Lecture n°10 — Minnaert resonances and holomorphic Fredholm operators

Florian Feppon

December 6, 2021

In this lecture, we study the phenomenon of subwavelength resonances for acoustic waves in high-contrast media. we consider an acoustic medium $D \subset \mathbb{R}^3$ constituted of N smooth connected components B_i (the "bubbles" or acoustic resonators):

$$D = \bigcup_{i=1}^{N} B_i.$$

We refer to fig. 1 for an illustration of the setting. The background medium $\mathbb{R}^3\backslash\overline{D}$ is a homogeneous acoustic material characterized by a homogeneous density ρ and bulk modulus κ. The "bubbles" are acoustic heterogeneities with homogeneous density ρ_b and bulk modulus κ_b. We are interested in the scattering of an incoming wave u_{in} propagating through the bulk material with frequency ω. We denote by

$$v = \sqrt{\frac{\kappa}{\rho}}, \quad v_b = \sqrt{\frac{\kappa_b}{\rho_b}}, \quad k = \frac{\omega}{v}, \quad k_b = \frac{\omega}{v_b}$$

the sound velocities v and v_b and the wave numbers k and k_b in respectively the background medium and the acoustic obstacles. We consider the high-contrast regime whereby the quantity

$$\delta := \frac{\rho_b}{\rho}$$

is asymptotically small: $\delta \to 0$. The incoming sound wave u_{in} is the solution to the Helmholtz equation in the free space $\mathbb{R}^3$; it satisfies

$$\nabla \cdot \left(\frac{1}{\rho}\nabla u_{\text{in}}\right) + \frac{\omega^2}{\kappa} u_{\text{in}} = 0 \text{ in } \mathbb{R}^3 \setminus D.$$

The wave u_{in} generates a scattered field u_s, which is such that the total field $u_{\text{tot}} := u_{\text{in}} + u_s$ is

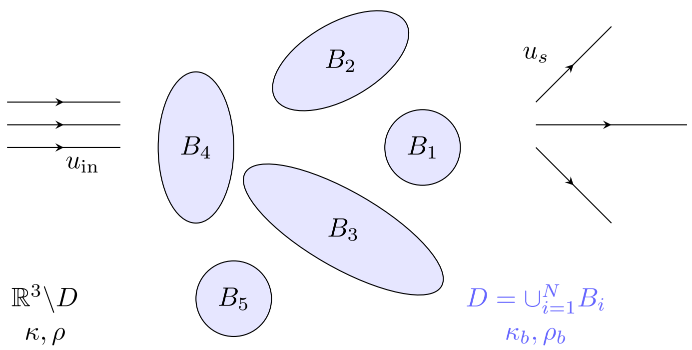

Figure 1: Distribution of acoustic obstacles in the three-dimensional space $\mathbb{R}^3$. An incident wave u_{in} is propagating with frequency ω and generates a total wave field u_{tot}.

the solution to the following scattering problem:

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\rho_b} \nabla u_{\mathrm{tot}} \right) + \dfrac{\omega^2}{\kappa_b} u_{\mathrm{tot}} = 0 \text{ in } D, \\ \nabla \cdot \left(\dfrac{1}{\rho} \nabla u_{\mathrm{tot}} \right) + \dfrac{\omega^2}{\kappa} u_{\mathrm{tot}} = 0 \text{ in } \mathbb{R}^3 \backslash D, \\ u_{\mathrm{tot},+} - u_{\mathrm{tot},-} = 0 \text{ on } \partial D, \\ \dfrac{1}{\rho_b} \dfrac{\partial u_{\mathrm{tot}}}{\partial n} \bigg|_- = \dfrac{1}{\rho} \dfrac{\partial u_{\mathrm{tot}}}{\partial n} \bigg|_+ \text{ on } \partial D, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k \right) (u_{\mathrm{tot}} - u_{\mathrm{in}}) = O\left(\dfrac{1}{|x|^2} \right) \text{ as } |x| \to +\infty, \end{cases} \tag{1}$$

where $u_{\mathrm{tot},+}$ and $u_{\mathrm{tot},-}$ denote the trace of u_{tot} on respectively the outer and the inner boundaries of the obstacles ∂D, and $\partial u_{tot}/\partial n|_-$ and $\partial u_{tot}/\partial n|_+$ the inner and outer normal derivatives with the normal vector $\boldsymbol{n}$ pointing outward D. The last equation is the outgoing Sommerfeld radiation condition for the scattered field u_s.

The goal of these notes is to show that as $\delta \to 0$, there exists $2N$ complex resonant frequencies $(\omega_i^{\pm}(\delta))_{1 \leqslant i \leqslant N}$ satisfying $\omega_i^{\pm}(\delta) = O(\delta^{\frac{1}{2}})$. For a real frequency ω satisfying $\omega \to \omega_i^{+}(\delta)$, we shall establish a point scatterer approximation of the form

$$u_{\mathrm{tot}}(x) - u_{\mathrm{in}}(x) \simeq \frac{\alpha}{\frac{\omega^2}{\Re(\omega_i^+(\delta))} - 1 + \mathrm{i}\tau\omega} u_{\mathrm{in}}(x) \Gamma^k(x) \text{ as } |x| \to +\infty, \tag{2}$$

for some constants α and τ, showing that there is a magnification of the total field around the resonant frequencies. The formula (2) shows the effect of a single group of bubble on the incident wave field. Since the magnification occurs at wavelength $2\pi v/\omega$ much larger than the size of the bubbles, the system D manipulates the incident wave at subwavelength scales. We can construct a metamaterial by filling a bounded domain with many small packets of such resonators and it is possible to guess the physical properties of the effective medium from (2), we will discuss this in the next lecture.

The material of the present lecture is strongly inspired from my recent work [6], which is a sequel to several previous analyses [3, 2].

1 Integral formulation of the scattering problem

We had seen in the first lecture that the solution u_{tot} can be represented as single layer potentials in D and $\mathbb{R}^3 \backslash D$:

$$u_{\mathrm{tot}}(x) = \begin{cases} \mathcal{S}_D^{k_b}[\phi](x) \text{ if } x \in \overline{D}, \\ u_{\mathrm{in}}(x) + \mathcal{S}_D^{k}[\psi](x) \text{ if } x \in \mathbb{R}^3 \setminus D, \end{cases} \tag{3}$$

where the functions $(\phi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ solve the integral equation

$$\mathcal{A}(\omega, \delta) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} u_{\mathrm{in}} \\ \delta \frac{\partial u_{\mathrm{in}}}{\partial n} \end{bmatrix}, \tag{4}$$

with the operator $\mathcal{A}(\omega, \delta)$ being given by

$$\mathcal{A}(\omega, \delta) = \begin{bmatrix} \mathcal{S}_D^{k_b} & -\mathcal{S}_D^{k} \\ -\frac{1}{2}I + \mathcal{K}_D^{k_b *} & -\delta\left(\frac{1}{2}I + \mathcal{K}_D^{k*}\right) \end{bmatrix}.$$

Due to the Sommerfeld radiation condition, the problem (4) can be shown to admit a unique solution for any real frequency provided the wave number $k = \omega/v$ is not a Dirichlet eigenvalue of the domain D [4]. This assumption is naturally satisfied in the regime $\omega \to 0$.

In order to compute the inverse of $\mathcal{A}(\omega,\delta)$, we solve the following linear system (4) which reads explicitly

$$\begin{cases} \mathcal{S}_D^{k_b}[\phi] - \mathcal{S}_D^{k}[\psi] = u_{\mathrm{in}}, \\ \left(-\frac{1}{2}I + \mathcal{K}_D^{k_b*}\right)[\phi] - \delta\left(\frac{1}{2}I + \mathcal{K}_D^{k*}\right)[\psi] = \delta\frac{\partial u_{\mathrm{in}}}{\partial n}. \end{cases} \tag{5}$$

Reducing (5) to a single equation by using the invertibility of $\mathcal{S}_D^k$ we are left with

$$\begin{cases} \psi = (\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}[\phi] - (\mathcal{S}_D^k)^{-1}[u_{\mathrm{in}}], \\ \left(-\frac{1}{2}I + \mathcal{K}_D^{k_b*} - \delta\left(\frac{1}{2}I + \mathcal{K}_D^{k*}\right)(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}\right)[\phi] = \delta\frac{\partial u_{\mathrm{in}}}{\partial n} - \delta\left(\frac{1}{2}I + \mathcal{K}_D^{k*}\right)(\mathcal{S}_D^k)^{-1}[u_{\mathrm{in}}]. \end{cases} \tag{6}$$

So the invertibility of $\mathcal{A}(\omega,\delta)$ is equivalent to that of the operator

$$\mathcal{L}(\omega,\delta) := -\frac{1}{2}I + \mathcal{K}_D^{k_b*} - \delta\left(\frac{1}{2}I + \mathcal{K}_D^{k*}\right)(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}. \tag{7}$$

The operator $\mathcal{L}(\omega,\delta)$ is holomorphic in the variables ω and δ. Indeed, we recall the following classical expansions of the potential (see e.g. [4]).

Proposition 1. *The following expansions hold for the single layer potential and the Neumann-Poincaré operator as $k = \omega/v \to 0$:*

$$\mathcal{S}_D^k = \sum_{p=0}^{+\infty} k^p \mathcal{S}_{D,p} = \mathcal{S}_D + k\mathcal{S}_{D,1} + k^2\mathcal{S}_{D,2} + \dots, \tag{8}$$

$$\mathcal{K}_D^{k*} = \sum_{p=0}^{+\infty} k^p \mathcal{K}_{D,p}^* = \mathcal{K}_D^* + k^2\mathcal{K}_{D,2}^* + k^3\mathcal{K}_{D,3}^* + \dots, \tag{9}$$

where the series converges in operator norms, and where the operators $\mathcal{S}_{D,p}$ and $\mathcal{K}_{D,p}^*$ are defined by

$$\mathcal{S}_{D,p}[\phi] := -\frac{\mathrm{i}^p}{4\pi p!}\int_{\partial D}|x-y|^{p-1}\phi(y)\mathrm{d}\sigma(y), \qquad \phi \in L^2(\partial D),\ p \in \mathbb{N}, \tag{10}$$

$$\mathcal{K}_{D,p}^*[\phi] := -\frac{\mathrm{i}^p}{4\pi p!}\int_{\partial D}\boldsymbol{n}(x)\cdot\nabla_x|x-y|^{p-1}\phi(y)\mathrm{d}\sigma(y), \qquad \phi \in L^2(\partial D),\ p \in \mathbb{N}. \tag{11}$$

Furthermore, we have the identities

(i) $\Delta\mathcal{S}_{D,0}[\phi] = \Delta\mathcal{S}_{D,1}[\phi] = 0$ *and* $\Delta\mathcal{S}_{D,p}[\phi] = -\mathcal{S}_{D,p-2}[\phi]$ *for any* $p \geqslant 2$,

(ii) $\mathcal{K}_{D,p}[\phi](x) = \boldsymbol{n}(x)\cdot\nabla_x\mathcal{S}_{D,p}[\phi]$ *for* $p \geqslant 1$, *and*

$$\int_{\partial B_i}\mathcal{K}_D^*[\phi]\mathrm{d}\sigma = \frac{1}{2}\int_{\partial B_i}\phi\mathrm{d}\sigma \ \textit{and} \ \int_{\partial B_i}\mathcal{K}_{D,p}^*[\phi]\mathrm{d}\sigma = -\int_{B_i}\mathcal{S}_{D,p-2}[\phi]\mathrm{d}\sigma \ \textit{for}\ p \geqslant 2.$$

In view of (9) we find that (7) can be rewritten as

$$\mathcal{L}(\omega,\delta) = -\frac{1}{2}I + \mathcal{K}_D^* + \omega^2\mathcal{B}_1(\omega) + \delta\mathcal{B}_2(\omega), \tag{12}$$

where $\mathcal{B}_1(\omega)$ and $\mathcal{B}_2(\omega)$ are the holomorphic and compact operators defined by

$$\mathcal{B}_1(\omega) := \sum_{p=2}^{+\infty}\frac{\omega^p}{v_b^p}\mathcal{K}_{D,p}^*, \qquad \mathcal{B}_2(\omega) := -\left(\frac{1}{2}I + \mathcal{K}_D^{k*}\right)(\mathcal{S}_D^k)^{-1}\mathcal{S}_D^{k_b}. \tag{13}$$

Classically, the computation of the inverse of the holomorphic Fredholm operator $\mathcal{L}(\omega,\delta)$ reduces to that of a finite dimensional holomorphic Schur complement matrix after introducing suitable projections on the kernel and coimage [8, 9]. In our context, we compute $\mathcal{L}(\omega,\delta)^{-1}$ by using a method inspired from [5] which consists in introducing a constant finite-range operator $\mathcal{H}$ making the operator $-\frac{1}{2}I + \mathcal{K}_D^* + \mathcal{H}$ invertible.

2 Inverse of holomorphic Fredholm operators

In order to study the properties of $\mathcal{L}(\omega,\delta)^{-1}$, we establish in this part a result regarding the inverse of holomorphic Fredholm operators. These results are to be related to Gohberg and Sigal theory, e.g. [8].

Let $\mathcal{A}(z) := \mathcal{A}_0 + z\mathcal{B}(z) : V \to V$ a parameterized Fredholm operator of index 0 holomorphic with respect to z. Assume that $\dim(\mathrm{Ker}(\mathcal{A}(z))) = N$ and denote by H and G two N-dimensional complements of $\mathrm{Ker}\mathcal{A}_0$ and $\mathrm{Ran}\mathcal{A}_0$:

$$V = \mathrm{Ker}\mathcal{A}_0 \oplus H = G \oplus \mathrm{Ran}\mathcal{A}_0.$$

Let $\mathcal{H}$ be any operator of the form

$$\mathcal{H} = \begin{array}{cc} \begin{array}{cc} \overbrace{\quad}^{\mathrm{Ker}\mathcal{A}_0} & \overbrace{\qquad\qquad}^{H} \end{array} & \\ \left[\begin{array}{c|c} \widehat{\mathcal{H}} & 0 \\ \hline 0 & 0 \end{array}\right] & \begin{array}{l} \}\,G \\ \}\,\mathrm{Ran}\mathcal{A}_0 \end{array} \end{array}$$

where $\widetilde{\mathcal{H}}$ is an invertible operator $\mathrm{Ker}\mathcal{A}_0 \to G$.

Lemma 1. *(i) $\mathcal{A}_0 + \mathcal{H}$ is invertible and its inverse reads*

$$(\mathcal{A}_0 + \mathcal{H})^{-1} = \begin{array}{cc} \begin{array}{cc} \overbrace{\quad}^{G} & \overbrace{\qquad\qquad}^{\mathrm{Ran}\mathcal{A}_0} \end{array} & \\ \left[\begin{array}{c|c} \widehat{\mathcal{H}}^{-1} & 0 \\ \hline 0 & \mathcal{A}_0^{-1} \end{array}\right] & \begin{array}{l} \}\,\mathrm{Ker}\mathcal{A}_0 \\ \}\,H \end{array} \end{array}$$

where $\mathcal{A}_0^{-1}$ denotes the inverse of $\mathcal{A}_0 : H \to \mathrm{Ran}\mathcal{A}_0$.

(ii) The linear operator $\mathcal{A}_0 + \mathcal{H} + zB(z) : V \to V$ is invertible for small z and the inverse reads

$$(\mathcal{A}_0 + \mathcal{H} + zB(z))^{-1} = (\mathcal{A}_0 + \mathcal{H})^{-1} - \mathcal{C}(z),$$

where $\mathcal{C}(z)$ is the operator defined by the Neumann series

$$\mathcal{C}(z) = \sum_{p\geqslant 1} (-1)^{p+1} z^p (\mathcal{A}_0 + \mathcal{H})^{-1} (B(z)(\mathcal{A}_0 + \mathcal{H})^{-1})^p.$$

Proposition 2. *The linear system*

$$\mathcal{A}(z)[\phi] = f \tag{14}$$

is invertible if and only if the N-dimensional linear system $\mathrm{Ker}\mathcal{A}_0 \to G$,

$$\mathcal{H}\mathcal{C}(z)\psi = \mathcal{H}(\mathcal{A}_0 + \mathcal{H} + zB(z))^{-1}[f], \qquad \psi \in \mathrm{Ker}\mathcal{A}_0, \tag{15}$$

is invertible. When this is the case, the unique solution to (14) is given by

$$\phi = (\mathcal{A}_0 + \mathcal{H} + zB(z))^{-1}[\psi + f] \tag{16}$$

and it holds $\psi = \mathcal{H}[\phi]$.

Proof. We rewrite (14) as

$$(\mathcal{A}_0+\mathcal{H}+zB(z))[\phi]-\mathcal{H}[\phi]=f,$$

which is equivalent to

$$\phi-(\mathcal{A}_0+\mathcal{H}+zB(z))^{-1}\mathcal{H}[\phi]=(\mathcal{A}_0+\mathcal{H}+zB(z))^{-1}[f]. \tag{17}$$

If this equation has a solution, then left multiplying by $\mathcal{H}$ and setting $\psi=\mathcal{H}[\phi]$, we obtain

$$\psi-\mathcal{H}(\mathcal{A}_0+\mathcal{H})^{-1}\psi+\mathcal{H}\mathcal{C}(z)[\psi]=\mathcal{H}(\mathcal{A}_0+\mathcal{H}+zB(z))^{-1}[f].$$

The linear system (15) follows since for $\psi\in\operatorname{Ker}\mathcal{A}_0$, $\mathcal{H}(\mathcal{A}_0+\mathcal{H})^{-1}[\psi]=\psi$. Then (16) is obtained from (17).

Reciprocally, if (15) has a unique solution, one verifies that setting (16) yields a solution to (14). □

The result of proposition 2 is very powerful: it shows that the inversion of the infinite dimensional system (14) can be reduced to a finite dimensional one (15), if one knows the inverse of $\mathcal{A}_0+\mathcal{H}$.

2.1 Minnaert resonances

The operator $\mathcal{L}(\omega,\delta)$ is a compact perturbation of the Fredholm operator $-\frac{1}{2}I+\mathcal{K}_D^*$, which has a finite dimensional kernel, as recalled in the following proposition (see e.g. [10, 7, 4]):

Proposition 3. *The kernel of the operator $-\frac{1}{2}I+\mathcal{K}_D^*$ is the N-dimensional space*

$$\operatorname{Ker}\left(-\frac{1}{2}I+\mathcal{K}_D^*\right)=\operatorname{span}((\psi_i^*)_{1\leqslant i\leqslant N}),$$

where $(\psi_i^*)_{1\leqslant i\leqslant N}$ are the functions defined by

$$\psi_i^*=\mathcal{S}_D^{-1}[1_{\partial B_i}],\qquad 1\leqslant i\leqslant N.$$

The range of the operator $-\frac{1}{2}I+\mathcal{K}_D^*$ is the space of zero average square integrable functions $L_0^2(\partial D)$:

$$\operatorname{Ran}\left(-\frac{1}{2}I+\mathcal{K}_D^*\right)=L_0^2(\partial D),$$

where $L_0^2(\partial D):=\left\{\phi\in L^2(\partial D)\,\middle|\,\int_{\partial B_i}\phi\mathrm{d}\sigma=0 \text{ for any } 1\leqslant i\leqslant N\right\}$. Furthermore, we have the direct sum decomposition

$$L^2(\partial D)=L_0^2(\partial D)\oplus\operatorname{Ker}\left(-\frac{1}{2}I+\mathcal{K}_D^*\right),$$

and $-\frac{1}{2}I+\mathcal{K}_D^*$ is invertible as an operator $L_0^2(\partial D)\to L_0^2(\partial D)$.

In order to introduce the operator $\mathcal{H}$ in the context of the operator $\mathcal{L}(\omega,\delta)$ of (7), we introduce a new basis of functions $(\phi_i^*)_{1\leqslant i\leqslant N}$ of $\operatorname{Ker}(-\frac{1}{2}I+\mathcal{K}^*)$ defined by

$$\phi_i^*:=-\sum_{j=1}^N(C^{-1})_{ij}\psi_j^*,\qquad 1\leqslant i\leqslant N, \tag{18}$$

where C is the *capacitance* matrix

$$C_{ij}:=-\int_{\partial B_i}\psi_j^*\mathrm{d}\sigma. \tag{19}$$

The definition (18) ensures the property

$$\int_{\partial B_i}\phi_j^*\mathrm{d}\sigma=\delta_{ij}\text{ for any }1\leqslant i,j\leqslant N. \tag{20}$$

Definition 1. *We choose* $\mathcal{H} : L^2(\partial D) \to L^2(\partial D)$ *to be the unique projection operator satisfying* $\mathrm{Ker}(\mathcal{H}) = L^2_0(\partial D)$ *and* $\mathrm{Ran}(\mathcal{H}) = \mathrm{Ker}(-\frac{1}{2}I + \mathcal{K}_D^*)$*. For any* $\phi \in L^2(\partial D)$*, the value of* $\mathcal{H}[\phi]$ *reads explicitly*

$$\mathcal{H}[\phi] = \sum_{i=1}^{N} \left(\int_{\partial B_i} \phi \mathrm{d}\sigma \right) \phi_i^*. \tag{21}$$

The operator $\mathcal{L}(\omega, \delta)$ reads

$$\mathcal{L}(\omega, \delta) = \mathcal{G}(\omega, \delta) - \mathcal{H}, \tag{22}$$

with $\mathcal{L}_0 := (-\frac{1}{2} + \mathcal{K}_D^*) + \mathcal{H}$. and where $\mathcal{G}(\omega, \delta)$ is the operator

$$\mathcal{G}(\omega, \delta) := \mathcal{L}_0 + \omega^2 \mathcal{B}_1(\omega) + \delta \mathcal{B}_2(\omega).$$

Let us introduce the operator $\mathcal{C}(\omega, \delta)$

$$\mathcal{C}(\omega, \delta) := \sum_{p=1}^{+\infty} (-1)^{p+1} \mathcal{L}_0^{-1} ((\omega^2 \mathcal{B}_1(\omega) + \delta \mathcal{B}_2(\omega)) \mathcal{L}_0^{-1})^p. \tag{23}$$

The result of proposition 2 reads

Proposition 4. *The operator* $\mathcal{A}(\omega, \delta)$ *is invertible if and only if the* $N \times N$ *matrix* $A(\omega, \delta) \equiv (A(\omega, \delta)_{ij})_{1 \leqslant i,j \leqslant N}$ *defined by*

$$A(\omega, \delta)_{ij} := \int_{\partial B_i} \mathcal{C}(\omega, \delta)[\phi_j^*] \mathrm{d}\sigma, \qquad 1 \leqslant i, j \leqslant N, \tag{24}$$

is invertible. When it is the case, the solution (ϕ, ψ) *to the problem* (5) *reads*

$$\begin{cases} \phi = \sum_{i=1}^{N} x_i \mathcal{G}^{-1}(\omega, \delta)[\phi_i^*] + \mathcal{G}^{-1}(\omega, \delta)[f], \\ \psi = \sum_{i=1}^{N} x_i (\mathcal{S}_D^k)^{-1} \mathcal{S}_D^{k_b} \mathcal{G}^{-1}(\omega, \delta)[\phi_i^*] + (\mathcal{S}_D^k)^{-1} \mathcal{S}_D^{k_b} \mathcal{G}^{-1}(\omega, \delta)[f] - (\mathcal{S}_D^k)^{-1}[u_{\mathrm{in}}], \end{cases} \tag{25}$$

where $f \in L^2(\partial D)$ *is the function*

$$f := \delta \frac{\partial u_{\mathrm{in}}}{\partial n} - \delta \left(\frac{1}{2} I + \mathcal{K}_D^{k*} \right) (\mathcal{S}_D^k)^{-1}[u_{\mathrm{in}}], \tag{26}$$

and where the coefficients $\boldsymbol{x} := (x_i)_{1 \leqslant i \leqslant N}$ *are the solutions to the finite dimensional problem*

$$A(\omega, \delta)\boldsymbol{x} = \boldsymbol{F} \text{ with } \boldsymbol{F} := \left(\int_{\partial B_i} \mathcal{G}^{-1}(\omega, \delta)[f] \mathrm{d}\sigma \right)_{1 \leqslant i \leqslant N}. \tag{27}$$

Let V be the (positive definite) diagonal matrix whose entries are the volumes of the resonators $(B_i)_{1 \leqslant i \leqslant N}$:

$$V := \mathrm{diag}((|B_i|)_{1 \leqslant i \leqslant N}). \tag{28}$$

In [1, 6], we prove the following result:

Proposition 5. *The following asymptotic holds true as* $\omega \to 0$ *and* $\delta \to 0$*:*

$$A(\omega, \delta) = \frac{\omega^2}{v_b^2} V C^{-1} - \delta I + O(\omega(\omega^2 + \delta)), \tag{29}$$

where C *is the capacitance matrix* (19)*,* V *is the volume matrix* (28) *and* $\mathbf{1} = (1)_{1 \leqslant i \leqslant N}$ *is the vector of ones.*

The expansion (29) shows that we can expect $2N$ resonances $(\omega_i^{\pm}(\delta))_{1<i\leqslant N}$ such that $A(\omega,\delta)$ has a non-trivial kernel, and hence $\mathcal{A}(\omega,\delta)$, these resonances satisfy

$$\omega_i^{\pm}(\delta) \sim v_b \lambda_i^{\frac{1}{2}} \delta^{\frac{1}{2}}$$

where $(\lambda_i)_{1\leqslant i\leqslant N}$ are the eigenvalues of the generalized eigenvalue problem

$$C\boldsymbol{a}_i = \lambda_i V \boldsymbol{a}_i. \tag{30}$$

Notably, the eigenvalues $(\lambda_i)_{1\leqslant i\leqslant N}$ enable to predict the frequencies of the subwavelength resonances thanks to the asymptotic $\omega_i^{\pm}(\delta) \sim \pm v_b \lambda_i^{\frac{1}{2}} \delta^{\frac{1}{2}}$ as $\delta \to 0$. When the eigenvalues of the capacitance matrix are simple, we can prove the following result.

Corollary 1. *Assume that the eigenvalues of the (weighted) capacitance matrix are simple. The subwavelength resonances $\omega_i^{\pm}(\delta)$ admit the following asymptotic expansions:*

$$\omega_i^{\pm}(\delta) = \pm\delta^{\frac{1}{2}} v_b \lambda_i^{\frac{1}{2}} - \frac{\mathrm{i} v_b^2 \lambda_i^2}{8\pi v} (\boldsymbol{a}_i^T V \mathbf{1})^2 \delta + O(\delta^{\frac{3}{2}}). \tag{31}$$

Remark 1. *If D is a single resonator $D \equiv B$ ($N = 1$), we have $C = \mathrm{cap}\,(B)$ and $V = |B|$ hence $\lambda_1 = \mathrm{cap}\,(B)/|B|$ and $\boldsymbol{a}_1 = |B|^{-1/2}$. Then* (31) *reads more explicitly as*

$$\omega_1^{\pm}(\delta) = \pm\delta^{\frac{1}{2}} v_b \sqrt{\frac{\mathrm{cap}\,(B)}{|B|}} - \frac{\mathrm{i} v_b^2 \mathrm{cap}\,(B)^2}{8\pi v |B|}\delta + O(\delta^{\frac{3}{2}}). \tag{32}$$

3 Point scatterer approximation

Lemma 2. *The following expansion holds for the vector $\boldsymbol{F}$ of* (27)*:*

$$\boldsymbol{F} = \delta u_{\mathrm{in}}(0) C \mathbf{1} + O(\omega\delta). \tag{33}$$

For $\omega \to \omega_i^{\pm}(\delta)$, the solution $\boldsymbol{x}$ to (27) *reads approximately*

$$\boldsymbol{x} \simeq u_{\mathrm{in}}(0) \left(\delta \frac{\omega^2}{v_b^2} V C^{-1} - I\right)^{-1} \mathbf{1} = \sum_{i=1}^{N} \frac{u_{\mathrm{in}}(0)}{\frac{\omega^2}{\omega_{M,i}^2} - 1} (\mathbf{1}^T V \boldsymbol{a}_i) V \boldsymbol{a}_i$$

where we denote

$$\omega_{M,i} := \delta^{\frac{1}{2}} \lambda_i^{\frac{1}{2}} v_b.$$

Therefore, we find that as $\omega \to \omega_{i,M}^{+}$,

$$\psi \simeq \sum_{i=1}^{N} x_i \phi_i^* \simeq \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{(\boldsymbol{e}_i^T V \boldsymbol{a}_j)(\mathbf{1}^T V \boldsymbol{a}_j)}{\frac{\omega^2}{\omega_{M,j}^2} - 1} u_{\mathrm{in}}(0) \phi_i^*.$$

As a result, as $|x| \to +\infty$,

$$u_{\mathrm{tot}}(x) - u_{\mathrm{in}}(x) = \mathcal{S}_D^k[\psi](x) \simeq \left(\int_{\partial D} \psi \mathrm{d}\sigma\right) \Gamma^k(x) \simeq \frac{1}{\frac{\omega^2}{\omega_{M,i}^2} - 1} (\mathbf{1}^T V \boldsymbol{a}_i)^2 u_{\mathrm{in}}(0) \Gamma^k(x). \tag{34}$$

Remark 2. *The approximation* (34) *is called "point-scatterer approximation". It shows that if $\mathbf{1}^T V \boldsymbol{a}_i \neq 0$, then the scattered field behaves as a monopole with a resonant amplification factor. If $\mathbf{1}^T V \boldsymbol{a}_i = 0$, then the resonance is of "dipole type" and higher order terms need to be taken into account.*

We have neglected the terms of order $O(\omega\delta)$ for computing $A(\omega,\delta)^{-1}$. Therefore, the approximation (34) *is valid only for ω slightly away from $\omega_{M,i}$. A more accurate expansion with quantitative error terms is computed in [6].*

References

[1] H. Ammari and B. Davies, *Mimicking the active cochlea with a fluid-coupled array of subwavelength Hopf resonators*, Proceedings of the Royal Society A, 476 (2020), p. 20190870.

[2] H. Ammari, B. Davies, E. O. Hiltunen, H. Lee, and S. Yu, *Wave interaction with subwavelength resonators*, arXiv preprint arXiv:2011.03575, (2020).

[3] H. Ammari, B. Fitzpatrick, D. Gontier, H. Lee, and H. Zhang, *Minnaert resonances for acoustic waves in bubbly media*, Annales de l'Institut Henri Poincare (C) Analyse Non Lineaire, 35 (2018), pp. 1975–1998.

[4] H. Ammari, B. Fitzpatrick, H. Kang, M. Ruiz, S. Yu, and H. Zhang, *Mathematical and Computational Methods in Photonics and Phononics*, vol. 235 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, Rhode Island, oct 2018.

[5] H. Ammari and H. Zhang, *A mathematical theory of super-resolution by using a system of sub-wavelength Helmholtz resonators*, Communications in Mathematical Physics, 337 (2015), pp. 379–428.

[6] F. Feppon and H. Ammari, *Modal decompositions and point scatterer approximations near the Minnaert resonance frequencies*, Hal preprint hal-03319394, (2021).

[7] W. C. H. McLean, *Strongly elliptic systems and boundary integral equations*, vol. 86, Cambridge university press, 2000.

[8] R. Mennicken and M. Möller, *Root functions, eigenvectors, associated vectors and the inverse of a holomorphic operator function*, Archiv der Mathematik, 42 (1984), pp. 455–463.

[9] ——, *Non-self-adjoint boundary eigenvalue problems*, vol. 192, Gulf Professional Publishing, 2003.

[10] J.-C. Nédélec, *Acoustic and electromagnetic equations: integral representations for harmonic problems*, vol. 144, Springer Science \& Business Media, 2013.