Null Space Gradient Flows for Shape Optimization of Multiphysics Systems

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New trends in PDE constrained optimization
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Outline

1. Shape derivatives for a weakly coupled multiphysics system
2. Null space gradient flows for constrained optimization
3. Numerical illustrations on 2-d and 3-d test cases
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We are interested in multiphysics systems featuring

- fluids: velocity–pressure ($\mathbf{v}, p$)
- thermal exchanges: temperature field $T$, convected by $\mathbf{v}$
- mechanical structures: displacement $\mathbf{u}$, subjected to fluid-structure interaction with $\mathbf{v}$ and thermoelasticity with $T$. 

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1. Shape derivatives for a multiphysics system

Proposed system

- Incompressible Navier-Stokes equations for \((\boldsymbol{v}, p)\) in \(\Omega_f\)

\[-\text{div}(\sigma_f(\boldsymbol{v}, p)) + \rho \nabla \boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{f}_f \text{ in } \Omega_f\]
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$$-\text{div}(\sigma_f(\mathbf{v}, p)) + \rho \nabla \mathbf{v} \cdot \mathbf{v} = \mathbf{f}_f \text{ in } \Omega_f$$

- Steady-state convection-diffusion for $T_f$ and $T_s$ in $\Omega_f$ and $\Omega_s$:

$$-\text{div}(k_f \nabla T_f) + \rho \mathbf{v} \cdot \nabla T_f = Q_f \text{ in } \Omega_f$$

$$-\text{div}(k_s \nabla T_s) = Q_s \text{ in } \Omega_s$$
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  \[-\text{div}(k_s \nabla T_s) = Q_s \text{ in } \Omega_s\]

- Linearized thermoelasticity with fluid-structure interaction for \(u\) in \(\Omega_s\):
  \[-\text{div}(\sigma_s(u, T_s)) = f_s \text{ in } \Omega_s\]
  \[\sigma_s(u, T_s) \cdot \mathbf{n} = \sigma_f(\mathbf{v}, p) \cdot \mathbf{n} \text{ on } \Gamma.\]
1. Shape derivatives for a multiphysics system

The method of Hadamard

\[ \min_{\Gamma} J(\Gamma) \]

\[ \Gamma_\theta = (I + \theta)\Gamma, \]

where \( \theta \in W^{1,\infty}(D, \mathbb{R}^d) \), \( ||\theta||_{W^{1,\infty}(D, \mathbb{R}^d)} < 1 \).

\[ J(\Gamma_\theta) = J(\Gamma) + dJ(d\theta)(\theta) + o(\theta) \]

\( \theta \rightarrow 0 \quad \Rightarrow \quad 0 \).
1. Shape derivatives for a multiphysics system
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1. Shape derivatives for a multiphysics system
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\[J(\Gamma_{\theta}) = J(\Gamma) + \frac{dJ}{d\theta}(\theta) + o(\theta), \text{ where } \frac{|o(\theta)|}{\|\theta\|_{W^{1,\infty}(D, \mathbb{R}^d)}} \xrightarrow{\theta \to 0} 0.\]
Proposition

Let \( J(\Gamma, u, T, v, p) \) an arbitrary functional with continuous partial derivatives and \( u(\Gamma), T(\Gamma), v(\Gamma), p(\Gamma) \) the above state variables. Then, if these are smooth enough, \( \Gamma \mapsto J(\Gamma, u(\Gamma), T(\Gamma), v(\Gamma), p(\Gamma)) \) is shape differentiable and the derivative reads:
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$$
\frac{d}{d\theta} \left[ J(\Gamma_{\theta}, v(\Gamma_{\theta}), p(\Gamma_{\theta}), T(\Gamma_{\theta}), u(\Gamma_{\theta})) \right](\theta)
= \frac{\partial J}{\partial \theta}(\theta) + \int_{\Gamma} (f_f \cdot w - \sigma_f(v, p) : \nabla w + n \cdot \sigma_f(w, q) \nabla v \cdot n + n \cdot \sigma_f(v, p) \nabla w \cdot n)(\theta \cdot n)ds
+ \int_{\Gamma} \left( k_s \nabla T_s \cdot \nabla S_s - k_f \nabla T_f \cdot \nabla S_f + Q_f S_f - Q_s S_s - 2k_s \frac{\partial T_s}{\partial n} \frac{\partial S_s}{\partial n} + 2k_f \frac{\partial T_f}{\partial n} \frac{\partial S_f}{\partial n} \right)(\theta \cdot n)ds
+ \int_{\Gamma} (\sigma_s(u, T_s) : \nabla r - f_s \cdot r - n \cdot Ae(r) \nabla u \cdot n - n \cdot \sigma_s(u, T_s) \nabla r \cdot n)(\theta \cdot n)ds
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Proposition

Let $J(\Gamma, \mathbf{u}, T, \mathbf{v}, p)$ an arbitrary functional with continuous partial derivatives and $\mathbf{u}(\Gamma), T(\Gamma), \mathbf{v}(\Gamma), p(\Gamma)$ the above state variables. Then, if these are smooth enough, $\Gamma \mapsto J(\Gamma, \mathbf{u}(\Gamma), T(\Gamma), \mathbf{v}(\Gamma), p(\Gamma))$ is shape differentiable and the derivative reads:

$$
\frac{d}{d\theta} \left[ J(\Gamma_\theta, \mathbf{v}(\Gamma_\theta), p(\Gamma_\theta), T(\Gamma_\theta), \mathbf{u}(\Gamma_\theta)) \right](\theta) = \frac{\partial \tilde{J}}{\partial \theta}(\theta) + \int_\Gamma (f_f \cdot \mathbf{w} - \sigma_f(\mathbf{v}, p) : \nabla \mathbf{w} + \mathbf{n} \cdot \sigma_f(\mathbf{w}, \mathbf{q}) \nabla \mathbf{v} \cdot \mathbf{n} + \mathbf{n} \cdot \sigma_f(\mathbf{v}, p) \nabla \mathbf{w} \cdot \mathbf{n}) (\theta \cdot \mathbf{n}) \, ds
$$

$$
+ \int_\Gamma \left( k_s \nabla T_s \cdot \nabla S_s - k_f \nabla T_f \cdot \nabla S_f + Q_f S_f - Q_s S_s - 2k_s \frac{\partial T_s}{\partial \mathbf{n}} \frac{\partial S_s}{\partial \mathbf{n}} + 2k_f \frac{\partial T_f}{\partial \mathbf{n}} \frac{\partial S_f}{\partial \mathbf{n}} \right) (\theta \cdot \mathbf{n}) \, ds
$$

$$
+ \int_\Gamma (\sigma_s(u, T_s) : \nabla \mathbf{r} - f_s \cdot \mathbf{r} - \mathbf{n} \cdot \mathbf{Ae}(\mathbf{r}) \nabla \mathbf{u} \cdot \mathbf{n} - \mathbf{n} \cdot \sigma_s(\mathbf{u}, T_s) \nabla \mathbf{r} \cdot \mathbf{n}) (\theta \cdot \mathbf{n}) \, ds
$$

$
\tilde{J}$ is a "transported" functional:

$$
\tilde{J}(\theta, \hat{\mathbf{v}}, \hat{\mathbf{p}}, \hat{T}, \hat{\mathbf{u}}) := J(\Gamma_\theta, \hat{\mathbf{v}} \circ (I + \theta)^{-1}, \hat{\mathbf{p}} \circ (I + \theta)^{-1}, \hat{T} \circ (I + \theta)^{-1}, \hat{\mathbf{u}} \circ (I + \theta)^{-1}).
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$$
\frac{d}{d\theta} \left[ J(\Gamma_\theta, v(\Gamma_\theta), p(\Gamma_\theta), T(\Gamma_\theta), u(\Gamma_\theta)) \right](\theta) = \frac{\partial J}{\partial \theta}(\theta) + \int_\Gamma \left( f_f \cdot w - \sigma_f(v, p) : \nabla w + n \cdot \sigma_f(w, q) \nabla v \cdot n + n \cdot \sigma_f(v, p) \nabla w \cdot n \right)(\theta \cdot n) ds
$$

$$
+ \int_\Gamma \left( k_s \nabla T_s \cdot \nabla S_s - k_f \nabla T_f \cdot \nabla S_f + Q_f S_f - Q_s S_s - 2k_s \frac{\partial T_s}{\partial n} \frac{\partial S_s}{\partial n} + 2k_f \frac{\partial T_f}{\partial n} \frac{\partial S_f}{\partial n} \right)(\theta \cdot n) ds
$$

$$
+ \int_\Gamma \left( \sigma_s(u, T_s) : \nabla r - f_s \cdot r - n \cdot Ae(r) \nabla u \cdot n - n \cdot \sigma_s(u, T_s) \nabla r \cdot n \right)(\theta \cdot n) ds
$$

Partial derivative for $J$ with respect to the shape.
Proposition

Let \( J(\Gamma, \mathbf{u}(\Gamma), T(\Gamma), \mathbf{v}(\Gamma), p(\Gamma)) \) an arbitrary functional with continuous partial derivatives and \( \mathbf{u}(\Gamma), T(\Gamma), \mathbf{v}(\Gamma), p(\Gamma) \) the above state variables. Then, if these are smooth enough, \( \Gamma \mapsto J(\Gamma, \mathbf{u}(\Gamma), T(\Gamma), \mathbf{v}(\Gamma), p(\Gamma)) \) is shape differentiable and the derivative reads:

\[
\frac{d}{d\theta} \left[ J(\Gamma, \mathbf{v}(\Gamma), p(\Gamma), T(\Gamma), \mathbf{u}(\Gamma)) \right](\theta) = \frac{\partial J}{\partial \theta}(\theta) + \int_{\Gamma} (\mathbf{f} \cdot \mathbf{w} - \sigma_f(\mathbf{v}, p) : \nabla \mathbf{w} + \mathbf{n} \cdot \sigma_f(\mathbf{w}, q) \nabla \mathbf{v} \cdot \mathbf{n} + \mathbf{n} \cdot \sigma_f(\mathbf{v}, p) \nabla \mathbf{w} \cdot \mathbf{n})(\theta \cdot \mathbf{n}) d\mathbf{s}
\]

\[
+ \int_{\Gamma} \left( k_s \nabla T_s \cdot \nabla S_s - k_f \nabla T_f \cdot \nabla S_f + Q_f \nabla S_f - Q_s \nabla S_s - 2k_s \frac{\partial T_s}{\partial \mathbf{n}} \frac{\partial S_s}{\partial \mathbf{n}} + 2k_f \frac{\partial T_f}{\partial \mathbf{n}} \frac{\partial S_f}{\partial \mathbf{n}} \right)(\theta \cdot \mathbf{n}) d\mathbf{s}
\]

\[
+ \int_{\Gamma} (\sigma_s(\mathbf{u}, T_s) : \nabla \mathbf{r} - \mathbf{f}_s \cdot \mathbf{r} - \mathbf{n} \cdot \mathbf{Ae}(\mathbf{r}) \nabla \mathbf{u} \cdot \mathbf{n} - \mathbf{n} \cdot \sigma_s(\mathbf{u}, T_s) \nabla \mathbf{r} \cdot \mathbf{n})(\theta \cdot \mathbf{n}) d\mathbf{s}
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Three “adjoint” terms for each of the three physics.
Proposition

Let \( J(\Gamma, u, T, v, p) \) an arbitrary functional with continuous partial derivatives and \( u(\Gamma), T(\Gamma), v(\Gamma), p(\Gamma) \) the above state variables. Then, if these are smooth enough, \( \Gamma \mapsto J(\Gamma, u(\Gamma), T(\Gamma), v(\Gamma), p(\Gamma)) \) is shape differentiable and the derivative reads:

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\[
+ \int_\Gamma (\sigma_s(u, T_s) : \nabla r - f_s \cdot r - n \cdot Ae(r) \nabla u \cdot n - n \cdot \sigma_s(u, T_s) \nabla r \cdot n)(\theta \cdot n)ds
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\]

Adjoint variables \( w, q, S_f, S_s, r \) are solved in a reversed cascade.
Outline

1. Shape derivatives for a weakly coupled multiphysics system
2. Null space gradient flows for constrained optimization
3. Numerical illustrations on 2-d and 3-d test cases
2. Null space gradient flows for constrained optimization

- Our goal: solve constrained shape optimization problems

\[
\min_{\Gamma} \ J(\Gamma, v(\Gamma), p(\Gamma), T(\Gamma), u(\Gamma))
\]

s.t. \[ g_i(\Gamma, v(\Gamma), p(\Gamma), T(\Gamma), u(\Gamma)) = 0, \ 1 \leq i \leq p \cdot \]
\[ h_i(\Gamma, v(\Gamma), p(\Gamma), T(\Gamma), u(\Gamma)) \leq 0, \ 1 \leq i \leq q \]

with arbitrary functionals \( J, g_i, h_i; \)

- if possible, no fine tunings of optimization algorithm parameters;

- must deal with unfeasible initializations.
For the exposure of our method, let us consider

$$\min_{x \in V} J(x)$$

subject to

$$\begin{cases} g(x) = 0 \\ h(x) \leq 0, \end{cases}$$

with

- $J : V \to \mathbb{R}$, $g : V \to \mathbb{R}^p$ and $h : V \to \mathbb{R}^q$ Fréchet differentiable
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\]

with

- $J : V \to \mathbb{R}$, $g : V \to \mathbb{R}^p$ and $h : V \to \mathbb{R}^q$ Fréchet differentiable
- $V$ a Hilbert space equipped with a scalar product $(\cdot, \cdot)_V$. 

2. Null space gradient flows for constrained optimization

\[
\min_{(x_1, x_2) \in \mathbb{R}^2} J(x_1, x_2) = x_1^2 + (x_2 + 3)^2
\]

s.t.

\[
\begin{align*}
  h_1(x_1, x_2) &= -x_1^2 + x_2 & \leq 0 \\
  h_2(x_1, x_2) &= -x_1 - x_2 - 2 & \leq 0
\end{align*}
\]
2. Null space gradient flows for constrained optimization

We extend classical dynamical systems approaches to constrained optimization:
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We extend classical dynamical systems approaches to constrained optimization:

▶ For unconstrained optimization, the celebrated gradient flow:

\[ \dot{x} = -\nabla J(x) \]

▶ For equality constrained optimization, projected gradient flow (Tanabe (1980)):

\[ \dot{x} = -\left( I - Dg^T(DgDg^T)^{-1}Dg \right) \nabla J(x) \]

Then Yamashita (1980) added a Gauss-Newton direction:

\[ \dot{x} = -\alpha J\left( I - Dg^T(DgDg^T)^{-1}Dg \right) \nabla J(x) - \alpha CDg^T(DgDg^T)^{-1}g(x) \]

\[ g(x(t)) = g(x(0)) e^{-\alpha Ct} \text{ and } J(x(t)) \text{ decreases if } g(x(t)) = 0. \]
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$$\dot{x} = -(I - \mathbf{D}g^T(\mathbf{D}g\mathbf{D}g^T)^{-1}\mathbf{D}g)\nabla J(x)$$

(gradient flow on $V = \{x \in V \mid \mathbf{g}(x) = 0\}$)
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(gradient flow on \( V = \{ x \in V \mid g(x) = 0 \} \)) Then Yamashita (1980) added a Gauss-Newton direction:

\[ \dot{x} = -\alpha_J(I - Dg^T(DgDg^T)^{-1}Dg)\nabla J(x) - \alpha_C Dg^T(DgDg^T)^{-1}g(x) \]
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\[ \dot{x} = -\alpha J(l - Dg^T (Dg Dg^T)^{-1} Dg) \nabla J(x) - \alpha_c Dg^T (Dg Dg^T)^{-1} g(x) \]

\( g(x(t)) = g(x(0)) e^{-\alpha_c t} \) and \( J(x(t)) \) decreases if \( g(x(t)) = 0 \).
2. Null space gradient flows for constrained optimization

For both equality constraints $g(x) = 0$ and inequality constraints $h(x) \leq 0$, we consider:

$$\dot{x} = -\alpha_J \xi_J(x(t)) - \alpha_C \xi_C(x(t))$$

with

$$\xi_J(x) := (I - DC_{\tilde{I}(x)}^T (DC_{\tilde{I}(x)} DC_{\tilde{I}(x)}^T)^{-1} DC_{\tilde{I}(x)})(\nabla J(x))$$

$$\xi_C(x) = DC_{\tilde{I}(x)}^T (DC_{\tilde{I}(x)} DC_{\tilde{I}(x)}^T)^{-1} C_{\tilde{I}(x)}(x),$$
For both equality constraints $g(x) = 0$ and inequality constraints $h(x) \leq 0$, we consider:

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$$\xi_C(x) = DC_{\tilde{I}(x)}^T (DC_{\tilde{I}(x)} DC_{\tilde{I}(x)}^T )^{-1} C_{\tilde{I}(x)}(x),$$

$\tilde{I}(x)$ the set of violated constraints:

$$\tilde{I}(x) = \{ i \in \{1, \ldots, q\} \mid h_i(x) \geq 0 \}.$$

$$C_{\tilde{I}(x)} = \left[ \begin{array}{c} g(x) \\ (h_i(x))_{i \in \tilde{I}(x)} \end{array} \right]^T$$
2. Null space gradient flows for constrained optimization

For both equality constraints $g(x) = 0$ and inequality constraints $h(x) \leq 0$, we consider:

$$\dot{x} = -\alpha_J \xi_J(x(t)) - \alpha_C \xi_C(x(t))$$

with

$$\xi_J(x) := (I - DC_{\hat{I}(x)}(DC_{\hat{I}(x)} DC_{\hat{I}(x)}^T)^{-1}DC_{\hat{I}(x)})(\nabla J(x))$$

$$\xi_C(x) = DC_{\tilde{I}(x)}(DC_{\tilde{I}(x)} DC_{\tilde{I}(x)}^T)^{-1}C_{\tilde{I}(x)}(x),$$

$\hat{I}(x) \subset \tilde{I}(x)$ is an “optimal” subset of the active or violated constraints which can be computed by mean of a dual subproblem.

$$\hat{I}(x) := \{i \in \tilde{I}(x) | \mu_i^*(x) > 0\}.$$ 

$$C_{\hat{I}(x)} = \left[ g(x) \mid (h_i(x))_{i \in \hat{I}(x)} \right]^T$$
2. Null space gradient flows for constrained optimization

Definition

Let \((\lambda^*(x), \mu^*(x)) \in \mathbb{R}^p \times \mathbb{R}^{\text{Card} \tilde{I}(x)}\) the solutions of the following dual minimization problem:

\[
(\lambda^*(x), \mu^*(x)) := \arg \min_{\lambda \in \mathbb{R}^p} \min_{\mu \in \mathbb{R}^{\tilde{q}(x)}, \mu \geq 0} \| \nabla J(x) + Dg(x)^T \lambda + Dh_{\tilde{I}(x)}(x)^T \mu \|_V.
\]

Define \(\hat{I}(x)\) the set obtained by collecting the non zero components of \(\mu^*(x)\):

\[
\hat{I}(x) := \{ i \in \tilde{I}(x) \mid \mu^*_i(x) > 0 \}.
\]
The best descent direction \(-\xi_J(x)\) must be proportional to

\[
\xi^* = \arg \min_{\xi \in V} DJ(x)\xi
\]

subject to

\[
\begin{align*}
DG(x)\xi &= 0 \\
Dh_{\tilde{I}(x)}(x)\xi &\leq 0 \\
\|\xi\|_V &\leq 1.
\end{align*}
\]

where \(h_{\tilde{I}(x)}(x) = (h_i(x))_{i \in \tilde{I}(x)}\)
Proposition

$\xi^*(x)$ is explicitly given by:

$$\xi^*(x) = -\frac{\Pi_C \hat{I}(x) (\nabla J(x))}{\|\Pi_C \hat{I}(x) (\nabla J(x))\|_V}$$

with

$$\Pi_C \hat{I}(x) (\nabla J(x)) = (I - DC^T_{\hat{I}(x)} (DC_{\hat{I}(x)} DC^T_{\hat{I}(x)})^{-1} DC_{\hat{I}(x)}) (\nabla J(x))$$
Proposition

\( \xi^*(x) \) is explicitly given by:

\[
\xi^*(x) = -\frac{\Pi_{C_{\hat{I}(x)}}(\nabla J(x))}{\|\Pi_{C_{\hat{I}(x)}}(\nabla J(x))\|_V},
\]

with

\[
\Pi_{C_{\hat{I}(x)}}(\nabla J(x)) = (I - D_{C_{\hat{I}(x)}}^T (D_{C_{\hat{I}(x)}} D_{C_{\hat{I}(x)}}^T)^{-1} D_{C_{\hat{I}(x)}})(\nabla J(x))
\]

whence our definition

\[
\dot{x} = -\alpha_J \xi_J(x(t)) - \alpha_C \xi_C(x(t))
\]

\[
\xi_J(x) := (I - D_{C_{\hat{I}(x)}}^T (D_{C_{\hat{I}(x)}} D_{C_{\hat{I}(x)}}^T)^{-1} D_{C_{\hat{I}(x)}})(\nabla J(x))
\]

\[
\xi_C(x) = D_{C_{\hat{I}(x)}}^T (D_{C_{\hat{I}(x)}} D_{C_{\hat{I}(x)}}^T)^{-1} C_{\hat{I}(x)}(x),
\]
2. Null space gradient flows for constrained optimization

We can prove:

1. Constraints are asymptotically satisfied:

\[ g(x(t)) = e^{-\alpha c t} g(x(0)) \text{ and } h_{\tilde{I}}(x(t)) \leq e^{-\alpha c t} h(x(0)) \]
2. Null space gradient flows for constrained optimization

We can prove:

1. Constraints are asymptotically satisfied:

   \[ g(x(t)) = e^{-\alpha ct} g(x(0)) \quad \text{and} \quad h_\tilde{l}(x(t)) \leq e^{-\alpha ct} h(x(0)) \]

2. \( J \) decreases as soon as the violation \( C_\tilde{l}(x(t)) \) is sufficiently small
2. Null space gradient flows for constrained optimization

We can prove:

1. Constraints are asymptotically satisfied:
   \[ g(x(t)) = e^{-\alpha c t} g(x(0)) \] \[ \text{and} \quad h_I(x(t)) \leq e^{-\alpha c t} h(x(0)) \]

2. \( J \) decreases as soon as the violation \( C_I(x(t)) \) is sufficiently small

3. All stationary points \( x^* \) of the ODE are KKT points
For shape optimization

\[ \dot{x} = -\alpha_J \xi_J(x(t)) - \alpha_C \xi_C(x(t)) \]

works the same with

\[ \xi_J(x) := (I - DC^T \hat{I}(x)(DC \hat{I}(x) DC^T \hat{I}(x))^{-1} DC \hat{I}(x)) (\nabla J(x)) \]

\[ \xi_C(x) = DC^T \hat{I}(x)(DC \hat{I}(x) DC^T \hat{I}(x))^{-1} C \hat{I}(x)(x), \]

where the transpose $^T$ and the gradient $\nabla$ must be computed with respect to $\langle \cdot, \cdot \rangle_V$ thanks to an identification problem.
For shape optimization

\[ \dot{x} = -\alpha J \xi_J(x(t)) - \alpha C \xi_C(x(t)) \]

works the same with

\[ \xi_J(x) := (I - D C_{\tilde{I}(x)}^T (D C_{\tilde{I}(x)} D C_{\tilde{I}(x)}^T)^{-1} D C_{\tilde{I}(x)})(\nabla J(x)) \]

\[ \xi_C(x) = D C_{\tilde{I}(x)}^T (D C_{\tilde{I}(x)} D C_{\tilde{I}(x)}^T)^{-1} C_{\tilde{I}(x)}(x), \]

where the transpose \( T \) and the gradient \( \nabla \) must be computed with respect to \( \langle \cdot, \cdot \rangle_V \) thanks to an identification problem. This encompasses the celebrated regularization and extension step of the shape derivative in numerical algorithms.
Outline

1. Shape derivatives for a weakly coupled multiphysics system
2. Null space gradient flows for constrained optimization
3. Numerical illustrations on 2-d and 3-d test cases
3. Numerical applications

Lift-drag minimization:

\[
\begin{align*}
\min & \quad - \text{Lift}(\Gamma, \mathbf{v}(\Gamma), p(\Gamma)) \\
\text{s.t.} & \quad \text{Drag}(\Gamma, \mathbf{v}(\Gamma), p(\Gamma)) \leq \text{DRAG}_0 \\
& \quad \text{Vol}(\Omega_f) = V_0 \\
& \quad \mathbf{X}(\Omega_s) := \frac{1}{|\Omega_s|} \int_{\Omega_s} \mathbf{x} \, dx = \mathbf{x}_0,
\end{align*}
\]

\[
\begin{align*}
\text{Lift}(\Gamma, \mathbf{v}(\Gamma), p(\Gamma)) & := - \int_{\Gamma} e_y \cdot \sigma_f(\mathbf{v}, p) n ds, \\
\text{Drag}(\Gamma, \mathbf{v}(\Gamma), p(\Gamma)) & := \int_{\Omega_f} \sigma_f(\mathbf{v}, p) : \nabla \mathbf{v} \, dx.
\end{align*}
\]

**Figure:** Optimized 2-d lift-drag flow profile.
3. Numerical applications

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\end{align*}
\]

\[
\begin{align*}
\text{Lift}(\Gamma, \mathbf{v}(\Gamma), p(\Gamma)) & := - \int_{\Gamma} \mathbf{e}_y \cdot \sigma_f(\mathbf{v}, p) \mathbf{n} \, ds, \\
\text{Drag}(\Gamma, \mathbf{v}(\Gamma), p(\Gamma)) & := \int_{\Omega_f} \sigma_f(\mathbf{v}, p) : \nabla \mathbf{v} \, d\mathbf{x}.
\end{align*}
\]

Figure: Optimized 2-d lift-drag flow profile.
3. Numerical applications

Lift-drag minimization in 3-d:

(a) Initial shape

(b) Optimized design

(c) Optimized design (other 3-d views)
3. Numerical applications

Lift-drag minimization in 3-d, convergence histories.

(a) Objective function.

(b) Volume constraint.

(c) Center of mass constraint.

(d) Drag constraint.
3. Numerical applications

Bi-tube heat exchanger with non penetrating constraint

\[
\min_{\Omega_f \subset D} J(\Omega_f) = - \left( \int_{\Omega_{f, \text{cold}}} \rho c_p v \cdot \nabla T \, dx - \int_{\Omega_{f, \text{hot}}} \rho c_p v \cdot \nabla T \, dx \right)
\]

\[
\text{s.t.} \begin{align*}
\text{DP}(\Omega_f) &= \int_{\partial \Omega_f^D} p \, ds - \int_{\partial \Omega_f^N} p \, ds \leq \text{DP}_0 \\
Q_{\text{hot} \leftrightarrow \text{cold}}(\Omega_f) &\geq d_{\text{min}}.
\end{align*}
\]
3. Numerical applications

Bi-tube heat exchanger with non penetrating constraint

(a) Initial temperature field

(b) Final temperature field.

(c) Intermediate iterations
3. Numerical applications

3-d compliance minimization problem with fluid-structure interaction
3. Numerical applications

3-d compliance minimization problem with fluid-structure interaction

(a) Initial shape

(b) Final design
3. Numerical applications

3-d compliance minimization problem with fluid-structure interaction

(a) Initial shape

(b) Final design

Approx. 2 millions elements.
3. Numerical applications

(a) Final design.

Figure: Linear elastic deformation.
3-d compliance minimization problem with fluid-structure interaction

Figure: Intermediate iterations 0, 40, 100, 125, 175 and 300.

Feppon, F., Allaire, G., and Dapogny, C. Null space gradient flows for constrained optimization with applications to shape optimization. 
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Many thanks for your attention.