The geometric interpretation of dynamical model order reduction:

Some recent developpements for continuous time matrix algorithms

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multidisciplinary simulation, estimation, and assimilation systems

Dynamical model order reduction

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If ℓ and m are both large, one rather seeks for a (low) rank r approximation $R(t)$:

$$\tilde{R}(t) \simeq R(t) = U(t)Z(t)^T,$$

with $U(t) \in \mathcal{M}_{\ell,r}$, $Z(t) \in \mathcal{M}_{m,r}$ and $r << \min(\ell, m)$.

Dynamical model order reduction

How to track efficiently the best rank r approximation $R(t) = U(t)Z(t)^T$ of a (full rank) time matrix $\tilde{R}(t) \in \mathcal{M}_{\ell,m}$?

Outline

1. The DO approximation: a geometric approach to dynamical model order reduction
2. Extrinsic curvatures on the fixed rank manifold: DO error analysis and dynamical systems computing the truncated SVD
3. Numerical application: ROM for 2D convection dominated problem.

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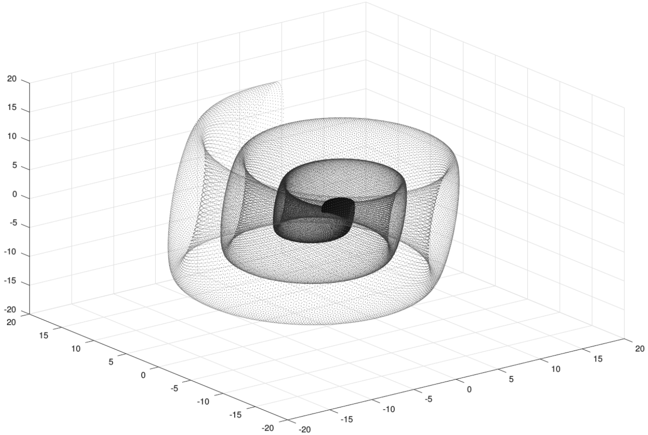

Figure: The manifold $\mathcal{M}$ of 2-by-2 rank 1 matrices

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Any rank preserving reduced order model is a dynamical system

$$\dot{R} = L(t, R(t)) \ \in \mathcal{T}(R(t))$$

where $\mathcal{T}(R(t))$ is the tangent space of $\mathcal{M}$ at $R(t)$.

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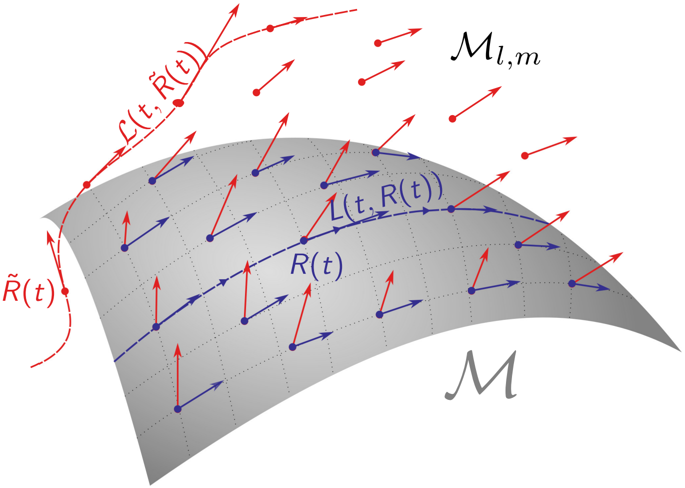

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The best approximation of the $\ell \times r$ matrix $\tilde{R}(t)$ is given by the truncated SVD:

$$\Pi_{\mathcal{M}}(\tilde{R}(t)) := \sum_{i=1}^{r} \sigma_i u_i v_i^T \text{ satisfies } \Pi_{\mathcal{M}}(\tilde{R}(t)) = \arg\min_{R \in \mathcal{M}} ||\tilde{R}(t) - R||,$$

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Finding a “good reduced order model” $\Leftrightarrow$ finding a tangent ODE

$$\dot{R} = L(t, R) \ \in \mathcal{T}(R)$$

such that $R(t) \simeq \Pi_{\mathcal{M}}(\tilde{R}(t))$ **.**

1. The DO approximation

The Dynamically Orthogonal (Sapsis and Lermusiaux (2009)), or Dynamical low-rank approximation (Koch and Lubich 2007):

$$\begin{cases} \dot{R} = \Pi_{\mathcal{T}(R)}(\mathcal{L}(t, R(t))) \\ R(0) = \Pi_{\mathcal{M}}(\tilde{R}(0)) \end{cases}$$

where $\Pi_{\mathcal{T}(R)}$ is the projection onto the tangent space of $\mathcal{M}$ at R.

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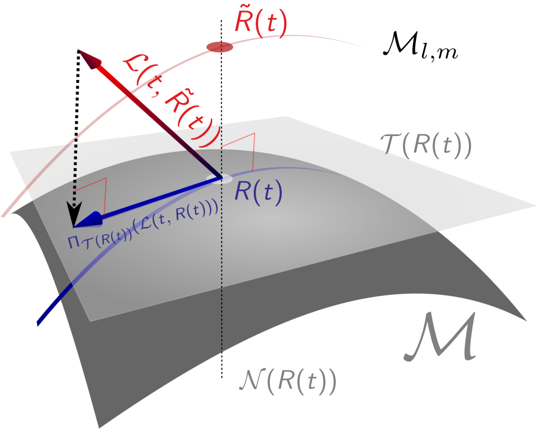

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1. The DO approximation: a geometric approach to dynamical model order reduction
2. Extrinsic curvatures on the fixed rank manifold: DO error analysis and dynamical systems computing the truncated SVD
3. Numerical application: ROM for 2D convection dominated problem.

The DO method:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{R}(t) = \mathcal{L}(t, \tilde{R}(t)) \longrightarrow \begin{cases} \dot{R} = \Pi_{\mathcal{T}(R)}(\mathcal{L}(t, R(t))) \\ R(0) = \Pi_{\mathcal{M}}(\tilde{R}(0)) \end{cases}$$

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We can prove the following approximation error bound:

$$\forall t \in [0, T],\ ||R(t) - \Pi_{\mathcal{M}}(\tilde{R}(t))|| \leq$$

$$\int_0^t \underbrace{||\tilde{R}(s) - \Pi_{\mathcal{M}}(\tilde{R}(s))||}_{\text{best approximation error}} \left(K + \frac{||\mathcal{L}(s, \tilde{R}(s))||}{\sigma_r(\tilde{R}(s)) - \sigma_{r+1}(\tilde{R}(s))} \right) \underbrace{e^{\eta(t-s)}}_{\text{exponential growth}} \mathrm{d}s,$$

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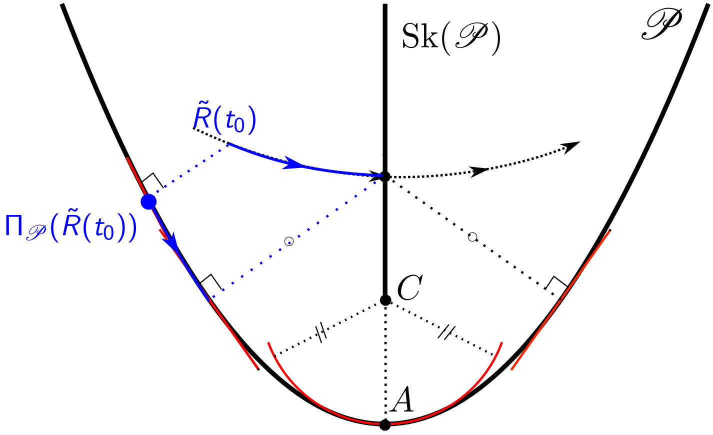
The approximation is "good" as long as there is no crossing of the singular value of order r, i.e. if

$$\sigma_r(\tilde{R}(t)) > \sigma_{r+1}(\tilde{R}(t))$$

2. Extrinsic curvatures on the fixed rank manifold

a. Error analysis of the DO approximation



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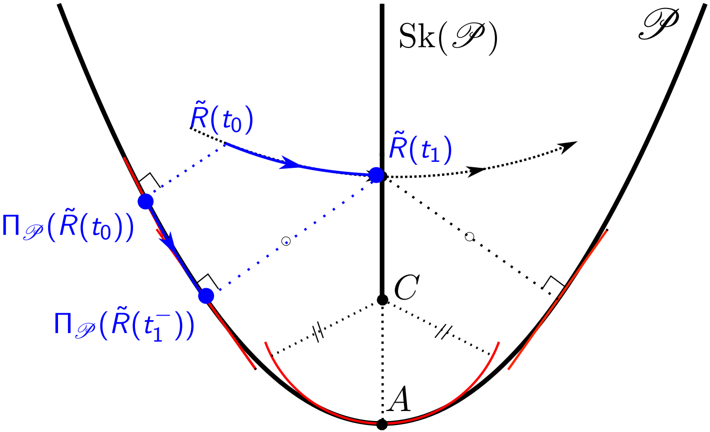

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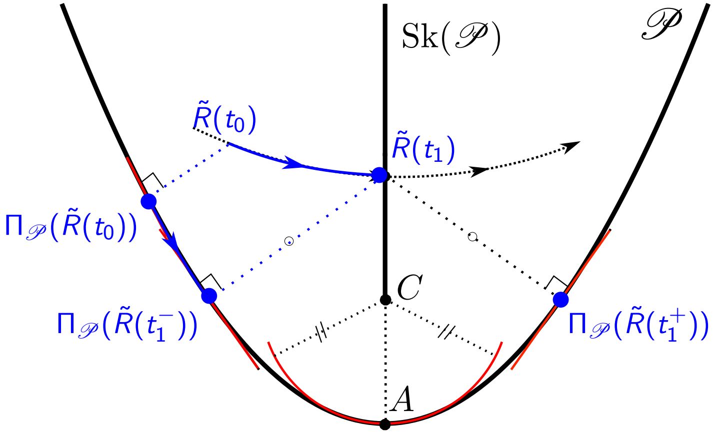

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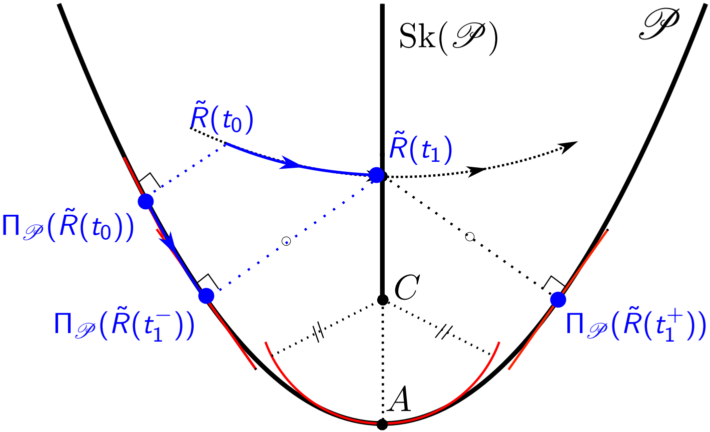

The skeleton set $\mathrm{Sk}(\mathcal{M})$ is exactly

$$\mathrm{Sk}(\mathcal{M}) = \{\tilde{R} \in \mathcal{M}_{\ell,m} \,|\, \sigma_r(\tilde{R}) = \sigma_{r+1}(\tilde{R})\}.$$

The DO method:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{R}(t) = \mathcal{L}(t, \tilde{R}(t)) \longrightarrow \begin{cases} \dot{R} = \Pi_{\mathcal{T}(R)}(\mathcal{L}(t, R(t))) \\ R(0) = \Pi_{\mathcal{M}}(\tilde{R}(0)) \end{cases}$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_{\mathcal{M}}(\tilde{R}(t)) \quad ?$$

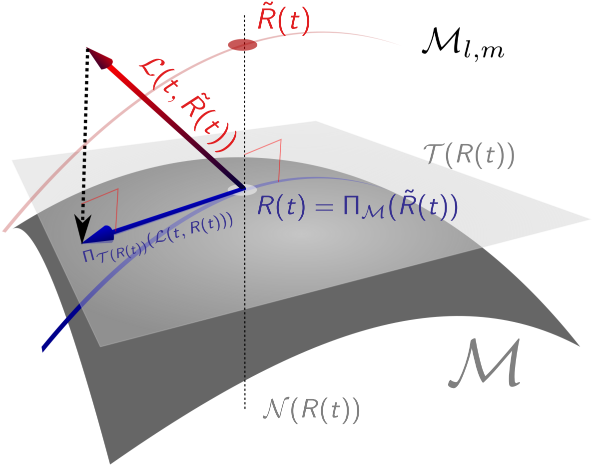
The answer is given by the computation of the extrinsic curvatures of $\mathcal{M}$.

The truncated SVD, $\Pi_{\mathcal{M}}$, is an orthogonal projection onto the manifold $\mathcal{M}$.

$$\mathcal{M} := \{R \in \mathcal{M}_{\ell,m} | \mathrm{rank}(R) = r\}.$$

2. Extrinsic curvatures on the fixed rank manifold

b. Derivative of the truncated SVD



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For a **co-dimension one** surface $\mathcal{M} \subset E = \mathbb{R}^n$, the differential of $\Pi_{\mathcal{M}}$ reads in terms of principal curvatures κ_i and directions Φ_i:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Pi_{\mathcal{M}}(\tilde{R}(t)) = \sum_{i=1}^{n-1} \frac{1}{1-\kappa_i}\Phi_i\Phi_i^T \mathrm{d}\tilde{R}/\mathrm{d}t.$$

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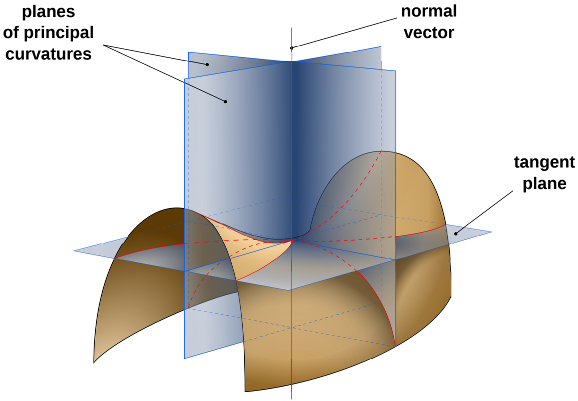
κ_i and Φ_i are the eigenvalues and eigenvectors of the *Weingarten map* at $R(t) = \Pi_{\mathcal{M}}(\tilde{R}(t))$:

$$L_{R(t)} := -\nabla \boldsymbol{n} = \sum_{i=1}^{n-1} \kappa_i \Phi_i \Phi_i^T$$

where $\boldsymbol{n}$ is the outward normal at $R(t)$.

1. Geometry of the fixed rank manifold

b. Derivative of the truncated SVD



This extends for arbitrary extrinsic submanifolds:

- the Weingerten map $L_{R(t)}(N)$ depends on the normal vector $N = \tilde{R}(t) - \Pi_{\mathcal{M}}(\tilde{R}(t))$.

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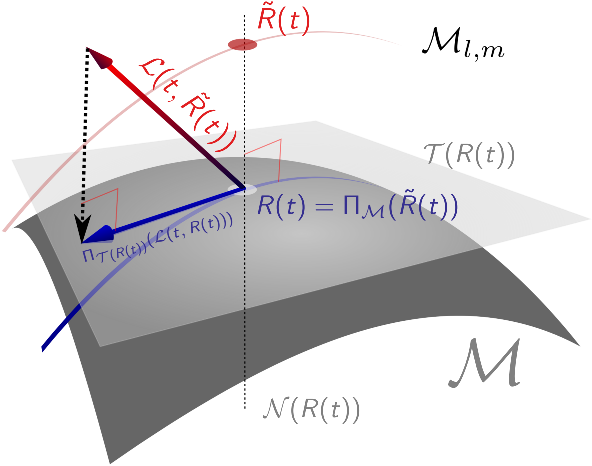

$$\tilde{R}(t) - \Pi_{\mathcal{M}}(\tilde{R}(t)) \in \mathcal{N}(R(t)).$$

This extends for arbitrary extrinsic submanifolds:

- the Weingerten map $L_{R(t)}(N)$ depends on the normal vector $N = \tilde{R}(t) - \Pi_{\mathcal{M}}(\tilde{R}(t))$.
- the derivative or $\Pi_{\mathcal{M}}(\tilde{R}(t))$ now reads

$$\mathrm{d}\Pi_{\mathcal{M}}(\tilde{R}(t))/\mathrm{d}t_{|t=0} = \sum_{i=1}^{\dim(\mathcal{M})} \frac{1}{1 - \kappa_i(N)} \Phi_i \Phi_i^T (\mathrm{d}\tilde{R}(t)/\mathrm{d}t)$$

$\kappa_i(N)$ and Φ_i depend on $N = \tilde{R}(t) - \Pi_{\mathcal{M}}(\tilde{R}(t))$.

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For the fixed rank manifold, it turns out that the spectral decomposition $\kappa_i(N), \Phi_i$ can be computed explicitly!!

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$$\tilde{R}(t) = \sum_{i=1}^{\mathrm{rank}\tilde{R}} \sigma_i u_i v_i^T,$$

$$\kappa_{i,r+j}^{\pm}(N) = \pm\frac{\sigma_{r+j}}{\sigma_i},\ \Phi_{i,r+j}^{\pm} = \frac{1}{\sqrt{2}}(u_{r+j} v_i^T \pm u_i v_{r+j}^T)$$

Let $\tilde{R}(t) = \sum_{i=1}^{r+k} \sigma_i(t) u_i(t) v_i(t)^T \in \mathcal{M}_{\ell,m}$ the SVD of $\tilde{R}(t)$ with $\sigma_r(t) > \sigma_{r+1}(t)$ for all time.

Let $\tilde{R}(t) = \sum_{i=1}^{r+k} \sigma_i(t) u_i(t) v_i(t)^T \in \mathcal{M}_{\ell,m}$ the SVD of $\tilde{R}(t)$ with $\sigma_r(t) > \sigma_{r+1}(t)$ for all time. Then a dynamical system for $\Pi_{\mathcal{M}}(\tilde{R}(t)) = U(t)Z(t)^T$ is given by:

$$
\begin{cases}
\dot{U} = (I - UU^T)\dot{\tilde{R}}Z(Z^TZ)^{-1} \\
\qquad + \left[\sum_{\substack{1 \le i \le r \\ 1 \le j \le k}} \frac{\sigma_{r+j}}{\sigma_i^2 - \sigma_{r+j}^2} (\sigma_i u_{r+j}^T \dot{\tilde{R}} v_i + \sigma_{r+j} u_i^T \dot{\tilde{R}} v_{r+j}) u_{r+j} v_i^T \right] Z(Z^TZ)^{-1} \\
\dot{Z} = \dot{\tilde{R}}^T U + \left[\sum_{\substack{1 \le i \le r \\ 1 \le j \le k}} \frac{\sigma_{r+j}}{\sigma_i^2 - \sigma_{r+j}^2} (\sigma_{r+j} u_{r+j}^T \dot{\tilde{R}} v_i + \sigma_i u_i^T \dot{\tilde{R}} v_{r+j}) v_{r+j} u_i^T \right] U.
\end{cases}
$$

Let $\tilde{R} \in \mathcal{M}_{\ell,m}$ be fixed and $U(t), Z(t)$ solving the ODE:

$$\begin{cases} \dot{U} = (I - UU^T)\tilde{R}Z(Z^TZ)^{-1} \\ \dot{Z} = \tilde{R}^TU - Z \end{cases}$$

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Then for almost *any* initial data $U(0), Z(0)$, $R(t) = U(t)Z(t)^T$ converges to $\Pi_{\mathcal{M}}(\tilde{R})$.

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  - Stiefel manifold $\leftrightarrow$ polar decomposition
  - Isospectral manifold $\leftrightarrow$ linear eigenprojectors of symmetric matrices
  - Grassmann manifold $\leftrightarrow$ linear eigenprojectors of *non symmetric* matrices

3. ROM for 2D convection

We applied the DO method for stochastic advection:

How to solve numerically the stochastic PDE in $\psi(t, x; \omega)$

$$\partial_t \psi + \boldsymbol{v}(t, x; \omega) \cdot \nabla \ \psi = 0$$

for a huge number of realizations ω?

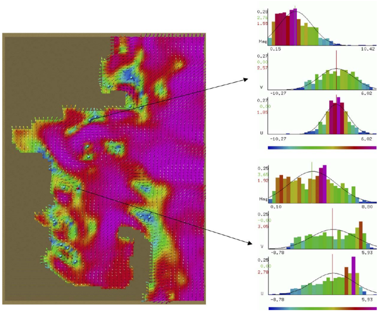

Figure: A “real-life” uncertain velocity field $\boldsymbol{v}(t, x; \omega)$ (Lermusiaux 2006)

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Methodology:

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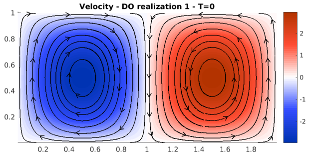
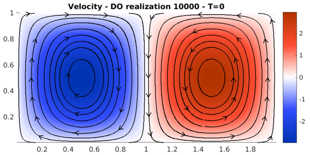
- We evolve time-dependent modes and coefficients $U(t)$ and $Z(t)$ with the DO approximation
- we use **fully linear central** finite difference schemes for advection.

3. ROM for 2D convection

Random oscillation frequency for $\boldsymbol{v}(t, x; \omega)$





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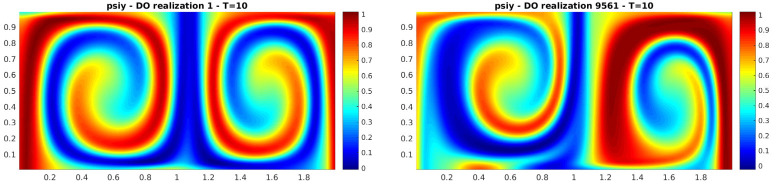
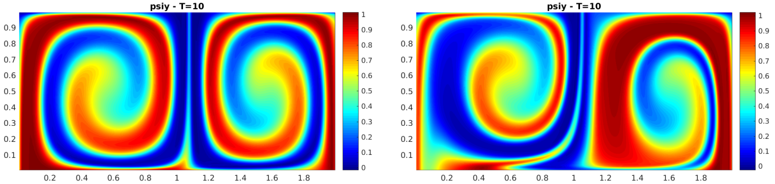
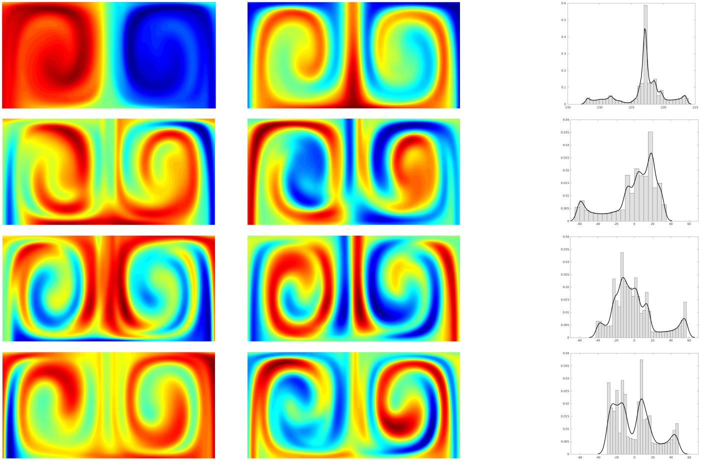


DO solutions



True realizations

3. ROM for 2D convection



$(\psi_i)_x$ $(\psi_i)_y$ Coefficient distribution

Figure: 4 first dominant modes and coefficients $U(T)$, $Z(T)$

References

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Thank you!