High order homogenized Stokes models capture all three regimes

Florian Feppon

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Outline

1. Motivations from topology optimization
2. Overview of our results
3. Higher order homogenized models for the Stokes problem: summary of the derivation
4. Higher order models capture all three regimes: low volume fraction asymptotics.

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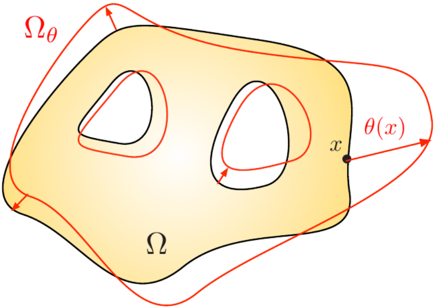
Topology optimization with the method of Hadamard



$$\min_{\Omega \subset D} \quad J(\Omega, \boldsymbol{u}(\Omega))$$

$$s.t. \begin{cases} -\mathrm{div}(A(1_\Omega)\nabla \boldsymbol{u}) = \boldsymbol{f} \text{ in } D \\ \boldsymbol{u} = 0 \text{ on } \partial D. \end{cases}$$

Topology optimization with the method of Hadamard

Optimal "shapes" are not shapes but composite structures.

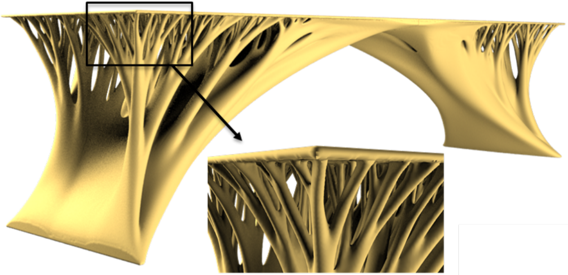
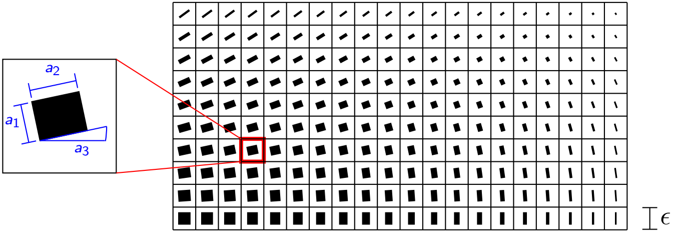


Figure: Kambampati et al., *"Fast level set topology optimization using a hierarchical data structure"* (2018).

Inverse homogenization method

$\Omega \equiv \Omega_\epsilon(\boldsymbol{a})$ is a composite material with parameterized microstructure:



$$\min_{\boldsymbol{a}=(a_1,a_2,a_3)} \quad J(\Omega_\epsilon(\boldsymbol{a}), \boldsymbol{u}_\epsilon(\Omega_\epsilon(\boldsymbol{a})))$$

$$s.t. \begin{cases} -\mathrm{div}(A(1_{\Omega_\epsilon(\boldsymbol{a})})\nabla \boldsymbol{u}_\epsilon) = \boldsymbol{f} \text{ in } D \\ \boldsymbol{u} = 0 \text{ on } \partial D. \end{cases}$$

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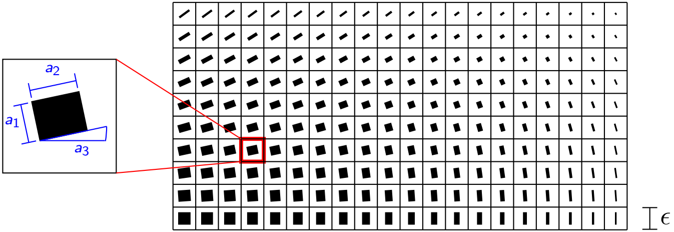


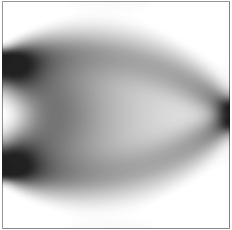
$$\min_{\boldsymbol{a}=(a_1,a_2,a_3)} \quad J^*(\boldsymbol{a}, \boldsymbol{u}(\boldsymbol{a}))$$

$$s.t. \begin{cases} -\mathrm{div}(A^*(\boldsymbol{a})\nabla \boldsymbol{u}) = \boldsymbol{f} \text{ in } D \\ \boldsymbol{u} = 0 \text{ on } \partial D, \end{cases}$$

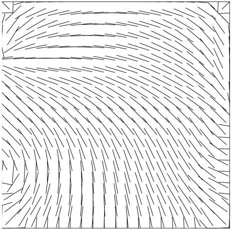
$A^*(\boldsymbol{a})$ is an effective material tensor and $\boldsymbol{u}_\epsilon(\Omega_\epsilon(\boldsymbol{a})) \to \boldsymbol{u}(\boldsymbol{a})$.

Optimize $a_1(x), \ldots a_3(x)$ instead of Ω!

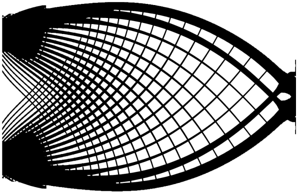
Inverse homogenization method



(a) Optimized density



(b) Optimized orientation



(c) De-homogenized shape

Figure: Topology optimization of a 2-d cantilever beam by a homogenization method.
Geoffroy Donders, *"Homogenization method for topology optmization of structures built with lattice materials."* (2018).

Fluid applications

We would like to extend the method for fluid applications:

$$\min_{\Omega \subset D} \quad J(\Omega, \boldsymbol{u}(\Omega), p(\Omega)),$$

$$s.t. \begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } \Omega \\ \operatorname{div}(\boldsymbol{u}) = 0 \text{ in } \Omega, \\ \boldsymbol{u} = 0 \text{ on } \partial\Omega. \end{cases}$$

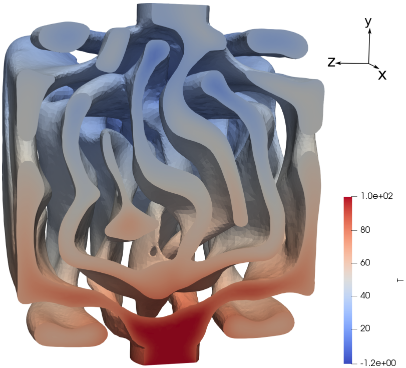


Figure: 2-fluid heat exchanger optimized with the method of Hadamard[1].

[1] Feppon et al., *"Body-fitted topology optimization of 2D and 3D fluid-to-fluid heat exchangers"* (2021)

Fluid applications

Several industrial systems such as multiphase heat exchangers involve complex fluid systems with numerous fins and pipes.

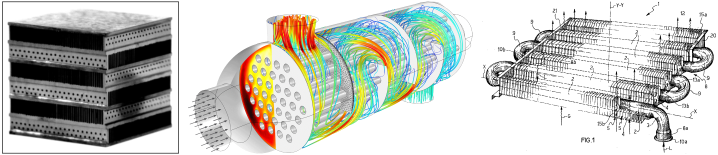


Figure: Figures from [2][3][4].

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[2] Material Innovation Inc., *Composite Heat Exchangers* (2009)

[3] Multiphysics, “” (1994)

[4] Barry, Gregory, and Abuaf, *Turbine blade with enhanced cooling and profile optimization* (1999)

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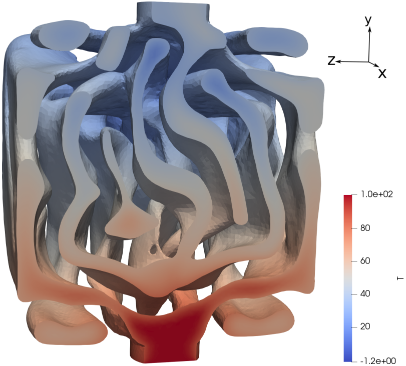

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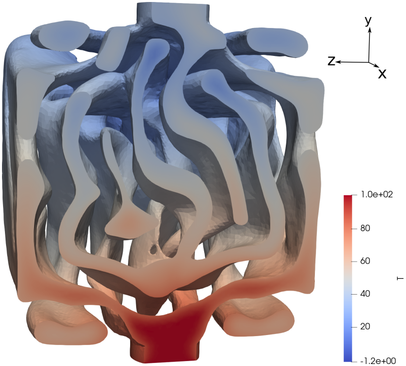

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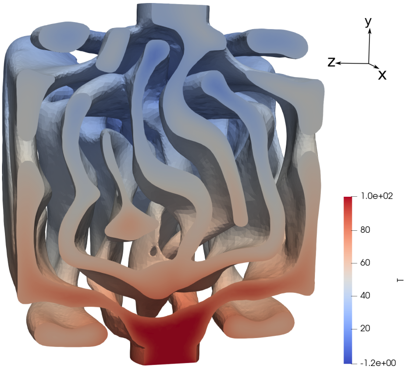
The heterogeneity of $\partial\Omega$ lies in the boundary condition.
The homogenization theory is different.



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Periodic setting considered

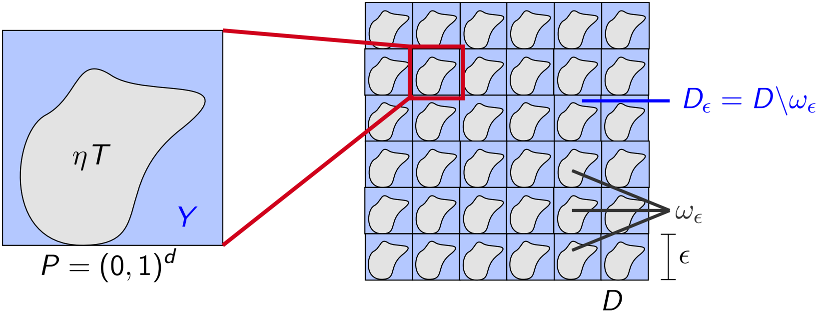

Figure: The perforated domain D_ϵ and the unit cell Y. $\Omega =$"the blue domain".

[5] Allaire, *"Homogenization of the Stokes flow in a connected porous medium"* (1989)

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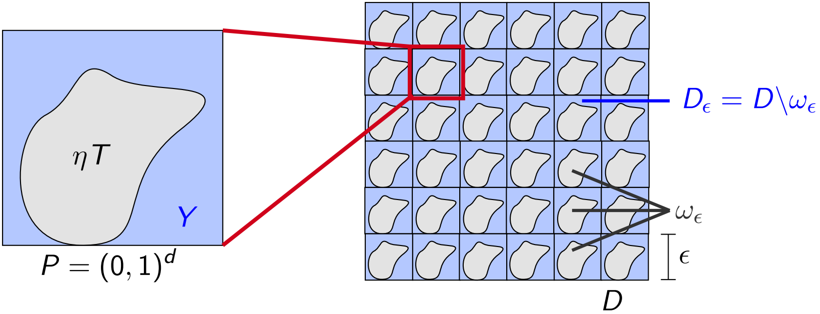

Figure: The perforated domain D_ϵ and the unit cell Y. $\Omega =$"the blue domain".

Depending on how η scales with ϵ, there are three known homogenized models[5].

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The three homogenized regimes

$$\begin{cases} -\Delta \boldsymbol{u}_\epsilon + \nabla p_\epsilon = \boldsymbol{f} \text{ in } D_\epsilon \\ \operatorname{div}(\boldsymbol{u}_\epsilon) = 0 \\ \boldsymbol{u}_\epsilon = 0 \text{ on } \partial\omega_\epsilon \\ \boldsymbol{u}_\epsilon \text{ is } D\text{–periodic,} \end{cases}$$

Let $a_\epsilon := \eta\epsilon$ the size of the holes ω_ϵ.
Let $\sigma_\epsilon = \epsilon^{d/(d-2)}$ (if $d \geqslant 3$).

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- if $a_\epsilon = o(\sigma_\epsilon)$, then $(\boldsymbol{u}_\epsilon, p_\epsilon) \to (\boldsymbol{u}, p)$ with

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}) = 0. \end{cases}$$

This is the "Stokes" regime.

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- if $a_\epsilon = \sigma_\epsilon$, then $(\boldsymbol{u}_\epsilon, p_\epsilon) \to (\boldsymbol{u}, p)$ with $(\boldsymbol{u}, p)$ solving the Brinkman 's equation

$$\begin{cases} -\Delta \boldsymbol{u} + F\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}) = 0. \end{cases}$$

where $F \equiv (F_{ij})_{1 \leq i,j \leq d}$ is a $d \times d$ symmetric positive matrix.

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Let $a_\epsilon := \eta\epsilon$ the size of the holes ω_ϵ.
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- if $a_\epsilon >> \sigma_\epsilon$ while $\epsilon \to 0$ then $(a_\epsilon^{d-2}\epsilon^{-d}\boldsymbol{u}_\epsilon, p_\epsilon) \to (\boldsymbol{u}, p)$ where

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$$\begin{cases} F\boldsymbol{u} + \nabla p = \boldsymbol{f} \text{ in } D \\ \mathrm{div}(\boldsymbol{u}) = 0 \text{ in } D \end{cases}$$

This rewrites as the Darcy 's law

$$\boldsymbol{u}_\epsilon \simeq \epsilon^d a_\epsilon^{2-d} F^{-1}(\boldsymbol{f} - \nabla p), \text{ with } \mathrm{div}(\boldsymbol{u}_\epsilon) = 0.$$

The three homogenized regimes

Not clear how to use this for inverse homogenization

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Can we derive a unified effective model which could encompass all three regimes?

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Higher order homogenized models for the Stokes problem

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- Feppon and Jing, *"High order homogenized Stokes models capture all three regimes"* (2021)

Outline of the results for the Stokes system

- We derive high order homogenized equations for the periodic Stokes problem with fixed $\eta > 0$:

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- We derive first a formal, "infinite-order" homogenized equation:

$$\begin{cases} \sum_{k=0}^{+\infty} \epsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{u}_\epsilon^* + \nabla p_\epsilon^* = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}_\epsilon^*) = 0 \text{ in } D. \end{cases}$$

where $(M^k \cdot \nabla^k \boldsymbol{u}_\epsilon^*)_l = M^k_{i_1 \dots i_k, lm} \partial^k_{i_1 \dots i_k} u^*_{\epsilon,m}$.

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- We have formally

$$\boldsymbol{u}_\epsilon(x) = \sum_{k=0}^{+\infty} \epsilon^k N^k(x/\epsilon) \cdot \nabla^k \boldsymbol{u}_\epsilon^*(x), \; p_\epsilon(x) = p_\epsilon^*(x) + \sum_{k=0}^{+\infty} \epsilon^{k-1} \boldsymbol{\beta}^k(x/\epsilon) \cdot \nabla^k \boldsymbol{u}_\epsilon^*(x).$$

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- We propose a truncation procedure to obtain well-posed homogenized model of order $2K+2$ for any $K \in \mathbb{N}$:

$$\begin{cases} \displaystyle\sum_{k=0}^{2K+2} \epsilon^{k-2} \mathbb{D}_K^k \cdot \nabla^k \boldsymbol{v}_K^* + \nabla p_K^* = \boldsymbol{f} \text{ in } D, \\ \operatorname{div}(\boldsymbol{v}_K^*) = 0 \text{ in } D, \\ \boldsymbol{v}_K^* \text{ is } D\text{–periodic.} \end{cases}$$

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- The first half coefficients coincide: $\mathbb{D}_K^k = M^k$ for $0 \leq k \leq K$.
- We have the following error bounds (recall $\boldsymbol{u}_\epsilon = O(\epsilon^2)$, $p_\epsilon = O(1)$):

$$\left\| \boldsymbol{u}_\epsilon - \sum_{k=0}^{K} \epsilon^k N^k(\cdot/\epsilon) \cdot \nabla^k \boldsymbol{v}_K^* \right\|_{L^2(D,\mathbb{R}^d)} \leq C_K(\boldsymbol{f}) \epsilon^{K+3}$$

$$\left\| p_\epsilon - \left(p_K^* + \sum_{k=0}^{K-1} \epsilon^{k-1} \beta^k(\cdot/\epsilon) \cdot \nabla^k \boldsymbol{v}_K^* \right) \right\|_{L^2(D)} \leq C_K(\boldsymbol{f}) \epsilon^{K+1}.$$

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- Finally, we compute asymptotics for the coefficients M^k in the regime $\eta \to 0$.

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- We find:

$$\begin{cases} M^0 \sim \eta^{d-2} F \\ M^1 = o(\eta^{d-2}) \\ M^2 = -I + o(\eta^{d-2}) \\ M^k = o(\eta^{d-2}) \text{ for any } k > 2 \end{cases}$$

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$$\begin{cases} M^0 \sim \eta^{d-2} F \\ M^1 = o(\eta^{d-2}) \\ M^2 = -I + o(\eta^{d-2}) \\ M^k = o(\eta^{d-2}) \text{ for any } k > 2 \end{cases} \Longrightarrow \begin{cases} \epsilon^{-2} M^0 \sim (a_\epsilon/\sigma_\epsilon)^{d-2} F \\ \epsilon^{-1} M^1 = o(\epsilon (a_\epsilon/\sigma_\epsilon)^{d-2}) \\ \epsilon^0 M^2 \to -I \end{cases}$$

- Since $\epsilon^{k-2}\eta^{d-2} = \epsilon^k (a_\epsilon/\sigma_\epsilon)^{d-2}$, the high-order model converges coefficient-wise to either of the Stokes, Brinkman or Darcy equation as $\eta \to 0$.

Outline of the results for the Stokes system

- Finally, we compute asymptotics for the coefficients M^k in the regime $\eta \to 0$.

$$\begin{cases} \displaystyle\sum_{k=0}^{+\infty} \epsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{u}_\epsilon^* + \nabla p_\epsilon^* = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}_\epsilon^*) = 0 \text{ in } D. \end{cases}$$

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- Since $\epsilon^{k-2}\eta^{d-2} = \epsilon^k (a_\epsilon/\sigma_\epsilon)^{d-2}$, the high-order model converges coefficient-wise to either of the Stokes, Brinkman or Darcy equation as $\eta \to 0$.

Symmetry properties

Remark: the higher order models contain odd orders differential operators (e.g. $\epsilon^{-1} M^1 \cdot \nabla$):

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However the *very strange* terms $\epsilon^{2k-1} M^{2k+1} \cdot \nabla^{2k+1}$ vanish if the unit cell Y has enough symmetries.

Outline

1. Motivations from topology optimization
2. Overview of our results
3. Higher order homogenized models for the Stokes problem: summary of the derivation
4. Higher order models capture all three regimes: low volume fraction asymptotics.

Two-scale expansions

- The starting point is to postulate two-scale expansions for $\boldsymbol{u}_\epsilon$ and p_ϵ:

$$\boldsymbol{u}_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^{i+2} \boldsymbol{u}_i(x, x/\epsilon), \quad p_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^{i} (p_i^*(x) + \epsilon p_i(x, x/\epsilon)),$$

  where $\boldsymbol{u}_i(x, y)$ and $p_i(x, y)$ are P-periodic in y, with $\int_Y p_i(x, y)\mathrm{d}y = 0$.

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- We seek homogenized equations for the averaged variables

$$\boldsymbol{u}_\epsilon^*(x) := \sum_{i=0}^{+\infty} \epsilon^{i+2} \int_Y \boldsymbol{u}_i(x, y) \mathrm{d}y, \quad p_\epsilon^*(x) := \sum_{i=0}^{+\infty} \epsilon^i p_i^*(x)$$

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- We insert the ansatz for $\boldsymbol{u}_\epsilon$ and p_ϵ in the Stokes equation,

$$\begin{cases} -\Delta \boldsymbol{u}_\epsilon + \nabla p_\epsilon = \boldsymbol{f} \text{ in } D_\epsilon \\ \mathrm{div}(\boldsymbol{u}_\epsilon) = 0 \text{ in } D_\epsilon \end{cases} \text{with } \begin{cases} \boldsymbol{u}_\epsilon = 0 \text{ on } \partial\omega_\epsilon \\ \boldsymbol{u}_\epsilon \text{ is } D\text{–periodic,} \end{cases}$$

Two-scale expansions

- We find

$$\boldsymbol{u}_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^{i+2} \mathcal{X}^i(x/\epsilon) \cdot \nabla^i(\boldsymbol{f}(x) - \nabla p_\epsilon^*(x)),$$

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$$\mathrm{div}(\boldsymbol{u}_\epsilon^*(x)) = 0 \text{ where } \boldsymbol{u}_\epsilon^*(x) = \sum_{i=0}^{+\infty} \epsilon^{i+2} \mathcal{X}^{i*} \cdot \nabla^i(\boldsymbol{f}(x) - \nabla p_\epsilon^*(x)).$$

and

$$\mathcal{X}^{i*} = \int_Y \mathcal{X}^i(y) \mathrm{d}y.$$

Two-scale expansions

The tensors $(\mathcal{X}^k(y), \boldsymbol{\alpha}^k(y))$ are defined by

$$\mathcal{X}^k(y) := \begin{bmatrix} \boldsymbol{\mathcal{X}}_1^k(y) & \dots & \boldsymbol{\mathcal{X}}_d^k(y) \end{bmatrix}$$

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$$\begin{cases} -\Delta_{yy}\boldsymbol{\mathcal{X}}_j^0 + \nabla_y \alpha_j^0 = \boldsymbol{e}_j \text{ in } Y, \\ \qquad \mathrm{div}_y(\boldsymbol{\mathcal{X}}_j^0) = 0 \text{ in } Y \end{cases}$$

$$\begin{cases} -\Delta_{yy}\boldsymbol{\mathcal{X}}_j^1 + \nabla_y \alpha_j^1 = (2\partial_l \boldsymbol{\mathcal{X}}_j^0 - \alpha_j^0 \boldsymbol{e}_l) \otimes e_l \text{ in } Y \\ \qquad \mathrm{div}_y(\boldsymbol{\mathcal{X}}_j^1) = -(\boldsymbol{\mathcal{X}}_j^0 - \langle \boldsymbol{\mathcal{X}}_j^0 \rangle) \cdot \boldsymbol{e}_l \otimes e_l \text{ in } Y, \end{cases}$$

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The infinite order homogenized equation

We have obtained

$$\boldsymbol{u}_\epsilon(x) = \sum_{i=0}^{+\infty} \epsilon^{i+2} \mathcal{X}^i(x/\epsilon) \cdot \nabla^i(\boldsymbol{f}(x) - \nabla p_\epsilon^*(x)),$$

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Introducing a family of tensors $(M^k)_{k \in \mathbb{N}}$ such that

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we obtain the infinite order homogenized equation:

$$\sum_{k=0}^{+\infty} \epsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{u}_\epsilon^* = \boldsymbol{f} - \nabla p_\epsilon^*.$$

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$$\begin{cases} M^0 = (\mathcal{X}^{0*})^{-1} \\ M^k = -(\mathcal{X}^{0*})^{-1} \displaystyle\sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^p \end{cases}$$

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$$\sum_{k=0}^{+\infty} \epsilon^k N^k(x/\epsilon) \cdot \nabla^k := \left(\sum_{k=0}^{+\infty} \epsilon^{k+2} \mathcal{X}^k(x/\epsilon) \cdot \nabla^k\right) \left(\sum_{k=0}^{+\infty} \epsilon^{k-2} M^k \cdot \nabla^k\right)$$

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. . . we find the "criminal" ansatz

$$\begin{cases} \boldsymbol{u}_\epsilon(x) = \displaystyle\sum_{i=0}^{+\infty} \epsilon^i N^i(x/\epsilon) \cdot \nabla^i \boldsymbol{u}^*_\epsilon(x) \\ p_\epsilon(x) = p^*_\epsilon(x) + \displaystyle\sum_{i=0}^{+\infty} \epsilon^{i-1} \boldsymbol{\beta}^i(x/\epsilon) \cdot \nabla^i \boldsymbol{u}^*_\epsilon(x), \end{cases}.$$

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We now want a "practical" equation for computing $(\boldsymbol{u}_\epsilon^*, p_\epsilon^*)$.

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yields in general an ill-posed problem.

We truncate the criminal ansatz and construct a homogenized equation from a minimization principle.

Truncating the infinite order homogenized equation

Recall $\boldsymbol{u}_\epsilon$ is the minimizer of

$$\min_{\boldsymbol{w}\in H^1(D_\epsilon,\mathbb{R}^d)} \qquad J(\boldsymbol{w},\boldsymbol{f}) := \int_D \left(\frac{1}{2}\nabla\boldsymbol{w} : \nabla\boldsymbol{w} - \boldsymbol{f}\cdot\boldsymbol{w}\right)\mathrm{d}y$$

$$\text{s.t.}\begin{cases} \mathrm{div}(\boldsymbol{w}) = 0 \text{ in } D_\epsilon \\ \boldsymbol{w} = 0 \text{ on } \partial\omega_\epsilon \\ \boldsymbol{w} \text{ is } D\text{–periodic.} \end{cases}$$

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For $K \in \mathbb{N}$ and $\boldsymbol{v} \in H^{K+1}(D,\mathbb{R}^d)$, let $\boldsymbol{w}_{\epsilon,K}(v)$ be the truncated ansatz

$$\boldsymbol{w}_{\epsilon,K}(\boldsymbol{v})(x) := \sum_{k=0}^{K} \epsilon^k N^k(x/\epsilon)\cdot\nabla^k\boldsymbol{v}(x), \quad x \in D_\epsilon,$$

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We consider the approximate minimization problem

$$\min_{\boldsymbol{v}\in H^{K+1}(D,\mathbb{R}^d)} J(\boldsymbol{w}_{\epsilon,K}(\boldsymbol{v}),\boldsymbol{f}) \text{ s.t. } \begin{cases} \mathrm{div}(\boldsymbol{v}) = 0 \\ \boldsymbol{v} \text{ is } D\text{–periodic.} \end{cases}$$

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Truncating the infinite order homogenized equation

We consider the approximate minimization problem

$$\min_{\boldsymbol{v}\in H^{K+1}(D,\mathbb{R}^d)} J_K^*(\boldsymbol{v},\boldsymbol{f}) \qquad \text{s.t.} \quad \begin{cases} \operatorname{div}(\boldsymbol{v}) = 0 \\ \qquad \boldsymbol{v} \text{ is } D\text{–periodic.} \end{cases}$$

after averaging with respect to x/ϵ.

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The first order optimality condition for J_K^* yields a well-posed homogenized equation of order $2K+2$:

$$\begin{cases} \displaystyle\sum_{k=0}^{2K+2} \epsilon^{k-2}\mathbb{D}_K^k \cdot \nabla^k \boldsymbol{v}_K^* + \nabla p_K^* = \boldsymbol{f} \text{ in } D, \\ \operatorname{div}(\boldsymbol{v}_K^*) = 0 \text{ in } D, \\ \boldsymbol{v}_K^* \text{ is } D\text{–periodic.} \end{cases}$$

Truncating the infinite order homogenized equation

Infinite-order equation:

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Because it turns out $\mathbb{D}_K^k = M^k$ for $0 \le k \le K$, one can prove the error bound

$$\left\| \boldsymbol{u}_\epsilon - \sum_{k=0}^{K} \epsilon^k N^k(\cdot/\epsilon) \cdot \nabla^k \boldsymbol{v}_K^* \right\|_{L^2(D,\mathbb{R}^d)} \le C_K(\boldsymbol{f}) \epsilon^{K+3}$$

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$$\left\| p_\epsilon - \left(p_K^* + \sum_{k=0}^{K-1} \epsilon^{k-1} \boldsymbol{\beta}^k(\cdot/\epsilon) \cdot \nabla^k \boldsymbol{v}_K^* \right) \right\|_{L^2(D)} \le C_K(\boldsymbol{f}) \epsilon^{K+1}.$$

Outline

1. Motivations from topology optimization
2. Overview of our results
3. Higher order homogenized models for the Stokes problem: summary of the derivation
4. Higher order models capture all three regimes: low volume fraction asymptotics.

Periodic setting considered

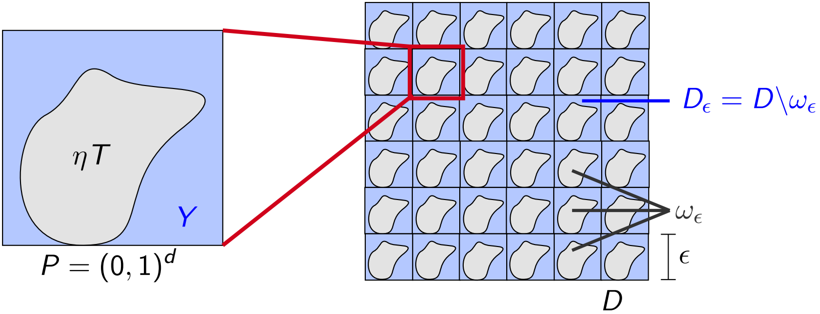

Figure: The perforated domain D_ϵ and the unit cell Y. $\Omega =$"the blue domain".

Depending on how η scales with ϵ, there are three known homogenized models[5].

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[5] Allaire, *"Homogenization of the Stokes flow in a connected porous medium"* (1989)

Low volume fraction asymptotics

Infinite-order equation:

$$\begin{cases} \displaystyle\sum_{k=0}^{+\infty} \epsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{u}_\epsilon^* + \nabla p_\epsilon^* = \boldsymbol{f} \text{ in } D \\ \text{div}(\boldsymbol{u}_\epsilon^*) = 0 \text{ in } D. \end{cases}$$

Well-posed homogenized equation of order $2K+2$:

$$\begin{cases} \displaystyle\sum_{k=0}^{2K+2} \epsilon^{k-2} \mathbb{D}_K^k \cdot \nabla^k \boldsymbol{v}_K^* + \nabla p_K^* = \boldsymbol{f} \text{ in } D, \\ \text{div}(\boldsymbol{v}_K^*) = 0 \text{ in } D, \\ \boldsymbol{v}_K^* \text{ is } D\text{–periodic.} \end{cases}$$

Low volume fraction asymptotics

Infinite-order equation:

$$\begin{cases} \displaystyle\sum_{k=0}^{+\infty} \epsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{u}_\epsilon^* + \nabla p_\epsilon^* = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}_\epsilon^*) = 0 \text{ in } D. \end{cases}$$

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We study the behavior of the coefficients M^k and $\mathbb{D}_K^k$ as $\eta \to 0$.

Low volume fraction asymptotics

$$\begin{cases} M^0 = (\mathcal{X}^{0*})^{-1} \\ M^k = -(\mathcal{X}^{0*})^{-1} \displaystyle\sum_{p=0}^{k-1} \mathcal{X}^{k-p*} \otimes M^p \end{cases}$$

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We estimate $\langle \boldsymbol{\mathcal{X}}_j^k \rangle$ as $\eta \to 0$.

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We estimate $\langle \boldsymbol{\mathcal{X}}_j^k \rangle$ as $\eta \to 0$.
Since $|\langle \boldsymbol{\mathcal{X}}_j^k \rangle| \leq C\eta^{1-d/2} ||\nabla \boldsymbol{\mathcal{X}}_j^k||_{L^2(Y,\mathbb{R}^{d\times d})}$, we need to estimate $||\nabla \boldsymbol{\mathcal{X}}_j^k||_{L^2(Y,\mathbb{R}^{d\times d})}$.

Energy inequality in the perforated cell

Lemma

Consider $\boldsymbol{h}$ and g satisfying $\int_{P\setminus(\eta T)} g \mathrm{d}x = 0$. Let $(\boldsymbol{v}, \phi)$ be the unique solution to the following Stokes system:

$$\begin{cases} -\Delta \boldsymbol{v} + \nabla\phi = \boldsymbol{h} \ \textit{in } P \setminus (\eta T) \\ \mathrm{div}(\boldsymbol{v}) = g \ \textit{in } P \setminus (\eta T) \\ \boldsymbol{v} = 0 \ \textit{on } \partial(\eta T) \\ \boldsymbol{v} \ \textit{is } P\textit{–periodic.} \end{cases}$$

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There exists a constant $C > 0$ *independent of* $(\boldsymbol{v}, \phi)$*,* η*,* $\boldsymbol{h}$ *and* g *such that*

$$||\nabla \boldsymbol{v}||_{L^2(P\setminus(\eta T),\mathbb{R}^{d\times d})} + ||\phi||_{L^2(P\setminus(\eta T))} \leq C(||\boldsymbol{h} - \langle \boldsymbol{h}\rangle||_{L^2(P\setminus(\eta T),\mathbb{R}^d)} + \eta^{1-d/2}|\langle \boldsymbol{h}\rangle| + ||g||_{L^2(P\setminus(\eta T))}).$$

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$$||\nabla \boldsymbol{v}||_{L^2(P\setminus(\eta T),\mathbb{R}^{d\times d})} + ||\phi||_{L^2(P\setminus(\eta T))} \\ \leq C(||\boldsymbol{h} - \langle \boldsymbol{h}\rangle||_{L^2(P\setminus(\eta T),\mathbb{R}^d)} + \eta^{1-d/2}|\langle \boldsymbol{h}\rangle| + ||g||_{L^2(P\setminus(\eta T))}).$$

$\nabla \boldsymbol{v}$ grows as $\eta \to 0$ only if $|\langle \boldsymbol{h}\rangle| \neq 0$.

A modified cascade of equations

$$\begin{cases} -\Delta_{yy}\boldsymbol{\mathcal{X}}_j^{k+2} + \nabla_y \alpha_j^{k+2} = (2\partial_l \boldsymbol{\mathcal{X}}_j^{k+1} - \alpha_j^{k+1}\boldsymbol{e}_l) \otimes e_l + \boldsymbol{\mathcal{X}}_j^k \otimes I \text{ in } Y \\ \mathrm{div}_y(\boldsymbol{\mathcal{X}}_j^{k+2}) = -(\boldsymbol{\mathcal{X}}_j^{k+1} - \langle \boldsymbol{\mathcal{X}}_j^{k+1} \rangle) \cdot \boldsymbol{e}_l \otimes e_l \text{ in } Y \\ \boldsymbol{\mathcal{X}}_j^{k+2} = 0 \text{ on } \partial(\eta T) \end{cases}$$

$||\nabla \boldsymbol{\mathcal{X}}_j^{k+2}||$ grows because of $\langle \boldsymbol{\mathcal{X}}_j^k \rangle \otimes I \neq 0$.

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$||\nabla \boldsymbol{\mathcal{X}}_j^{k+2}||$ grows because of $\langle \boldsymbol{\mathcal{X}}_j^k \rangle \otimes I \neq 0$.
We define a modified cascade of equations with zero-mean right hand sides:

$$\begin{cases} -\Delta \boldsymbol{\mathcal{Y}}_j^{k+2} + \nabla \omega_j^{k+2} = (2\partial_l \boldsymbol{\mathcal{Y}}_j^{k+1} - \omega_j^{k+1}\boldsymbol{e}_l) \otimes e_l + (\boldsymbol{\mathcal{Y}}_j^k - \langle \boldsymbol{\mathcal{Y}}_j^k \rangle) \otimes I, \text{ in } Y \\ \mathrm{div}(\boldsymbol{\mathcal{Y}}_j^{k+2}) = -(\boldsymbol{\mathcal{Y}}_j^{k+1} - \langle \boldsymbol{\mathcal{Y}}_j^{k+1} \rangle) \cdot \boldsymbol{e}_l \otimes e_l \text{ in } Y, \end{cases}$$

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Thanks to the previous lemma,

$$||\nabla \boldsymbol{\mathcal{Y}}_j^k||_{L^2(P \setminus (\eta T), \mathbb{R}^{d \times d})} \leq C_k \eta^{1-d/2} \text{ for any } k \in \mathbb{N}$$

$$\langle \boldsymbol{\mathcal{Y}}_j^k \rangle \leq C_k \eta^{2-d} \text{ for any } k \in \mathbb{N}$$

Recursive formula for the coefficients M^k

Knowing this, we can prove with weak convergence techniques

$$\mathcal{Y}^{0*} \sim \eta^{2-d} F^{-1} \text{ and } \mathcal{Y}^{k*} = o(\eta^{2-d}) \text{ for } k \geqslant 1.$$

where F is a matrix tensor which can be explicited.

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$$\mathcal{Y}^k(y) = \mathcal{X}^k(y) - \sum_{l=0}^{k-2} \mathcal{Y}^l(y) \otimes \mathcal{X}^{k-l-2*} \otimes I.$$

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$$\Longrightarrow \mathcal{X}^{k*} = \mathcal{Y}^{k*} + \sum_{l=0}^{k-2} \mathcal{Y}^{l*} \otimes \mathcal{X}^{k-l-2*} \otimes I$$

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$$\implies \mathcal{X}^{k*} = \mathcal{Y}^{k*} + \underbrace{\sum_{l=0}^{k-2} \mathcal{Y}^{l*} \otimes \mathcal{X}^{k-l-2*} \otimes I}_{\text{Cauchy product !}}$$

Recursive formula for the coefficients M^k

$$\mathcal{X}^{k*} = \underbrace{\sum_{l=0}^{k-2} \mathcal{Y}^{l*} \otimes \mathcal{X}^{k-l-2*} \otimes I}_{\text{Cauchy product !}}$$

Since

$$\sum_{p=0}^{k} \mathcal{X}^{k-p*} \otimes M^p = \begin{cases} I, & \text{if } k = 0, \\ 0, & \text{if } k \geq 1. \end{cases}$$

we obtain

$$\sum_{p=0}^{k} \mathcal{Y}^{p*} \otimes M^{k-p} = -\mathcal{Y}^{k-2*} \otimes I \text{ for any } k \geqslant 2.$$

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we obtain

$$\sum_{p=0}^{k} \mathcal{Y}^{p*} \otimes M^{k-p} = -\mathcal{Y}^{k-2*} \otimes I \text{ for any } k \geqslant 2.$$

which is equivalent to

$$\mathcal{Y}^{0*} \otimes M^k + \cdots + \mathcal{Y}^{k-3*} \otimes M^3 + \mathcal{Y}^{k-2*} \otimes (M^2 + I) + \mathcal{Y}^{k-1*} \otimes M^1 + \mathcal{Y}^{0*} \otimes M^0 = 0.$$

for any $k \geqslant 0$.

Recursive formula for the coefficients M^k

$$\mathcal{Y}^{0*} \otimes M^k + \cdots + \mathcal{Y}^{k-3*} \otimes M^3 + \mathcal{Y}^{k-2-*} \otimes (M^2 + I) + \mathcal{Y}^{k-1*} \otimes M^1 + \mathcal{Y}^{0*} \otimes M^0 = 0.$$

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$$\mathcal{Y}^{0*} \sim \eta^{2-d} F^{-1} \text{ and } \mathcal{Y}^{k*} = o(\eta^{2-d}) \text{ for } k \geqslant 1.$$

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for any $k\geqslant 0$.

$$\mathcal{Y}^{0*}\sim\eta^{2-d}F^{-1} \text{ and } \mathcal{Y}^{k*}=o(\eta^{2-d}) \text{ for } k\geqslant 1.$$

Combining the two equations, we find

$$\begin{cases} M^0\sim\eta^{d-2}F \\ M^1=o(\eta^{d-2}) \\ M^2=-I+o(\eta^{d-2}) \\ M^k=o(\eta^{d-2}) \text{ for any } k>2 \end{cases}$$

as claimed.

Recursive formula for the coefficients M^k

- Finally, we compute asymptotics for the coefficients M^k in the regime $\eta \to 0$.

$$\begin{cases} \displaystyle\sum_{k=0}^{+\infty} \epsilon^{k-2} M^k \cdot \nabla^k \boldsymbol{u}_\epsilon^* + \nabla p_\epsilon^* = \boldsymbol{f} \text{ in } D \\ \operatorname{div}(\boldsymbol{u}_\epsilon^*) = 0 \text{ in } D. \end{cases}$$

- We find:

$$\begin{cases} M^0 \sim \eta^{d-2} F \\ M^1 = o(\eta^{d-2}) \\ M^2 = -I + o(\eta^{d-2}) \\ M^k = o(\eta^{d-2}) \text{ for any } k > 2 \end{cases}$$

- Since $\epsilon^{k-2}\eta^{d-2} = \epsilon^k (a_\epsilon/\sigma_\epsilon)^{d-2}$, the high-order model converges coefficient-wise to either of the Stokes, Brinkman or Darcy equation as $\eta \to 0$.

Thank you for your attention!