Shape optimization of a coupled thermal fluid-structure problem in a level set mesh evolution framework

Florian Feppon

Grégoire Allaire, Charles Dapogny
Julien Cortial, Felipe Bordeu

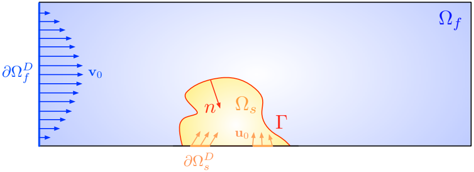
ECCM – June 12, 2018

Outline

1. Hadamard's boundary variation method for a simplified three-physics setting
2. Numerical implementation of various test cases with a mesh evolution algorithm

Simplified weakly coupled three-physics setting

$$\min_{\Gamma} J(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma), T(\Gamma), \boldsymbol{u}(\Gamma)).$$

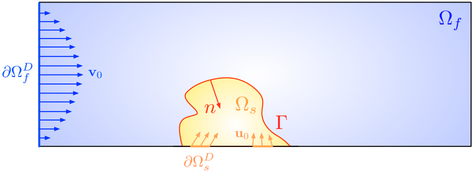


- Incompressible Navier-Stokes equations for $(\boldsymbol{v}, p)$ in Ω_f

$$-\mathrm{div}(\sigma_f(\boldsymbol{v}, p)) + \rho\nabla\boldsymbol{v}\,\boldsymbol{v} = \boldsymbol{f}_f \text{ in } \Omega_f$$

Simplified weakly coupled three-physics setting

$$\min_{\Gamma} J(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma), T(\Gamma), \boldsymbol{u}(\Gamma)).$$



- Incompressible Navier-Stokes equations for $(\boldsymbol{v}, p)$ in Ω_f

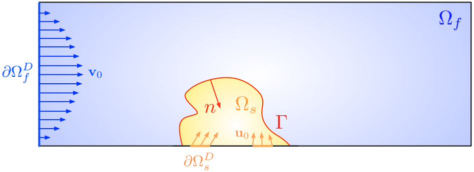
$$-\mathrm{div}(\sigma_f(\boldsymbol{v}, p)) + \rho \nabla \boldsymbol{v}\, \boldsymbol{v} = \boldsymbol{f}_f \text{ in } \Omega_f$$

- Steady-state convection-diffusion for T_f and T_s in Ω_f and Ω_s:

$$\begin{aligned} -\mathrm{div}(k_f \nabla T_f) + \rho \boldsymbol{v} \cdot \nabla T_f &= Q_f \quad \text{in } \Omega_f \\ -\mathrm{div}(k_s \nabla T_s) &= Q_s \quad \text{in } \Omega_s \end{aligned}$$

Simplified weakly coupled three-physics setting

$$\min_{\Gamma} J(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma), T(\Gamma), \boldsymbol{u}(\Gamma)).$$



- Incompressible Navier-Stokes equations for $(\boldsymbol{v}, p)$ in Ω_f

$$-\mathrm{div}(\sigma_f(\boldsymbol{v}, p)) + \rho\nabla\boldsymbol{v}\,\boldsymbol{v} = \boldsymbol{f}_f \text{ in } \Omega_f$$

- Steady-state convection-diffusion for T_f and T_s in Ω_f and Ω_s:

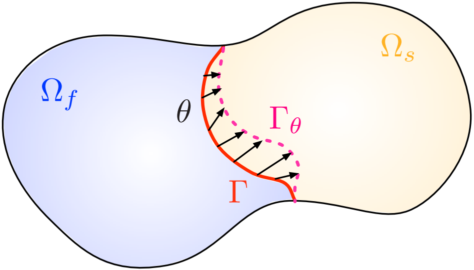
$$\begin{aligned} -\mathrm{div}(k_f\nabla T_f) + \rho\boldsymbol{v}\cdot\nabla T_f &= Q_f \quad \text{in } \Omega_f \\ -\mathrm{div}(k_s\nabla T_s) &= Q_s \quad \text{in } \Omega_s \end{aligned}$$

- Linearized thermoelasticity with fluid-structure interaction for $\boldsymbol{u}$ in Ω_s:

$$\begin{aligned} -\mathrm{div}(\sigma_s(\boldsymbol{u}, T_s)) &= \boldsymbol{f}_s && \text{in } \Omega_s \\ \sigma_s(\boldsymbol{u}, T_s)\cdot\boldsymbol{n} &= \sigma_f(\boldsymbol{v}, p)\cdot\boldsymbol{n} && \text{on } \Gamma. \end{aligned}$$

Hadamard's method of boundary variations

$$\min_{\Gamma} \; J(\Gamma)$$

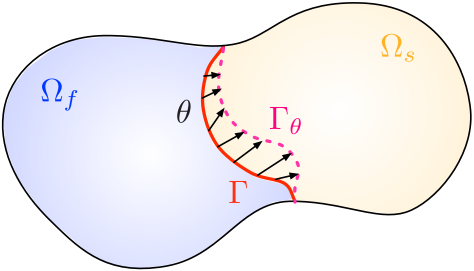


$$\Gamma_{\boldsymbol{\theta}} = (I + \boldsymbol{\theta})\Gamma, \text{ where } \boldsymbol{\theta} \in W_0^{1,\infty}(\Omega, \mathbb{R}^d), \; ||\boldsymbol{\theta}||_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)} < 1.$$

$$J(\Gamma_{\boldsymbol{\theta}}) = J(\Gamma) + \frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\theta}}(\boldsymbol{\theta}) + o(\boldsymbol{\theta}), \quad \text{where} \quad \frac{|o(\boldsymbol{\theta})|}{||\boldsymbol{\theta}||_{W^{1,\infty}(\Omega,\mathbb{R}^d)}} \xrightarrow{\boldsymbol{\theta}\to 0} 0,$$

Hadamard's method of boundary variations

$$\min_{\Gamma} \; J(\Gamma)$$

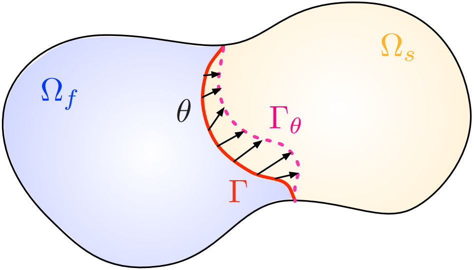


A descent direction $\boldsymbol{\theta} \in H^1(D)$ is obtained by solving an identification problem

$$\forall \boldsymbol{\theta}' \in H^1(D),\ a(\boldsymbol{\theta}, \boldsymbol{\theta}') = \frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\theta}}(\boldsymbol{\theta}').$$

Hadamard's method of boundary variations

$$\min_{\Gamma} \; J(\Gamma)$$



A descent direction $\boldsymbol{\theta} \in H^1(D)$ is obtained by solving an identification problem

$$\forall \boldsymbol{\theta}' \in H^1(D),\; a(\boldsymbol{\theta}, \boldsymbol{\theta}') = \frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\theta}}(\boldsymbol{\theta}').$$

Hadamard's structure theorem: if Γ, $\boldsymbol{\theta}$, and J are smooth enough, then there exists $v \in L^1(\Gamma)$ such that

$$\frac{\mathrm{d}J}{\mathrm{d}\boldsymbol{\theta}}(\boldsymbol{\theta}) = \int_\Gamma v \; \boldsymbol{\theta} \cdot \boldsymbol{n} \mathrm{d}s$$

Analytical shape derivative calculations

Outcomes:

- We propose a “pedestrian” method to compute shape derivatives in volumetric or surfacic form of **general** objective functionals in terms of its partial derivatives.

Analytical shape derivative calculations

Outcomes:

- We propose a "pedestrian" method to compute shape derivatives in volumetric or surfacic form of **general** objective functionals in terms of its partial derivatives.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}}\Big[J(\Gamma_{\boldsymbol{\theta}}, \boldsymbol{v}(\Gamma_{\boldsymbol{\theta}}), p(\Gamma_{\boldsymbol{\theta}}), T(\Gamma_{\boldsymbol{\theta}}), \boldsymbol{u}(\Gamma_{\boldsymbol{\theta}}))\Big](\boldsymbol{\theta}) \\
&= \overline{\frac{\partial \mathfrak{J}}{\partial \boldsymbol{\theta}}}(\boldsymbol{\theta}) + \int_\Gamma (\boldsymbol{f}_f \cdot \boldsymbol{w} - \sigma_f(\boldsymbol{v}, p) : \nabla \boldsymbol{w} + \boldsymbol{n} \cdot \sigma_f(\boldsymbol{w}, q) \nabla \boldsymbol{v} \cdot \boldsymbol{n} + \boldsymbol{n} \cdot \sigma_f(\boldsymbol{v}, p) \nabla \boldsymbol{w} \cdot \boldsymbol{n})(\boldsymbol{\theta} \cdot \boldsymbol{n}) \mathrm{d}s \\
&+ \int_\Gamma \left(k_s \nabla T_s \cdot \nabla S_s - k_f \nabla T_f \cdot \nabla S_f + Q_f S_f - Q_s S_s - 2k_s \frac{\partial T_s}{\partial n} \frac{\partial S_s}{\partial n} + 2k_f \frac{\partial T_f}{\partial n} \frac{\partial S_f}{\partial n} \right) (\boldsymbol{\theta} \cdot \boldsymbol{n}) \mathrm{d}s \\
&+ \int_\Gamma (\sigma_s(\boldsymbol{u}, T_s) : \nabla \boldsymbol{r} - \boldsymbol{f}_s \cdot \boldsymbol{r} - \boldsymbol{n} \cdot Ae(\boldsymbol{r}) \nabla \boldsymbol{u} \cdot \boldsymbol{n} - \boldsymbol{n} \cdot \sigma_s(\boldsymbol{u}, T_s) \nabla \boldsymbol{r} \cdot \boldsymbol{n}) (\boldsymbol{\theta} \cdot \boldsymbol{n}) \mathrm{d}s
\end{aligned}
$$

Analytical shape derivative calculations

Outcomes:

- We propose a "pedestrian" method to compute shape derivatives in volumetric or surfacic form of **general** objective functionals in terms of its partial derivatives.
- Adjoint variables $\boldsymbol{w}, q, S_f, S_s, \boldsymbol{r}$ are solved in a reversed cascade.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\theta}}\Big[J(\Gamma_{\boldsymbol{\theta}}, \boldsymbol{v}(\Gamma_{\boldsymbol{\theta}}), p(\Gamma_{\boldsymbol{\theta}}), T(\Gamma_{\boldsymbol{\theta}}), \boldsymbol{u}(\Gamma_{\boldsymbol{\theta}}))\Big](\boldsymbol{\theta}) \\
&= \overline{\frac{\partial \mathfrak{J}}{\partial \boldsymbol{\theta}}}(\boldsymbol{\theta}) + \int_\Gamma (\boldsymbol{f}_f \cdot \boldsymbol{w} - \sigma_f(\boldsymbol{v}, p) : \nabla \boldsymbol{w} + \boldsymbol{n} \cdot \sigma_f(\boldsymbol{w}, q) \nabla \boldsymbol{v} \cdot \boldsymbol{n} + \boldsymbol{n} \cdot \sigma_f(\boldsymbol{v}, p) \nabla \boldsymbol{w} \cdot \boldsymbol{n})(\boldsymbol{\theta} \cdot \boldsymbol{n}) \mathrm{d}s \\
&+ \int_\Gamma \left(k_s \nabla T_s \cdot \nabla S_s - k_f \nabla T_f \cdot \nabla S_f + Q_f S_f - Q_s S_s - 2k_s \frac{\partial T_s}{\partial n} \frac{\partial S_s}{\partial n} + 2k_f \frac{\partial T_f}{\partial n} \frac{\partial S_f}{\partial n} \right) (\boldsymbol{\theta} \cdot \boldsymbol{n}) \mathrm{d}s \\
&+ \int_\Gamma (\sigma_s(\boldsymbol{u}, T_s) : \nabla \boldsymbol{r} - \boldsymbol{f}_s \cdot \boldsymbol{r} - \boldsymbol{n} \cdot Ae(\boldsymbol{r}) \nabla \boldsymbol{u} \cdot \boldsymbol{n} - \boldsymbol{n} \cdot \sigma_s(\boldsymbol{u}, T_s) \nabla \boldsymbol{r} \cdot \boldsymbol{n}) (\boldsymbol{\theta} \cdot \boldsymbol{n}) \mathrm{d}s
\end{aligned}
$$

Adjoint system

$$\int_{\Omega_s} Ae(\boldsymbol{r}) : \nabla \boldsymbol{r}' \mathrm{d}\boldsymbol{x} = \frac{\partial \mathfrak{J}}{\partial \hat{\boldsymbol{u}}}(\boldsymbol{r}') \quad \forall \boldsymbol{r}' \in V_{\boldsymbol{u}}(\Gamma)\,.$$

Adjoint system

$$\int_{\Omega_s} Ae(\boldsymbol{r}) : \nabla \boldsymbol{r}' \mathrm{d}\boldsymbol{x} = \frac{\partial \mathfrak{J}}{\partial \hat{\boldsymbol{u}}}(\boldsymbol{r}') \quad \forall \boldsymbol{r}' \in V_{\boldsymbol{u}}(\Gamma)\,.$$

$$\downarrow$$

$$\int_{\Omega_s} k_s \nabla S \cdot \nabla S' \mathrm{d}\boldsymbol{x} + \int_{\Omega_f} (k_f \nabla S \cdot \nabla S' + \rho c_p S \boldsymbol{v} \cdot \nabla S') \mathrm{d}\boldsymbol{x} = \int_{\Omega_s} \alpha \operatorname{div}(\boldsymbol{r}) S' \mathrm{d}\boldsymbol{x} + \frac{\partial \mathfrak{J}}{\partial \hat{T}}(S) \quad \forall S' \in V_T(\Gamma)\,.$$

Adjoint system

$$\int_{\Omega_s} Ae(\boldsymbol{r}) : \nabla \boldsymbol{r}' \mathrm{d}\boldsymbol{x} = \frac{\partial \mathfrak{J}}{\partial \hat{\boldsymbol{u}}}(\boldsymbol{r}') \quad \forall \boldsymbol{r}' \in V_{\boldsymbol{u}}(\Gamma)\,.$$

$$\downarrow$$

$$\int_{\Omega_s} k_s \nabla S \cdot \nabla S' \mathrm{d}\boldsymbol{x} + \int_{\Omega_f} (k_f \nabla S \cdot \nabla S' + \rho c_p S \boldsymbol{v} \cdot \nabla S') \mathrm{d}\boldsymbol{x} = \int_{\Omega_s} \alpha \mathrm{div}(\boldsymbol{r}) S' \mathrm{d}\boldsymbol{x} + \frac{\partial \mathfrak{J}}{\partial \hat{T}}(S) \quad \forall S' \in V_T(\Gamma)\,.$$

$$\downarrow$$

$$\boldsymbol{w} = \boldsymbol{r} \text{ on } \Gamma \text{ and } \forall (\boldsymbol{w}', q') \in V_{\boldsymbol{v},p}(\Gamma)$$

$$\int_{\Omega_f} \Big(\sigma_f(\boldsymbol{w}, q) : \nabla \boldsymbol{w}' + \rho \boldsymbol{w} \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{v} + \rho \boldsymbol{w} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{w}' - q' \mathrm{div}(\boldsymbol{w}) \Big) \mathrm{d}\boldsymbol{x} =$$

$$\int_{\Omega_f} -\rho c_p S \nabla T \cdot \boldsymbol{w}' \mathrm{d}\boldsymbol{x} + \frac{\partial \mathfrak{J}}{\partial (\boldsymbol{v}', p')}(\boldsymbol{w}', q'),$$

Adjoint system

$$\int_{\Omega_s} Ae(\boldsymbol{r}) : \nabla \boldsymbol{r}' \mathrm{d}\boldsymbol{x} = \frac{\partial \mathfrak{J}}{\partial \hat{\boldsymbol{u}}}(\boldsymbol{r}') \quad \forall \boldsymbol{r}' \in V_{\boldsymbol{u}}(\Gamma) .$$

$$\downarrow$$

$$\int_{\Omega_s} k_s \nabla S \cdot \nabla S' \mathrm{d}\boldsymbol{x} + \int_{\Omega_f} (k_f \nabla S \cdot \nabla S' + \rho c_p S \boldsymbol{v} \cdot \nabla S') \mathrm{d}\boldsymbol{x} = \int_{\Omega_s} \alpha \mathrm{div}(\boldsymbol{r}) S' \mathrm{d}\boldsymbol{x} + \frac{\partial \mathfrak{J}}{\partial \hat{T}}(S) \quad \forall S' \in V_T(\Gamma) .$$

$$\downarrow$$

$$\boldsymbol{w} = \boldsymbol{r} \text{ on } \Gamma \text{ and } \forall (\boldsymbol{w}', q') \in V_{\boldsymbol{v},p}(\Gamma)$$

$$\int_{\Omega_f} \Big(\sigma_f(\boldsymbol{w}, q) : \nabla \boldsymbol{w}' + \rho \boldsymbol{w} \cdot \nabla \boldsymbol{w}' \cdot \boldsymbol{v} + \rho \boldsymbol{w} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{w}' - q' \mathrm{div}(\boldsymbol{w}) \Big) \mathrm{d}\boldsymbol{x} =$$

$$\int_{\Omega_f} -\rho c_p S \nabla T \cdot \boldsymbol{w}' \mathrm{d}\boldsymbol{x} + \frac{\partial \mathfrak{J}}{\partial (\boldsymbol{v}', p')}(\boldsymbol{w}', q'),$$

$\boldsymbol{w} = \boldsymbol{r}$ on Γ : "strange" boundary condition dual to the equality of normal stresses $\sigma_s(\boldsymbol{u}, T_s) \cdot \boldsymbol{n} = \sigma_f(\boldsymbol{v}, p) \cdot \boldsymbol{n}$ on Γ.

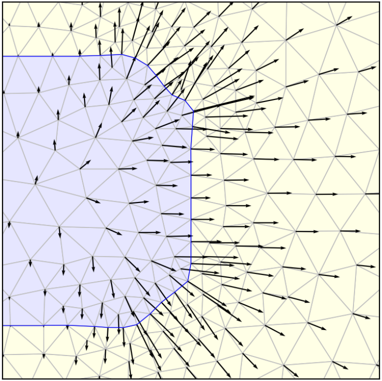
Outline

1. Hadamard's boundary variation method for a simplified three-physics setting
2. Numerical implementation of various test cases with a mesh evolution algorithm

Numerical implementation : mesh evolution algorithm

We consider the algorithm proposed by Allaire, Dapogny, Frey (2013):

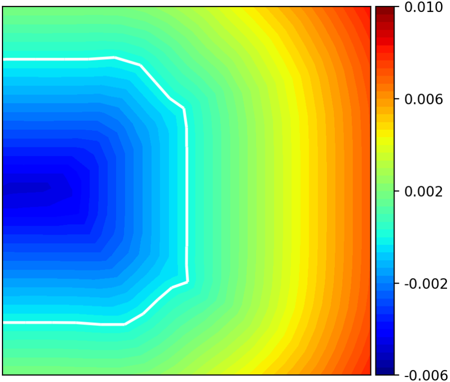
1. Given a mesh of $\Omega = \Omega_s \cup \Omega_f$ and a moving vector field $\boldsymbol{\theta}$



Numerical implementation : mesh evolution algorithm

We consider the algorithm proposed by Allaire, Dapogny, Frey (2013):

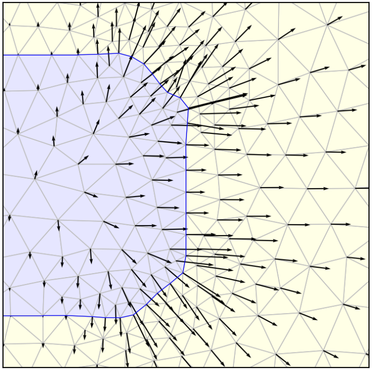
2. A level-set function ϕ associated to $\Omega = \Omega_s \cup \Omega_f$ is computed on the mesh.



Numerical implementation : mesh evolution algorithm

We consider the algorithm proposed by Allaire, Dapogny, Frey (2013):

3. The level-set function is avected on the computational domain which is then adaptively remeshed:

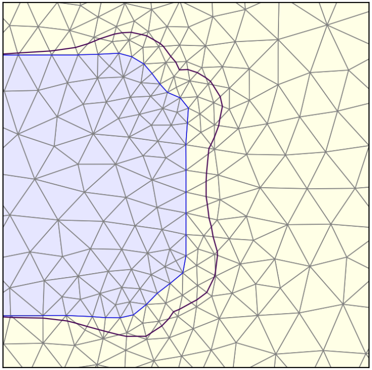


Numerical implementation : mesh evolution algorithm

We consider the algorithm proposed by Allaire, Dapogny, Frey (2013):

3. The level-set function is avected on the computational domain which is then adaptively remeshed:

Advection of a level set for Ω on the computational mesh.

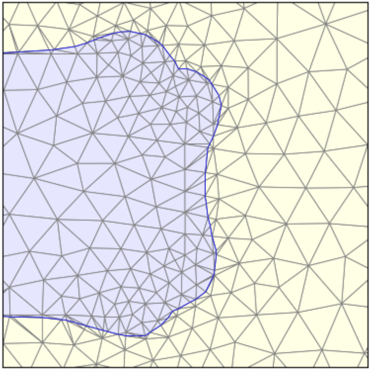


Numerical implementation : mesh evolution algorithm

We consider the algorithm proposed by Allaire, Dapogny, Frey (2013):

3. The level-set function is avected on the computational domain which is then adaptively remeshed:

Breaking the zero isoline of the level set.

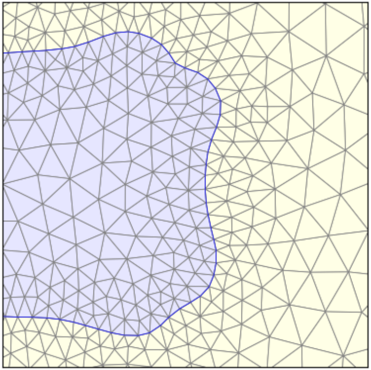


Numerical implementation : mesh evolution algorithm

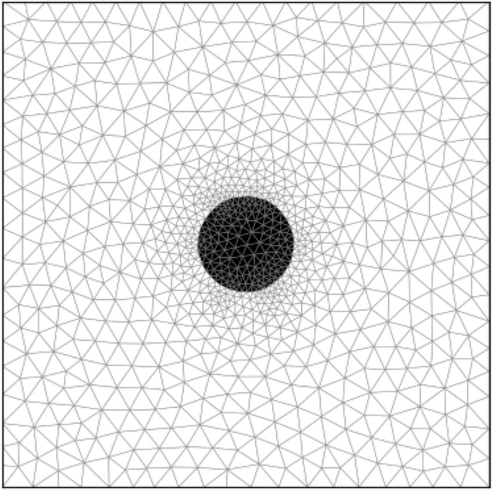
We consider the algorithm proposed by Allaire, Dapogny, Frey (2013):

3. The level-set function is avected on the computational domain which is then adaptively remeshed:

Remeshing adaptively the computational mesh.



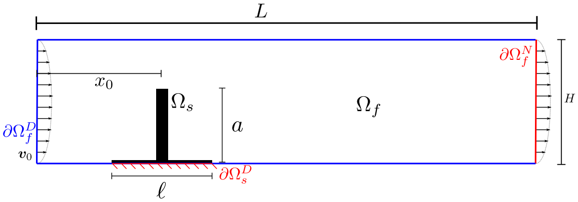
A numerical test case : shape optimization of an airfoil



Maximization of the lift and minimization of the viscous forces:

$$J(\Gamma) = -\omega \int_{\partial\Omega_f} \boldsymbol{e}_y \cdot \sigma_f(\boldsymbol{v}, p) \cdot \boldsymbol{n} \mathrm{d}s + (1-\omega) \int_{\Omega_f} 2\nu e(\boldsymbol{v}) : e(\boldsymbol{v}) \mathrm{d}x$$

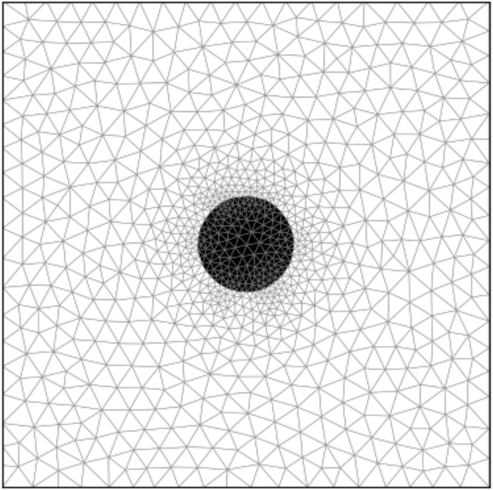
A numerical test case : fluid structure interaction problem



Minimization of the compliance:

$$J(\Gamma) = \int_{\Omega_s} Ae(\boldsymbol{u}) : e(\boldsymbol{u})\mathrm{d}x$$

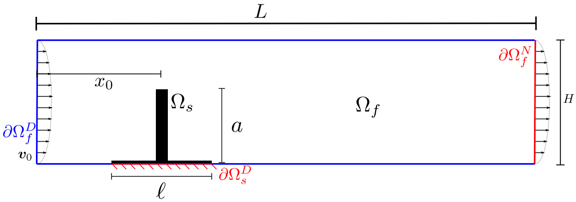
A numerical test case : fluid structure interaction problem

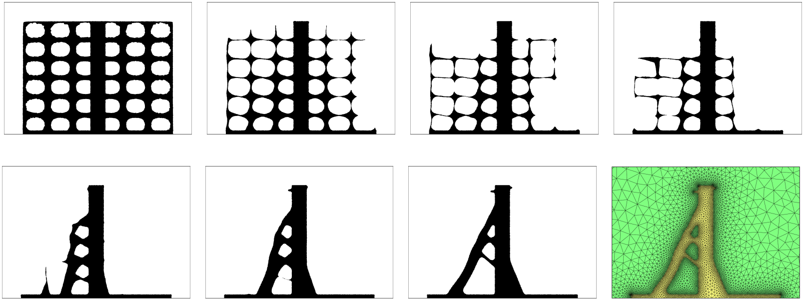


Minimization of the compliance:

$$J(\Gamma) = \int_{\Omega_s} Ae(\boldsymbol{u}) : e(\boldsymbol{u})\mathrm{d}x$$

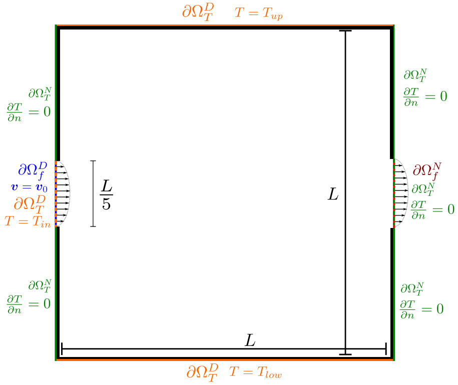
A numerical test case : fluid structure interaction problem





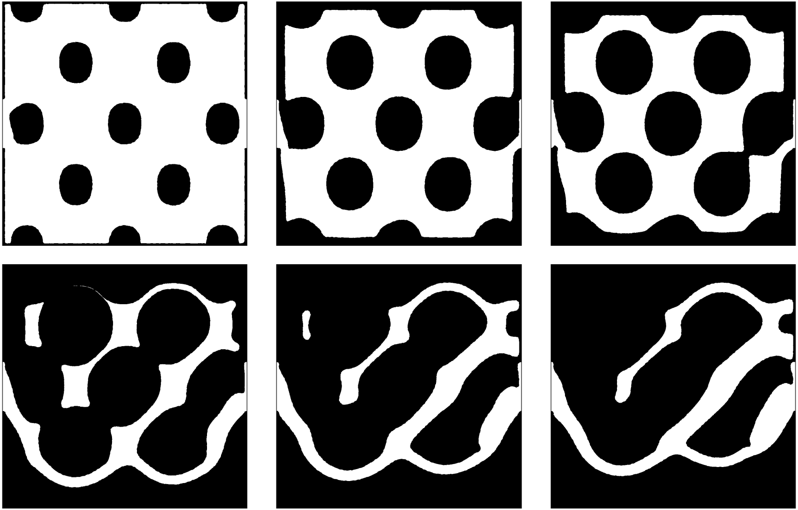
Heat transfer problem

Maximization of heat transfer and minimization of viscous dissipation.

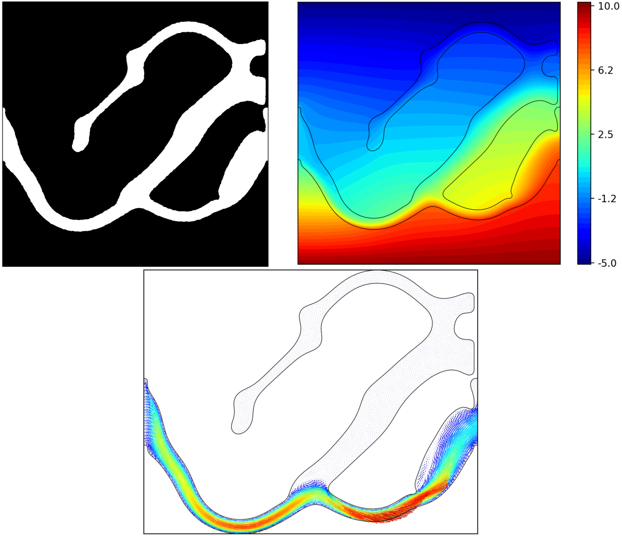


$$J(\Gamma) = \omega \int_{\Omega_f} 2\nu e(\boldsymbol{v}) : e(\boldsymbol{v}) \mathrm{d}x - (1-\omega) \int_{\partial\Omega_f^N} \rho c_p T_f \boldsymbol{v} \cdot \boldsymbol{n} \mathrm{d}s$$

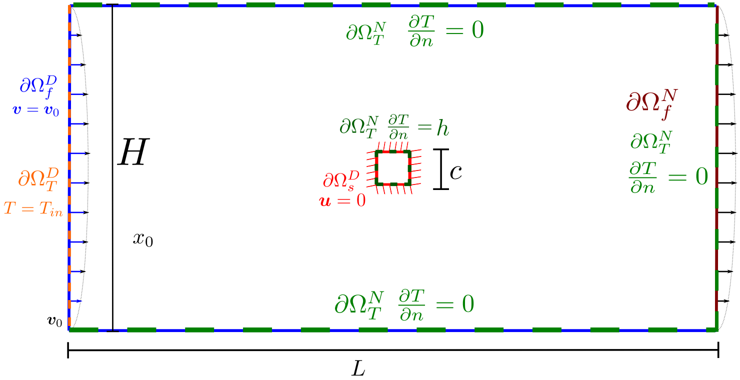
Heat transfer problem



Heat transfer problem



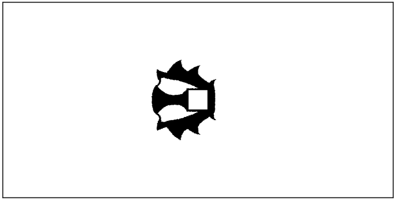
Three physics problem



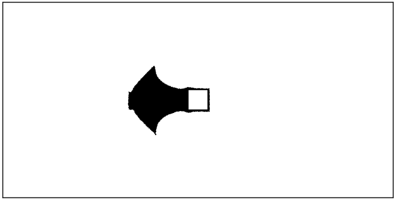
Minimization of the compliance:

$$J(\Gamma) = \int_{\Omega_s} \sigma_s(\boldsymbol{u}, T_s) : \nabla \boldsymbol{u} \mathrm{d}x$$

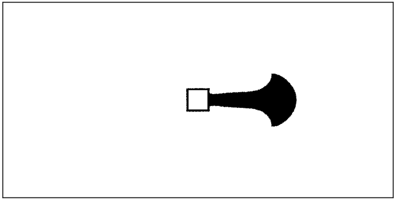
Three physics problem



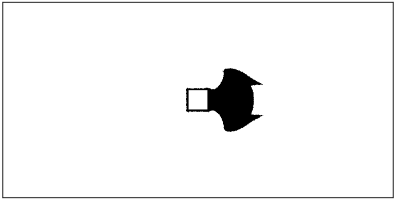
(a) $h > 0$ (Stokes)



(b) $h > 0$ (Navier-Stokes)



(c) $h < 0$ (Stokes)



(d) $h < 0$ (Navier-Stokes)

Current and future works

- Incorporating geometric constraints, e.g. enforcing a non penetrability condition between two pipes for heat exchangers designs.
- 3D test cases.
- Extending optimization algorithms to account for multiple equality and inequality constraints.

Submitted work and further references

Feppon, F., Allaire, G., Bordeu, F., Cortial, J., and Dapogny, C. Shape optimization of a coupled thermal fluid-structure problem in a level set mesh evolution framework.
Submitted to Applicable Analysis (2018).

Allaire, G., Dapogny, C., Frey, P. A mesh evolution algorithm based on the level set method for geometry and topology optimization.
Structural and Multidisciplinary Optimization (2013).