# Shape derivative of geometric constraints without integration along rays

Florian Feppon

Grégoire Allaire, Charles Dapogny
Julien Cortial, Felipe Bordeu

ENGOPT – September 18th, 2018

SAFRAN

l'X ÉCOLE POLYTECHNIQUE UNIVERSITÉ PARIS-SACLAY

# Thickness control in structural optimization

Some recent advances in level-set based shape optimization: geometric constraints.[1][2]:

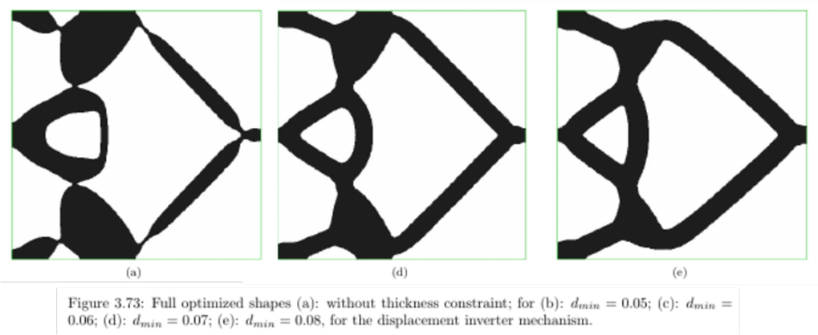



Figure: Michailidis (2014)

[1] **Michailidis2014Manufacturing**.

[2] **Allaire2016Thickness**.

# Outline

1. Shape derivatives of geometric constraints based on the signed distance function
2. A variational method for avoiding integration along rays
3. Numerical comparisons and applications to shape and topology optimization

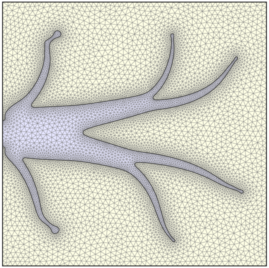
# Outline

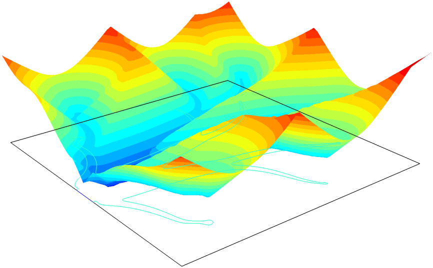
1. Shape derivatives of geometric constraints based on the signed distance function
2. A variational method for avoiding integration along rays
3. Numerical comparisons and applications to shape and topology optimization

# 1. Shape derivatives of geometric constraints

The signed distance function $d_\Omega$ to the domain $\Omega \subset D$ is defined by:

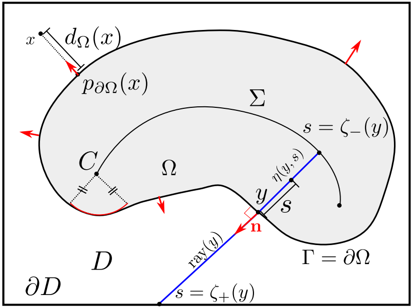
$$\forall x \in D, \ \ d_\Omega(x) = \begin{cases} -\min\limits_{y \in \partial\Omega} ||y - x|| & \text{if } x \in \Omega, \\ \min\limits_{y \in \partial\Omega} ||y - x|| & \text{if } x \in D \backslash \Omega. \end{cases}$$





# 1. Shape derivatives of geometric constraints

The signed distance function allows to formulate geometric constraints.




- Maximum thickness constraint :

$$\forall x \in \Omega,\ |d_\Omega(x)| \leq d_{\max}/2$$

- Minimum thickness constraint:

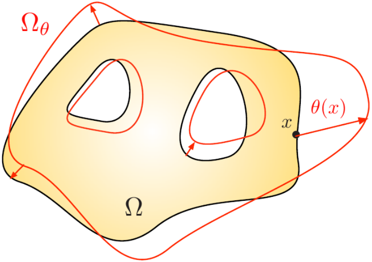
$$\forall y \in \partial\Omega,\ |\zeta_-(y)| \geq d_{\min}/2.$$

# 1. Shape derivatives of geometric constraints

For shape optimization, one formulates geometric constraints using penalty functionals $P(\Omega)$ as follows:

$$\min_{\Omega} J(\Omega), \text{ s.t. } P(\Omega) \leq 0, \text{ where } P(\Omega) := \int_D j(d_\Omega(x))\mathrm{d}x.$$

We rely on the method of Hadamard (figure from[3]):




[3] **dapogny2017geometrical**.

# 1. Shape derivatives of geometric constraints

For shape optimization, one formulates geometric constraints using penalty functionals $P(\Omega)$ as follows:

$$\min_{\Omega} J(\Omega), \text{ s.t. } P(\Omega) \leq 0, \text{ where } P(\Omega) := \int_D j(d_\Omega(x))\mathrm{d}x.$$

The shape derivative of $P(\Omega)$ reads

$$P'(\Omega)(\boldsymbol{\theta}) = \int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x = \int_{\partial\Omega} u\,\boldsymbol{\theta}\cdot\mathbf{n}\mathrm{d}y$$

with

$$\forall y \in \partial\Omega,\ u(y) = -\int_{x\in\mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1\leq i\leq n-1} (1+\kappa_i(y)d_\Omega(x))\mathrm{d}x.$$

# 1. Shape derivatives of geometric constraints

$$\forall y \in \partial\Omega,\ u(y) = -\int_{x \in \mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1 \leq i \leq n-1} (1 + \kappa_i(y) d_\Omega(x)) \mathrm{d}x.$$

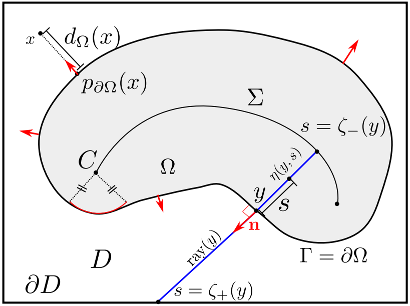
Computing $u$ requires:

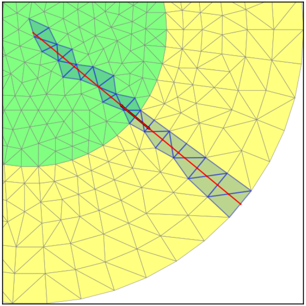
# 1. Shape derivatives of geometric constraints

$$\forall y \in \partial\Omega,\ u(y) = -\int_{x \in \mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1 \leq i \leq n-1} (1 + \kappa_i(y) d_\Omega(x)) \mathrm{d}x.$$

Computing $u$ requires:

1. Integrating along rays on the discretization mesh:



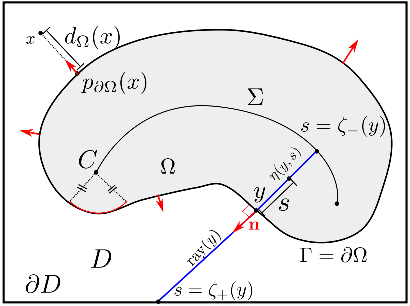



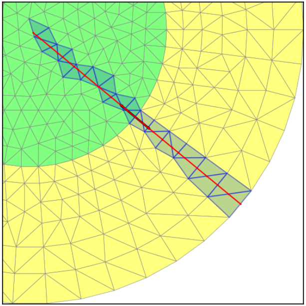
# 1. Shape derivatives of geometric constraints

$$\forall y \in \partial\Omega,\ u(y) = -\int_{x \in \mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1 \leq i \leq n-1} (1 + \kappa_i(y) d_\Omega(x)) \mathrm{d}x.$$

Computing $u$ requires:

1. Integrating along rays on the discretization mesh:



2. Estimating the principal curvatures $\kappa_i(y)$.

# Outline

1. Shape derivatives of geometric constraints based on the signed distance function
2. A variational method for avoiding integration along rays
3. Numerical comparisons and applications to shape and topology optimization

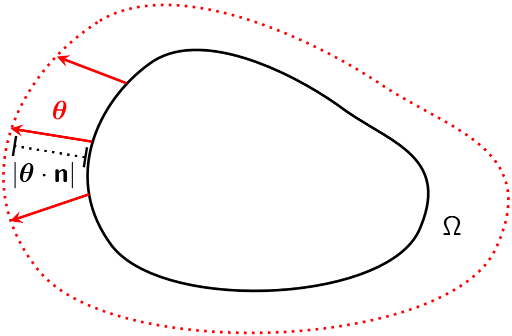
# 2. A variational method for avoiding integration along rays

More precisely, the shape derivative of $P(\Omega)$ reads

$$P'(\Omega)(\boldsymbol{\theta}) = \int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x = \int_{\partial\Omega} u\,\boldsymbol{\theta}\cdot\mathbf{n}\mathrm{d}y$$

with $d'_\Omega(\boldsymbol{\theta})$ satisfying

$$\begin{cases} \nabla d'_\Omega(\boldsymbol{\theta})\cdot\nabla d_\Omega = 0 \text{ in } D\setminus\overline{\Sigma} \\ d'_\Omega(\boldsymbol{\theta}) = -\boldsymbol{\theta}\cdot\mathbf{n} \text{ on } \partial\Omega. \end{cases}$$



# 2. A variational method for avoiding integration along rays

More precisely, the shape derivative of $P(\Omega)$ reads

$$P'(\Omega)(\boldsymbol{\theta}) = \int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x = \int_{\partial\Omega} u\,\boldsymbol{\theta}\cdot\mathbf{n}\mathrm{d}y$$

with $d'_\Omega(\boldsymbol{\theta})$ satisfying

$$\begin{cases} \nabla d'_\Omega(\boldsymbol{\theta})\cdot\nabla d_\Omega = 0 \text{ in } D\setminus\overline{\Sigma} \\ d'_\Omega(\boldsymbol{\theta}) = -\boldsymbol{\theta}\cdot\mathbf{n} \text{ on } \partial\Omega. \end{cases}$$

**Our method**: $u$ solves the following variational problem (with $\omega > 0$ rather arbitrary):

$$\text{Find } u \in V_\omega \text{ such that } \forall v \in V_\omega,$$
$$\int_{\partial\Omega} uv\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega\cdot\nabla u)(\nabla d_\Omega\cdot\nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x,$$

# 2. A variational method for avoiding integration along rays

More precisely, the shape derivative of $P(\Omega)$ reads

$$P'(\Omega)(\boldsymbol{\theta}) = \int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x = \int_{\partial\Omega} u\,\boldsymbol{\theta}\cdot\mathbf{n}\mathrm{d}y$$

with $d'_\Omega(\boldsymbol{\theta})$ satisfying

$$\begin{cases} \nabla d'_\Omega(\boldsymbol{\theta})\cdot\nabla d_\Omega = 0 \text{ in } D\setminus\overline{\Sigma} \\ d'_\Omega(\boldsymbol{\theta}) = -\boldsymbol{\theta}\cdot\mathbf{n} \text{ on } \partial\Omega. \end{cases}$$

**Our method**: Take $v = d'_\Omega(\boldsymbol{\theta})$:

Find $u \in V_\omega$ such that $\forall v \in V_\omega$,

$$\int_{\partial\Omega} uv\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega\cdot\nabla u)(\nabla d_\Omega\cdot\nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x,$$

# 2. A variational method for avoiding integration along rays

More precisely, the shape derivative of $P(\Omega)$ reads

$$P'(\Omega)(\boldsymbol{\theta}) = \int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x)) d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x = \int_{\partial\Omega} u\,\boldsymbol{\theta}\cdot\mathbf{n}\mathrm{d}y$$

with $d'_\Omega(\boldsymbol{\theta})$ satisfying

$$\begin{cases} \nabla d'_\Omega(\boldsymbol{\theta})\cdot\nabla d_\Omega = 0 \text{ in } D\setminus\overline{\Sigma} \\ d'_\Omega(\boldsymbol{\theta}) = -\boldsymbol{\theta}\cdot\mathbf{n} \text{ on } \partial\Omega. \end{cases}$$

**Our method**: Take $v = d'_\Omega(\boldsymbol{\theta})$:

Find $u \in V_\omega$ such that $\forall v \in V_\omega$,

$$\int_{\partial\Omega} uv\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega\cdot\nabla u)(\nabla d_\Omega\cdot\nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x,$$

$$\int_{\partial\Omega} u d'_\Omega(\boldsymbol{\theta})\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega\cdot\nabla u)(\nabla d_\Omega\cdot\nabla d'_\Omega(\boldsymbol{\theta}))\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x)) d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x,$$

# 2. A variational method for avoiding integration along rays

More precisely, the shape derivative of $P(\Omega)$ reads

$$P'(\Omega)(\boldsymbol{\theta}) = \int_{D\backslash\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x = \int_{\partial\Omega} u\,\boldsymbol{\theta}\cdot\mathbf{n}\mathrm{d}y$$

with $d'_\Omega(\boldsymbol{\theta})$ satisfying

$$\begin{cases} \nabla d'_\Omega(\boldsymbol{\theta})\cdot\nabla d_\Omega = 0 \text{ in } D\backslash\overline{\Sigma} \\ d'_\Omega(\boldsymbol{\theta}) = -\boldsymbol{\theta}\cdot\mathbf{n} \text{ on } \partial\Omega. \end{cases}$$

**Our method**: Take $v = d'_\Omega(\boldsymbol{\theta})$:

Find $u \in V_\omega$ such that $\forall v \in V_\omega$,

$$\int_{\partial\Omega} uv\mathrm{d}s + \int_{D\backslash\overline{\Sigma}} \omega(\nabla d_\Omega\cdot\nabla u)(\nabla d_\Omega\cdot\nabla v)\mathrm{d}x = -\int_{D\backslash\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x,$$

$$\int_{\partial\Omega} u d'_\Omega(\boldsymbol{\theta})\mathrm{d}s + \int_{D\backslash\overline{\Sigma}} \omega(\nabla d_\Omega\cdot\nabla u)(\nabla d_\Omega\cdot\nabla d'_\Omega(\boldsymbol{\theta}))\mathrm{d}x = -\int_{D\backslash\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x,$$

# 2. A variational method for avoiding integration along rays

More precisely, the shape derivative of $P(\Omega)$ reads

$$P'(\Omega)(\boldsymbol{\theta}) = \int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x = \int_{\partial\Omega} u\,\boldsymbol{\theta}\cdot\mathbf{n}\mathrm{d}y$$

with $d'_\Omega(\boldsymbol{\theta})$ satisfying

$$\begin{cases} \nabla d'_\Omega(\boldsymbol{\theta})\cdot\nabla d_\Omega = 0 \text{ in } D\setminus\overline{\Sigma} \\ d'_\Omega(\boldsymbol{\theta}) = -\boldsymbol{\theta}\cdot\mathbf{n} \text{ on } \partial\Omega. \end{cases}$$

**Our method**: Take $v = d'_\Omega(\boldsymbol{\theta})$:

Find $u \in V_\omega$ such that $\forall v \in V_\omega$,

$$\int_{\partial\Omega} uv\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega\cdot\nabla u)(\nabla d_\Omega\cdot\nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x,$$

$$\int_{\partial\Omega} ud'_\Omega(\boldsymbol{\theta})\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega\cdot\nabla u)(\nabla d_\Omega\cdot\nabla d'_\Omega(\boldsymbol{\theta}))\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x,$$

$$\int_{\partial\Omega} u\,(-\boldsymbol{\theta}\cdot\mathbf{n})\mathrm{d}s + 0 = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))d'_\Omega(\boldsymbol{\theta})(x)\mathrm{d}x.$$

# 2. A variational method for avoiding integration along rays

Our theoretical results for the variational problem:

$$\text{Find } u \in V_\omega \text{ such that } \forall v \in V_\omega,$$

$$\int_{\partial\Omega} u v \,\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega \cdot \nabla u)(\nabla d_\Omega \cdot \nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x \quad (1)$$

# 2. A variational method for avoiding integration along rays

Our theoretical results for the variational problem:

$$\text{Find } u \in V_\omega \text{ such that } \forall v \in V_\omega,$$
$$\int_{\partial\Omega} uv\,\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega \cdot \nabla u)(\nabla d_\Omega \cdot \nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x \quad (1)$$

1. Under rather unrestrictive assumptions, the trace of the solution $u$ is independent on the weight $\omega$ and is given by

$$\forall y \in \partial\Omega,\ u(y) = -\int_{x\in\mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1\leq i\leq n-1} (1 + \kappa_i(y)d_\Omega(x))\mathrm{d}x. \quad (2)$$

# 2. A variational method for avoiding integration along rays

Our theoretical results for the variational problem:

$$\text{Find } u \in V_\omega \text{ such that } \forall v \in V_\omega,$$
$$\int_{\partial\Omega} uv\,\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega \cdot \nabla u)(\nabla d_\Omega \cdot \nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x \quad (1)$$

1. Under rather unrestrictive assumptions, the trace of the solution $u$ is independent on the weight $\omega$ and is given by

$$\forall y \in \partial\Omega,\ u(y) = -\int_{x\in \mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1\leq i\leq n-1} (1+\kappa_i(y)d_\Omega(x))\mathrm{d}x. \quad (2)$$

(1) can be solved with FEM while (2) requires computing rays and curvatures!

# 2. A variational method for avoiding integration along rays

Our theoretical results for the variational problem:

$$\text{Find } u \in V_\omega \text{ such that } \forall v \in V_\omega,$$

$$\int_{\partial\Omega} uv\,\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega \cdot \nabla u)(\nabla d_\Omega \cdot \nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x \quad (1)$$

1. Under rather unrestrictive assumptions, the trace of the solution $u$ is independent on the weight $\omega$ and is given by

$$\forall y \in \partial\Omega,\ u(y) = -\int_{x\in\mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1\le i\le n-1}(1+\kappa_i(y)d_\Omega(x))\mathrm{d}x. \quad (2)$$

   (1) can be solved with FEM while (2) requires computing rays and curvatures!
2. It is possible to show the well-posedeness of (**??**) for a large class of weights $\omega$ in a suitable space $V_\omega$.

# 2. A variational method for avoiding integration along rays

Our theoretical results for the variational problem:

$$\text{Find } u \in V_\omega \text{ such that } \forall v \in V_\omega,$$

$$\int_{\partial\Omega} uv\,\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega \cdot \nabla u)(\nabla d_\Omega \cdot \nabla v)\,\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\,\mathrm{d}x \quad (1)$$

1. Under rather unrestrictive assumptions, the trace of the solution $u$ is independent on the weight $\omega$ and is given by

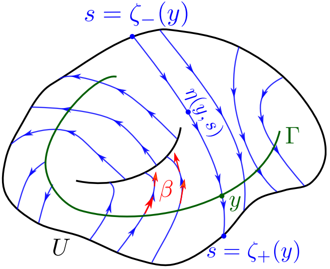
$$\forall y \in \partial\Omega,\ u(y) = -\int_{x\in \mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1\le i\le n-1} (1+\kappa_i(y)d_\Omega(x))\,\mathrm{d}x. \quad (2)$$

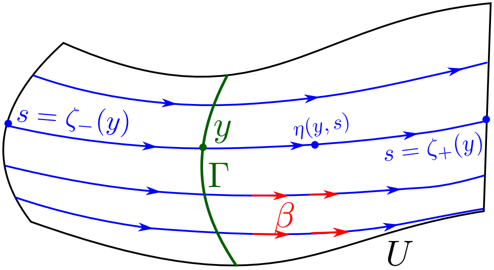
   (1) can be solved with FEM while (2) requires computing rays and curvatures!

2. It is possible to show the well-posedeness of (**??**) for a large class of weights $\omega$ in a suitable space $V_\omega$.
3. The framework extends to more general $\mathcal{C}^1$ vector field $\beta$ (without assuming $\mathrm{div}(\beta) \in L^\infty(D)$) than $\beta = \nabla d_\Omega$.

# 2. A variational method for avoiding integration along rays

Examples of more general settings:



# Outline

1. Shape derivatives of geometric constraints based on the signed distance function
2. A variational method for avoiding integration along rays
3. Numerical comparisons and applications to shape and topology optimization

# 3. Numerical comparisons and applications to shape and topology optimization

Does it really work?

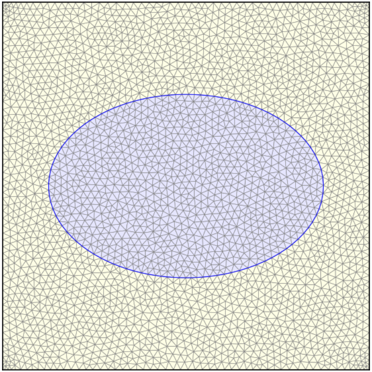
$$u(y) = -\int_{x\in \mathrm{ray}(y)} j'(d_\Omega(x)) \prod_{1\leq i\leq n-1} (1+\kappa_i(y)d_\Omega(x))\mathrm{d}x, \ \forall y \in \partial\Omega,$$

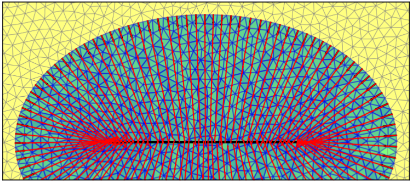
versus

$$\int_{\partial\Omega} uv\mathrm{d}s + \int_{D\setminus\overline{\Sigma}} \omega(\nabla d_\Omega \cdot \nabla u)(\nabla d_\Omega \cdot \nabla v)\mathrm{d}x = -\int_{D\setminus\overline{\Sigma}} j'(d_\Omega(x))v(x)\mathrm{d}x$$

# 3. Numerical comparisons and applications to shape and topology optimization

An analytic example...





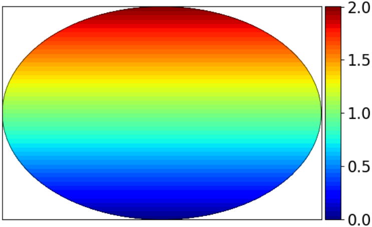
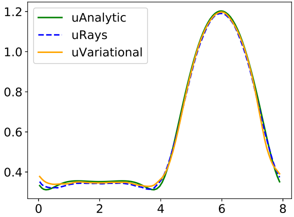


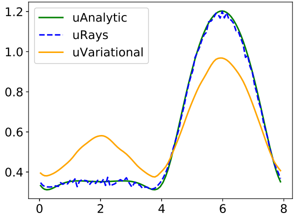

Figure: A prescribed $-j'(d_\Omega(x))$

# 3. Numerical comparisons and applications to shape and topology optimization

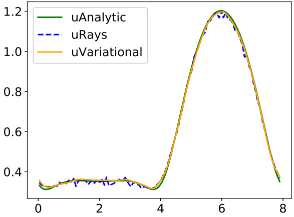
It works with weights $\omega$ vanishing near the skeleton.



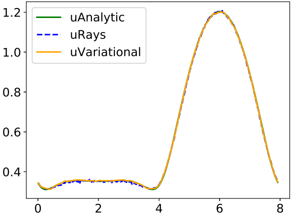

(a) Mesh $\mathcal{T}'$, $\omega = 1$




(b) Mesh $\mathcal{T}$, $\omega = 1$



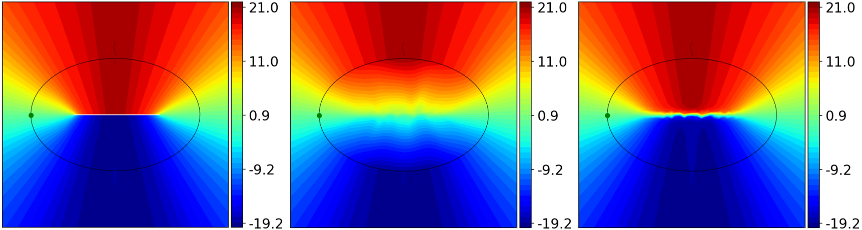

(c) Mesh $\mathcal{T}$, $\omega = 2/(1 + |\Delta d_\Omega|^{3.5})$




(d) Fine mesh $\mathcal{T}$, $\omega = 2/(1 + |\Delta d_\Omega|^{3.5})$

# 3. Numerical comparisons and applications to shape and topology optimization

It works with weights vanishing near the skeleton.




(a) Mesh $\mathcal{T}'$ (skeleton manually truncated), $\omega = 1$

(b) Mesh $\mathcal{T}$, $\omega = 1$.

(c) Mesh $\mathcal{T}$, $\omega = 2/(1 + |\Delta d_\Omega|^{3.5})$

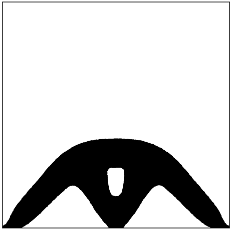
Figure: P1 elements with $\omega = 1$ do not allow discontinuities of test functions near the skeleton...

# 3. Numerical comparisons and applications to shape and topology optimization

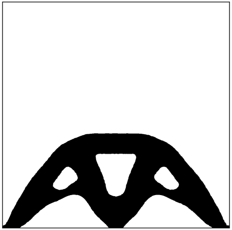
We were able to implement conveniently geometric constraints in level set based shape optimization.

# 3. Numerical comparisons and applications to shape and topology optimization

We were able to implement conveniently geometric constraints in level set based shape optimization.



(a) No maximum thickness constraint

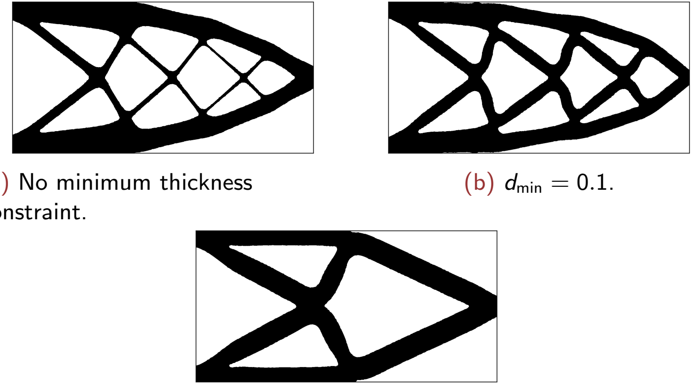


(b) $d_{\max} = 0.07$.

Figure: Maximum thickness constraint for 2D arch.

# 3. Numerical comparisons and applications to shape and topology optimization

We were able to implement conveniently geometric constraints in level set based shape optimization.



(a) No minimum thickness constraint.

(b) $d_{\text{min}} = 0.1$.

(c) $d_{\text{min}} = 0.2$

Figure: Minimum thickness.constraint for 2D cantilever.

# Preprint to appear

Thank you for your attention.

Much more details in the following preprint to appear.

FEPPON, F., ALLAIRE, AND DAPOGNY, C. A variational formulation for computing shape derivatives of geometric constraints along rays.
(2018).