigma(y). \tag{8}$$

The potential $\mathcal{S}_{\Omega,\omega_\epsilon}$ satisfies the following properties.

Proposition 2. *1. For any $\phi \in H^{-\frac{1}{2}}(\partial\omega_\epsilon)$, $\mathcal{S}_{\Omega,\omega_\epsilon}[\phi]$ satisfies*

$$\begin{cases} -\Delta\mathcal{S}_{\Omega,\omega_\epsilon}[\phi] = 0 \textit{ in } \Omega \setminus \partial\omega_\epsilon, \\ \mathcal{S}_{\Omega,\omega_\epsilon} = 0 \textit{ on } \partial\Omega. \end{cases}$$

2. $\mathcal{S}_{\Omega,\omega_\epsilon}$ *satisfies the jump relations*

$$[\![\mathcal{S}_{\Omega,\omega_\epsilon}[\phi]]\!] = 0, \qquad \left[\!\!\left[\frac{\partial \mathcal{S}_{\Omega,\omega_\epsilon}[\phi]}{\partial n}\right]\!\!\right] = \phi. \tag{9}$$

3. *The single layer potential* $\mathcal{S}_{\Omega,\omega_\epsilon}$ *is invertible when considered as an operator* $H^{-\frac{1}{2}}(\partial\omega_\epsilon) \to H^{\frac{1}{2}}(\partial\omega_\epsilon)$.

Proof. The point (i) is obtained from the definition (7). The point (ii) follows by using the jump relation on $\mathcal{S}_{\omega_\epsilon}$ and the fact that the perturbation in (8) is smoothing. Let us prove the point (iii). Recall that $\mathcal{S}_{\Omega,\omega_\epsilon}$ is a Fredholm operator of index 0 [5], as a compact perturbation of the Fredholm operator $\mathcal{S}_{\omega_\epsilon} : H^{-\frac{1}{2}}(\partial\omega_\epsilon) \to H^{\frac{1}{2}}(\partial\omega_\epsilon)$. Therefore, it is sufficient to show that this operator has a trivial kernel to prove its invertibility.
Let $\phi \in H^{-\frac{1}{2}}(\partial\omega_\epsilon)$ be such that $u := \mathcal{S}_{\Omega,\omega_\epsilon}[\phi] = 0$. The function u satisfies

$$\begin{cases} -\Delta u = 0 \text{ in } \Omega \setminus \partial\omega_\epsilon, \\ \quad u = 0 \text{ on } \partial\omega_\epsilon, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

which easily implies $u = 0$ on $\Omega\backslash\omega_\epsilon$ and on ω_ϵ. From the jump relation (ii), it holds $\phi = [\![\frac{\partial u}{\partial n}]\!] = 0$ hence $\mathcal{S}_{\Omega,\omega_\epsilon}$ is injective. □

It is now easy to infer the following integral representation for the solution u_ϵ to the perforated problem (1).

Proposition 3. *The following single layer potential representation holds for the solution* u_ϵ *to the perforated Poisson problem* (1)*:*

$$u_\epsilon(x) = u(x) - \mathcal{S}_{\Omega,\omega_\epsilon}[\mathcal{S}_{\Omega,\omega_\epsilon}^{-1}[u|_{\partial\omega_\epsilon}]], \tag{10}$$

where u *is the solution to the Poisson problem* (2) *without the hole.*

Proof. It is immediate to check from the properties of $\mathcal{S}_{\Omega,\omega_\epsilon}$ that the function

$$v_\epsilon := -\mathcal{S}_{\Omega,\omega_\epsilon}[\mathcal{S}_{\Omega,\omega_\epsilon}^{-1}[u|_{\partial\omega_\epsilon}]]$$

satisfies

$$\begin{cases} -\Delta v_\epsilon = 0 \text{ in } \Omega \setminus \overline{\omega}_\epsilon, \\ \quad v_\epsilon = -u|_{\partial\omega_\epsilon} \text{ on } \partial\omega_\epsilon. \end{cases}$$

Hence, by uniqueness of the solution to (1), we find that $u_\epsilon = u + v_\epsilon$. □

2 Factorization of the single layer representation

In what follows, we consider the mapping $\tau_{x_0,\epsilon}$ which rescales its argument by a factor ϵ around x_0:

$$\tau_{x_0,\epsilon}(t) := x_0 + \epsilon t, \qquad t \in \mathbb{R}^d, \tag{11}$$

and we introduce the (pull-back) operator $\mathcal{P}_{x_0,\epsilon} : H^s(\partial\omega_\epsilon) \to H^s(\partial\omega)$ defined for any $s \in \mathbb{R}$ by

$$\mathcal{P}_{x_0,\epsilon}[\phi] := \phi \circ \tau_{x_0,\epsilon} \text{ for any } \phi \in H^s(\partial\omega_\epsilon). \tag{12}$$

The operator $\mathcal{P}_{x_0,\epsilon}$ enables one to factorize $\mathcal{S}_{\Omega,\omega_\epsilon}$ in terms of an operator $\mathcal{S}_\omega(\epsilon) : H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega)$ which is defined on a space independent of ϵ, which is analytic in ϵ when $d \geqslant 3$, and in $\log\epsilon$ when $d = 2$.

Proposition 4. *The following factorizations holds:*

$$\mathcal{S}_{\Omega,\omega_\epsilon} = \epsilon \mathcal{P}_{x_0,\epsilon}^{-1} \mathcal{S}_\omega(\epsilon) \mathcal{P}_{x_0,\epsilon}, \tag{13}$$

where $\mathcal{S}_\omega(\epsilon) : H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega)$ *is the operator defined by*

$$\mathcal{S}_\omega(\epsilon)[\phi](t) = \frac{1}{2\pi} \log \epsilon \int_{\partial\omega} \phi \mathrm{d}\sigma \, \delta_{d=2} + \mathcal{S}_\omega[\phi](t) + \epsilon^{d-2} \int_{\partial\omega} R_\Omega(x_0 + \epsilon t, x_0 + \epsilon t')\phi(t')\mathrm{d}\sigma(t'), \quad t \in \partial\omega. \tag{14}$$

Proof. For $\phi \in H^{-\frac{1}{2}}(\partial\omega)$ and $t \in \partial\omega$, we compute

$$\begin{aligned} \mathcal{S}_\omega(\epsilon)[\phi](t) &:= \mathcal{P}_{x_0,\epsilon} \mathcal{S}_{\Omega,\omega_\epsilon} \mathcal{P}_{x_0,\epsilon}^{-1}[\phi](t) = \mathcal{S}_{\Omega,\omega_\epsilon}[\phi \circ \tau_{x_0,\epsilon}^{-1}] \circ \tau_{x_0,\epsilon}(t) \\ &= \int_{\partial\omega_\epsilon} G_\Omega(x_0 + \epsilon t, y)\phi \circ \tau_{x_0,\epsilon}^{-1}(y)\mathrm{d}\sigma(y) = \epsilon^{d-1} \int_{\partial\omega} G_\Omega(x_0 + \epsilon t, x_0 + \epsilon t')\phi(t')\mathrm{d}\sigma(t') \\ &= \epsilon^{d-1} \int_{\partial\omega} \Gamma(\epsilon(t - t'))\phi(t')\mathrm{d}\sigma(t') + \epsilon^{d-1} \int_{\partial\omega} R_\Omega(x_0 + \epsilon t, x_0 + \epsilon t')\phi(t')\mathrm{d}\sigma(t'). \end{aligned}$$

The identity (14) is obtained by using $\Gamma(\epsilon(t - t')) = \frac{1}{2\pi} \log \epsilon \delta_{d=2} + \epsilon^{2-d}\Gamma(t - t')$. ☐

3 Full asymptotic expansions in the case $d \geqslant 3$

In order to obtain full asymptotic expansions of the solution u_ϵ to (1), we rewrite (10) in terms of $\mathcal{S}_\omega(\epsilon)$. Using (13), it is clear that

$$u_\epsilon(x) = u(x) - \mathcal{S}_\omega(\epsilon)\Big[\mathcal{S}_\omega(\epsilon)^{-1}[u|_{\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon}]\Big] \left(\frac{x - x_0}{\epsilon}\right), \tag{15}$$

therefore it is enough to compute the inverse of $\mathcal{S}_\omega(\epsilon)$. When $d \geqslant 3$, we have

$$\mathcal{S}_\omega(\epsilon)[\phi] = \mathcal{S}_\omega[\phi] + \epsilon^{d-2} \int_{\partial\omega} R_\Omega(x_0 + \epsilon\cdot, x_0 + \epsilon t')\phi(t')\mathrm{d}\sigma(t').$$

Since $\mathcal{S}_\omega$ is invertible, it is straightforward to obtain that the inverse of $\mathcal{S}_\omega(\epsilon)$ is analytic in ϵ.

Proposition 5. *Assume* $d \geqslant 3$*. Then* $\mathcal{S}_\omega(\epsilon) : H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega)$ *is an analytic operator in* ϵ *and we have further*

$$\mathcal{S}_\omega(\epsilon)^{-1} = \mathcal{S}_\omega^{-1} + O(\epsilon^{d-2}), \tag{16}$$

where $O(\epsilon^{d-2})$ *is an analytic operators in* ϵ *estimated in operator norm.*

Inserting the asymptotic formula (16) into (15), we read a two-scale ansatz for u_ϵ.

Corollary 1. *Assume* $d \geqslant 3$*. There exist functions* $(v_p)_{p\geqslant 0}$ *and* $(w_p)_{p\geqslant d-2}$ *such that the following ansatz holds for the solution* u_ϵ *to the perforated Laplace problem* (1)*:*

$$u_\epsilon(x) = u(x) + \sum_{p=0}^{+\infty} \epsilon^p v_p\left(\frac{x - x_0}{\epsilon}\right) + \sum_{p=d-2}^{+\infty} \epsilon^p w_p(x), \qquad x \in \Omega \setminus \omega_\epsilon, \tag{17}$$

where:

1. *the series* (17) *converges for any fixed* $x \in \Omega \backslash \{x_0\}$*;*
2. *for any* $p \geqslant 0$*,* $v_p \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ *is the solution of an exterior Dirichlet problem of the form*

$$\begin{cases} -\Delta v_p = 0 \text{ in } \mathbb{R}^d \setminus \partial\omega, \\ v_p = g_p \text{ on } \partial\omega, \\ v_p \in \mathcal{D}^{1,2}(\mathbb{R}^d), \end{cases} \tag{18}$$

for some $g_p \in H^{\frac{1}{2}}(\partial\omega)$*.*

3. $w_p \in H^1(\Omega)$ *for any* $p \geqslant d-2$*;*

4. *the series* $\sum_{p=0}^{+\infty} \epsilon^p v_p$ *is convergent in* $\mathcal{D}^{1,2}(\mathbb{R}^d \setminus \overline{\omega})$*;*

5. *the series* $\sum_{p=d-2}^{+\infty} \epsilon^p w_p$ *is convergent in* $H^1(\Omega)$*;*

6. *the first term of the series is given by* $v_0 = -u(x_0)\mathcal{S}_\omega[\mathcal{S}_\omega^{-1}[1_{\partial\omega}]] = -u(x_0)\Phi$ *with* $\Phi := \mathcal{S}_\omega[\mathcal{S}_\omega^{-1}[1_{\partial\omega}]]$ *being the solution to the exterior problem* (19)*.*

$$\begin{cases} -\Delta\Phi = 0 \text{ in } \mathbb{R}^d \setminus \overline{\omega}, \\ \quad \Phi = 1 \text{ on } \partial\omega, \\ \Phi(x) \to 0 \text{ as } |x| \to +\infty. \end{cases} \tag{19}$$

Proof. We use the representation (15). Since $u_{|\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon} = u(x_0)1_{\partial\omega} + O(\epsilon)$, we find by using (16):

$$\mathcal{S}_\omega(\epsilon)^{-1}[u|_{\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon}] = u(x_0)\mathcal{S}_\omega^{-1}[1_{\partial\omega}] + O(\epsilon),$$

where $O(\epsilon)$ is analytic in ϵ. Noticing that (14) can be rewritten as

$$\mathcal{S}_\omega(\epsilon)[\phi]\left(\frac{x-x_0}{\epsilon}\right) = \mathcal{S}_\omega[\phi]\left(\frac{x-x_0}{\epsilon}\right) + \epsilon^{d-2}\int_{\partial\omega} R_\Omega(x, x_0+\epsilon t')\phi(t')\mathrm{d}\sigma(t'),$$

we obtain the ansatz (17) by inserting $\phi = \mathcal{S}_\omega(\epsilon)^{-1}[u_{|\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon}]$, using a Taylor series and by identifying powers of ϵ. The convergence of the series results from the convergence of the far field expansion (20) (proposition 6 below) for the fundamental solution. The properties (ii)-(v) are then easily verified. The property (vi) is obtained by computing explicitly the leading order asymptotic:

$$u_\epsilon(x) = u(x) - u(x_0)\mathcal{S}_\omega[\mathcal{S}_\omega^{-1}[1_{\partial\omega}]]\left(\frac{x-x_0}{\epsilon}\right) + O(\epsilon) = u(x) - u(x_0)\Phi\left(\frac{x-x_0}{\epsilon}\right) + O(\epsilon),$$

from where the value of v_0 is inferred. □

We recall the following far field asymptotic expansion result for the fundamental solution, used in the proof of the previous proposition.

Proposition 6. 1. *For any* $\phi \in H^{-\frac{1}{2}}(\partial D)$*,* $\mathcal{S}_D[\phi]$ *has the following asymptotic behavior at infinity:*

$$\mathcal{S}_D[\phi](x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!}\nabla^k\Gamma(x) \cdot \int_{\partial D} y^k\phi(y)\mathrm{d}\sigma(y), \tag{20}$$

where the series converge for any $x \in \mathbb{R}^d \setminus D$ *satisfying* $|x| > \sup_{y\in D}|y|$*.*

In (20)*,* y^k *and* $\nabla^k\Gamma$ *denote the* k*-th order tensors*

$$y^k = (y_{i_1}y_{i_2}\dots y_{i_k})_{1\leqslant i_1\dots i_k\leqslant d}, \quad \nabla^k\Gamma = (\partial^k_{i_1\dots i_k}\Gamma)_{1\leqslant i_1\dots i_k\leqslant d},$$

and $\nabla^k\Gamma(x)\cdot y^k$ *is their contraction:*

$$\nabla^k\Gamma(x)\cdot y^k := \sum_{1\leqslant i_1\dots i_k\leqslant d} \partial^k_{i_1\dots i_k}\Gamma(x)y_{i_1}\dots y_{i_k}.$$

2. *The solution* v_p *of an exterior Dirichlet problem of the form* (18) *admits the following far field expansion at infinity:*

$$v(x) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\nabla^k\Gamma(x)\cdot\int_{\partial\omega}\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] t^k\mathrm{d}\sigma(t). \tag{21}$$

Proof. This is a consequence of

$$\Gamma(x-y) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \nabla^k \Gamma(x) \cdot y^k$$

valid for $|x| > |y|$. □

Inserting the ansatz (17) into the original perforated Laplace problem (1) and identifying identical powers of ϵ, the following proposition shows that the functions $(v_p)_{p\geqslant 0}$ and $(w_p)_{p\geqslant d-2}$ are sequences of correctors, correcting successive errors on the boundary and in the vicinity of the hole committed by the previous correctors.

Proposition 7. *The functions $(v_p)_{p\geqslant 0}$ and $(w_p)_{p\geqslant d-2}$ of (17) are uniquely characterized by the following recursive system of exterior and interior problems:*

$$\begin{cases} -\Delta v_p = 0 & \text{in } \mathbb{R}^d \setminus \overline{\omega},\ p \geqslant 0, \\ v_p(t) = -\dfrac{1}{p!} \nabla^p u(x_0) \cdot t^p & \text{for } t \in \partial\omega,\ 0 \leqslant p < d-2, \\ v_p(t) = -\dfrac{1}{p!} \nabla^p u(x_0) \cdot t^p - \displaystyle\sum_{k=0}^{p-d+2} \frac{1}{k!} \nabla^k w_{p-k}(x_0) \cdot t^k & \text{for } t \in \partial\omega,\ p \geqslant d-2, \\ v_p(x) = O(|x|^{2-d}) & \text{as } |x| \to +\infty,\ p \geqslant 0, \end{cases} \tag{22}$$

and

$$\begin{cases} -\Delta w_p = 0 & \text{in } \Omega, \\ w_p(x) = -\displaystyle\sum_{k=d-2}^{p} v_{p-k}^{(k)}(x) & \text{for } x \in \partial\Omega, \end{cases} \qquad \text{for all } p \geqslant d-2, \tag{23}$$

where for any $p \geqslant 0$ and $k \geqslant d-2$, $v_p^{(k)}$ are the functions occuring in the far field expansion of v_p, i.e.:

$$v_p\left(\frac{x-x_0}{\epsilon}\right) = \sum_{k=d-2}^{+\infty} \epsilon^k v_p^{(k)}(x) \tag{24}$$

with

$$v_p^{(k)}(x) := \frac{(-1)^{k-d+2}}{(k-d+2)!} \nabla^{k-d+2}\Gamma(x-x_0) \cdot \int_{\partial\omega} \left[\!\left[\frac{\partial v_p}{\partial n} \right]\!\right] t^{k-d+2} \mathrm{d}\sigma(t), \qquad x \in \partial\Omega, k \geqslant d-2. \tag{25}$$

Proof. Inserting $x = x_0 + \epsilon t$ with $t \in \partial\omega$ in the ansatz (17), using the boundary condition satisfied by $u_\epsilon - u$ and some Taylor expansions in the vicinity of x_0, we obtain:

$$-\sum_{p=0}^{+\infty} \epsilon^p \frac{1}{p!} \nabla^p u(x_0) \cdot t^p = u_\epsilon(x_0+\epsilon t) - u(x_0+\epsilon t) = \sum_{p=0}^{+\infty} \epsilon^p v_p(t) + \sum_{p=d-2}^{+\infty} \sum_{k=0}^{+\infty} \epsilon^{p+k} \frac{1}{k!} \nabla^k w_p(x_0) \cdot t^k$$
$$= \sum_{p=0}^{+\infty} \epsilon^p v_p(t) + \sum_{p=d-2}^{+\infty} \sum_{k=0}^{p-d+2} \epsilon^p \frac{1}{k!} \nabla^k w_{p-k}(x_0) \cdot t^k.$$

Identifying identical powers of ϵ yields the system (22). Then, considering $x \in \partial\Omega$, we find by using the far field expansion (25):

$$0 = u_\epsilon(x) - u(x) = \sum_{p=0}^{+\infty} \sum_{k=d-2}^{+\infty} \epsilon^{p+k} v_p^{(k)}(x) + \sum_{p=d-2}^{+\infty} \epsilon^p w_p(x) = \sum_{p=d-2} \sum_{k=d-2}^{p} \epsilon^p v_{p-k}^{(k)}(x) + \sum_{p=d-2}^{+\infty} \epsilon^p w_p(x).$$

Hence, (23) follows by identifying identical powers of ϵ. □

Remark 1. *The system (22) and (23) determines completely the functions $(v_p)_{p\geqslant 0}$ and $(w_p)_{p\geqslant d-2}$. Indeed, the functions $(v_{p'})_{0\leqslant p'<d-2}$ are determined from (22). Then if $(v_{p'})_{0\leqslant p'\leqslant p-d+2}$ is determined for $p \geqslant d-2$, then w_p is determined from the boundary value problem (23), which determines in turn v_p through the exterior problem (22).*

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