Analysis of a Monte-Carlo Nystrom method and well-posedness of the Foldy-Lax approximation

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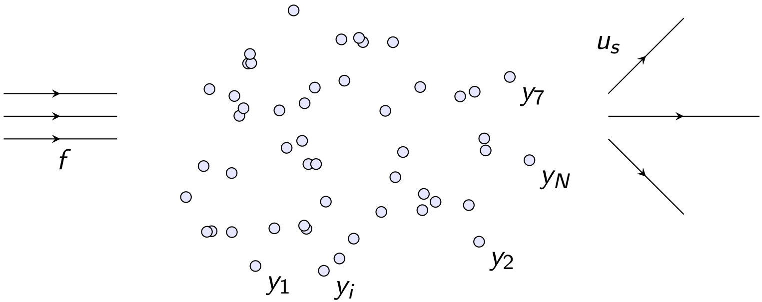
Seminar for Applied Mathematics

ETH zürich

Motivation: the Foldy-Lax approximation

Acoustic scattering of an incident field f through N obstacles $(B_i)_{1\leq i\leq N}$ located at $(y_i)_{1\leq i\leq N}$:



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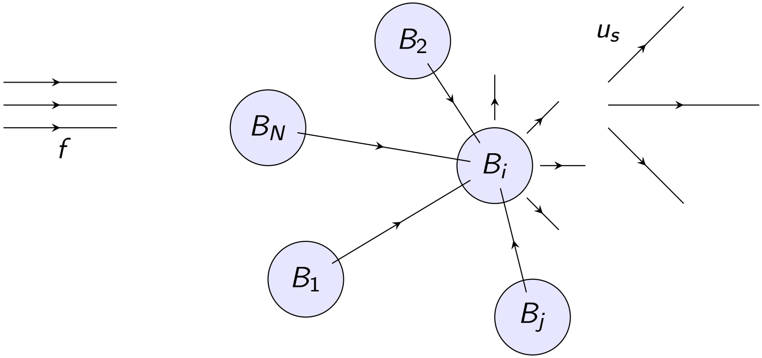

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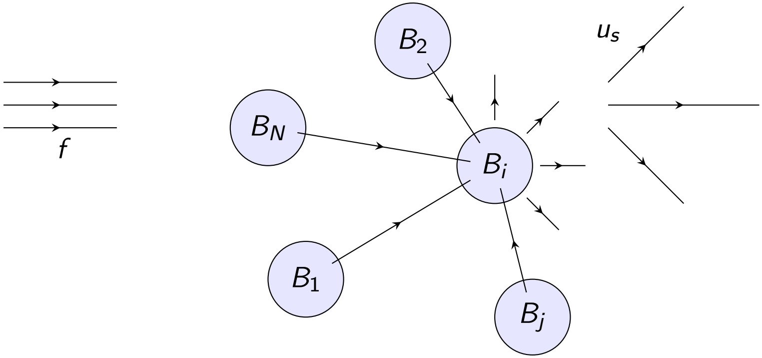

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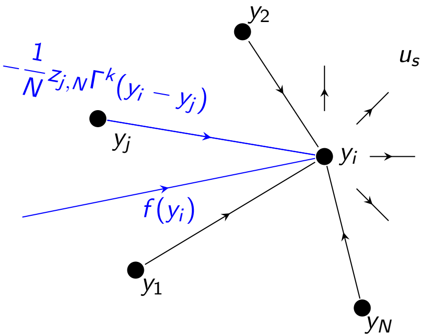

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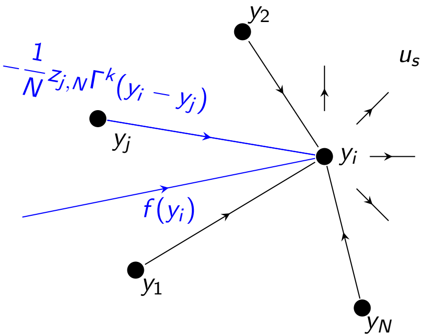

The Foldy-Lax approximation:

1. The scattered field can be approximated by the contribution of N point-sources located at the centers $(y_i)_{1 \leq i \leq N}$:

$$u_s(y) \simeq -\frac{1}{N} \sum_{i=1}^{N} z_{i,N} \Gamma^k(y - y_i)$$

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$\Gamma^k(y)$ is e.g. the (outgoing) fundamental solution to the Helmholtz equation:

$$(\Delta + k^2)\Gamma^k = \delta_0 \text{ in } \mathbb{R}^d,$$

$$\Gamma^k(y) = \begin{cases} -\frac{i}{4} H_0^{(1)}(k|y|) \text{ if } d = 2, \\ -\frac{e^{ik|y|}}{4\pi|y|} \text{ if } d = 3 \end{cases}$$

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$\Gamma^k(\cdot - y)$ is the wave pattern generated by a point source located at y.

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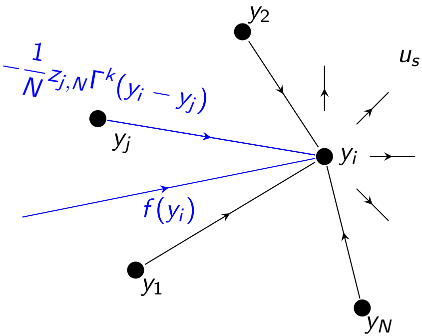

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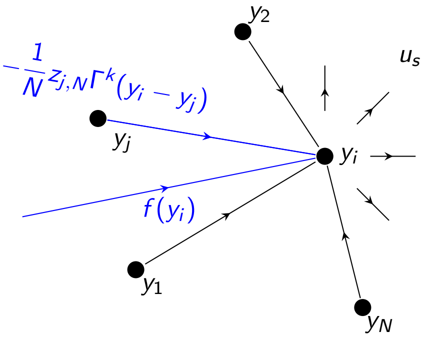

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2. The intensity $z_{i,N}$ of the wave field scattered by the source y_i is the contribution of the field scattered by the other sources $(y_j)_{1\leq j\neq i\leq N}$ and of the incident field $f(y_i)$:

$$z_{i,N} = f(y_i) - \frac{1}{N}\sum_{j\neq i} z_{j,N}\Gamma^k(y_j - y_i).$$

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We obtain the following linear system for the wave field intensity $(z_{i,N})_{1 \leq i \leq N}$:

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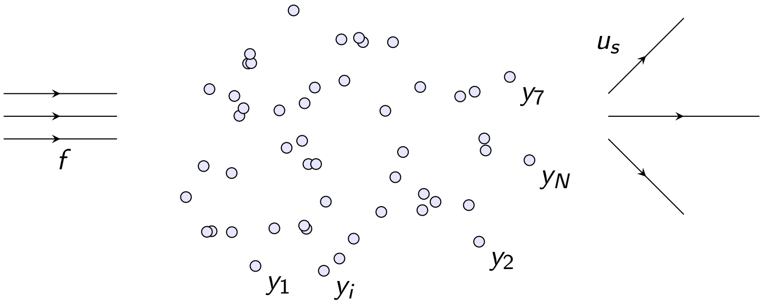

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3. i.e. can we prove a convergence result $z_{i,N} \to z(y_i)$ as $N \to +\infty$?
   In that case (2) is an equation characterizing the effective medium associated to the random point cloud $(y_i)_{1\leq i\leq N}$.

A Monte-Carlo Nystrom method

Replace Γ^k with a general kernel $k(y, y')$:

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If $z_{i,N} \to z(y_i)$ as $N \to +\infty$, then (1) can also be viewed as a Monte-Carlo method for solving (2).

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3. Sketch of the proofs: random operator theory

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(i) $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and

$$\sup_{y' \in \Omega} \int_\Omega |k(y,y')|^2 \mathrm{d}y < +\infty, \ \int_\Omega |f(y)|^2 \mathrm{d}y < +\infty.$$

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(ii) $(y_i)_{1\le i\le N}$ are independent samples of a probability distribution $\rho(y)\mathrm{d}y$ with density $\rho \in L^\infty(\Omega,\mathbb{R}^+)$ (satisfying $\int_\Omega \rho(y)\mathrm{d}y = 1$).

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(iii) The integral equation (2) is well-posed.

Result 1: well-conditioning

$$z_{i,N} + \frac{1}{N}\sum_{j=1}^{N} z_{j,N} k(y_i, y_j) = f(y_i), \quad 1 \leq i \leq N \quad \Longleftrightarrow (\mathrm{I} + A_N) z_N = F \qquad (1)$$

with $z_N = (z_{i,N})_{1\leq i\leq N}$ and $F = (f(y_i))_{1\leq i\leq N}$ and where $(A_N)_{1\leq i,j\leq N}$ is the random matrix defined by

$$A_{N,ij} = \begin{cases} \frac{1}{N} k(y_i, y_j) \text{ if } i \neq j, \\ 0 \text{ if } i = j. \end{cases}$$

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Proposition

Assume (i), (ii) and (iii). Then with probability one, there exists $N_0 \in \mathbb{N}$ such that the matrix $\mathrm{I} + A_N$ is invertible for any $N \geqslant N_0$, and there exists a constant $C > 0$ independent of N such that

$$\forall N \geqslant N_0, |||(\mathrm{I} + A_N)^{-1}|||_2 \leq C$$

where $||| \cdot |||_2$ is the operator norm ($|||A|||_2 := \sup_{||x||_2 = 1} ||Ax||_2$).

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So (1) is well-posed if the continuous problem is well-posed.

Result 2: convergence of the Nystrom interpolant

$$z_{i,N} + \frac{1}{N}\sum_{j=1}^{N} z_{j,N} k(y_i, y_j) = f(y_i), \qquad 1 \le i \le N, \tag{1}$$

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If $k(y,y') = \Gamma^k(y-y')$, then

$$z_N(y) = f(y) - \frac{1}{N}\sum_{i=1}^{N} \Gamma^k(y-y_i) z_{N,i} = f(y) + u_s(y), \qquad y \in \Omega.$$

is the total wave field, and $z_N - f$ is the scattered field.

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2. *z_N converges to z at rate $O(N^{-\frac{1}{2}})$ in a mean-square sense:*

$$\mathbb{E}[||z_N - z||^2_{L^2(\Omega)} | \mathcal{H}_{N_0}]^{\frac{1}{2}} \leq CN^{-\frac{1}{2}}.$$

Result 3: point-wise convergence

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3. Sketch of the proofs: random operator theory

Numerical 1D example

- We consider $k(y,y') := |y - y'|^{-\alpha}$ with $\alpha = 0.4 < 1/2$ on the interval $\Omega = (0,1)$ and the integral equation

$$z(y) + \int_0^1 k(y,y')z(y')\mathrm{d}y' = f(y), \qquad y \in (0,1). \tag{2}$$

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- We solve the M linear systems for $1 \le p \le M$:

$$z_{N,i}^p + \frac{1}{N} \sum_{j \neq i} k(y_i^p, y_j^p) z_{N,j}^p = f(y_i^p), \qquad 1 \le i \le N.$$

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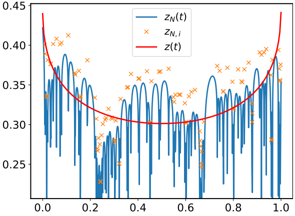
- We solve (2) accurately with a Nystrom method on a regular grid and we estimate the mean-square error:

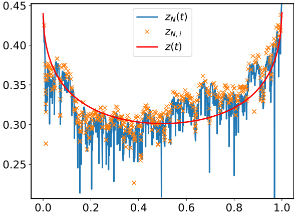
$$\mathtt{MSE} := \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N |z_{N,i} - z(y_i)|^2\right]^{\frac{1}{2}} \simeq \sqrt{\frac{1}{MN}\sum_{p=1}^M\sum_{i=1}^N |z_{N,i}^p - z(y_i^p)|^2}.$$

Numerical 1D example

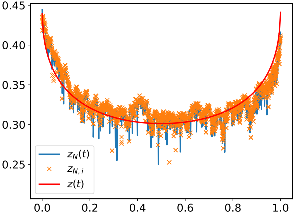
Case 1: $f(y) = 1$



(a) $N = 100$



(b) $N = 500$



(c) $N = 2,000$

Numerical 1D example

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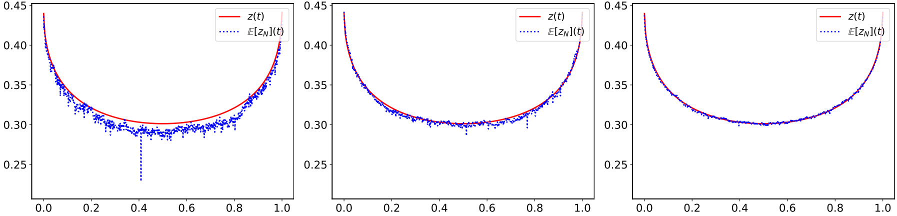

(a) $N = 100$ (b) $N = 500$ (c) $N = 2,000$

Figure: Empirical average of the Nystrom interpolant $\mathbb{E}[z_N]$.

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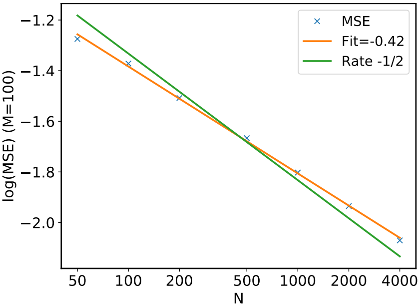
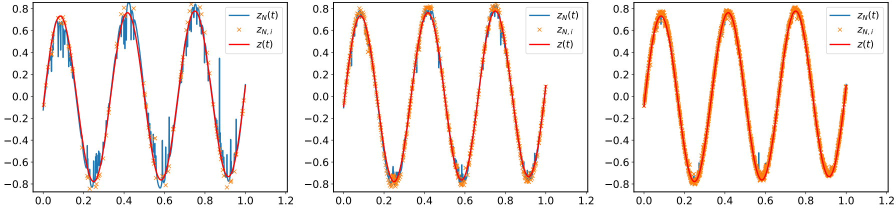


Figure: Mean-square error `MSE`.

Numerical 1D example

Case 2: $f(y) = \sin(6\pi y)$



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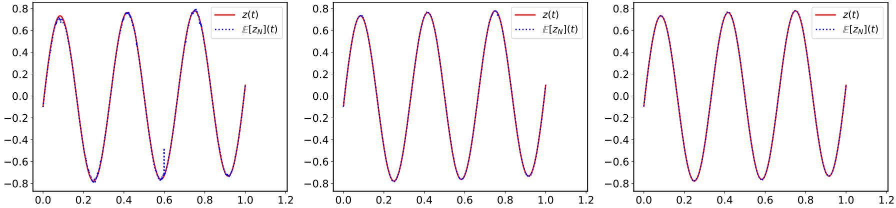

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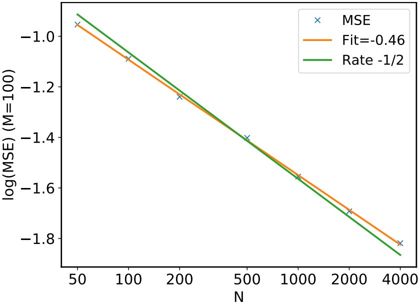

Figure: Mean-square error MSE.

Numerical 2D example

We solve with our Monte-Carlo method the following Lippmann-Schwinger equation:

$$\begin{cases} (\Delta + k^2 n_\Omega) z = 0 \text{ in } \mathbb{R}^2, \\ (\partial_r - ik)(z - u_{in}) = O(|x|^{-2}) \text{ as } r \to +\infty, \end{cases} \tag{3}$$

whose solution z is the scattered field produced by an incident wave u_{in} propagating through a material with refractive index $n_\Omega(x)$ given by

$$n_\Omega(x) = \begin{cases} m \text{ if } x \in \Omega, \\ 1 \text{ if } x \in \mathbb{R}^2 \setminus \Omega, \end{cases}$$

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The integral formulation of (3) is

$$z(y) + (m-1)k^2 \int_\Omega \Gamma^k(y - y') z(y') \mathrm{d}y' = u_{in}(y), \quad y \in \Omega, \tag{2}$$

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---

[1] Aussal and Alouges, *Gypsilab* (2018)

[2] Averseng, *Fast discrete convolution in $\mathbb{R}^2$ with radial kernels using non-uniform fast Fourier transform with nonequispaced frequencies* (2020)

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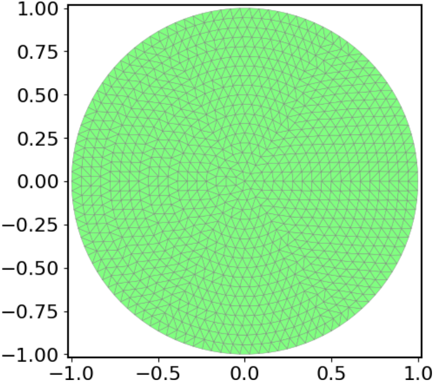
- We solve (2) with the finite-element method[1].
- We solve (1) for $500 \le N \le 40,000$ using the Efficient Bessel Decomposition method[2].

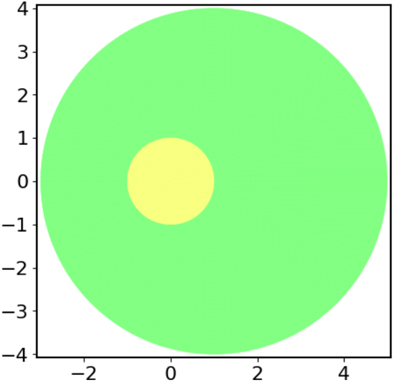
[1] Aussal and Alouges, *Gypsilab* (2018)

[2] Averseng, *Fast discrete convolution in* $\mathbb{R}^2$ *with radial kernels using non-uniform fast Fourier transform with nonequispaced frequencies* (2020)

Numerical 2D example

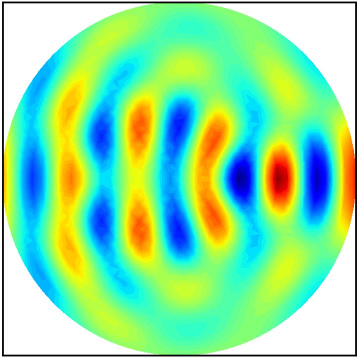


(a) The discretization mesh $\mathcal{T}$ considered for the acoustic obstacle Ω (the unit disk).

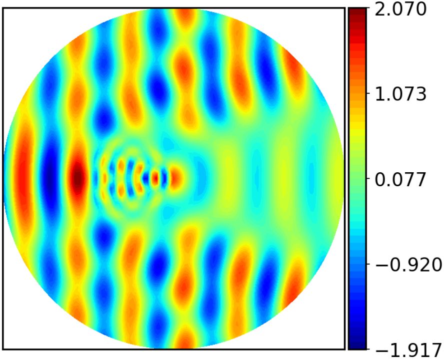


(b) The surrounding disk Ω' (the disk centered at $(1, 0)$ of radius 4, in green) containing the accoustic obstacle Ω (in yellow).

Numerical 2D example



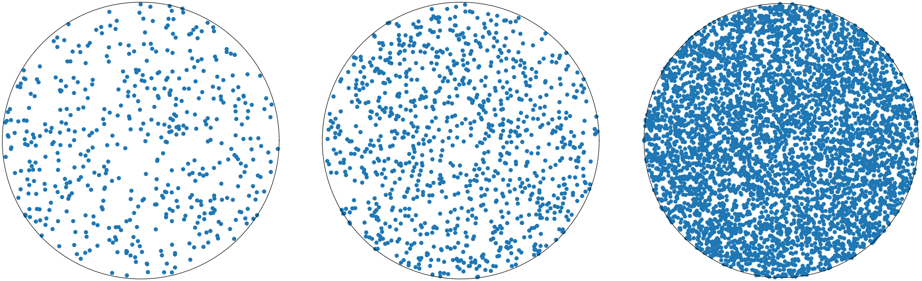
(a) Plot of the solution z in the interior domain Ω.



(b) Plot of the solution z in the exterior domain Ω'.

Thanks Martin Averseng and Ignacio Labarca.

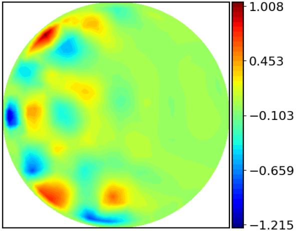
Numerical 2D example



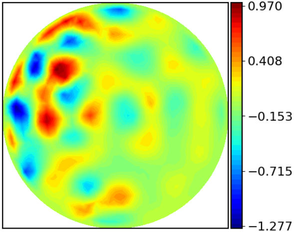
(a) $N = 500$ (b) $N = 1,000$ (c) $N = 5,000$

Figure: Samples of N random points drawn randomly and independently from the uniform distribution in the unit disk.

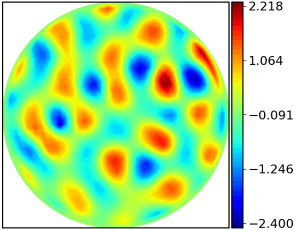
Numerical 2D example



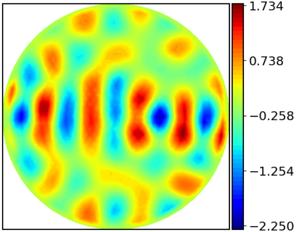
(a) $N = 500$



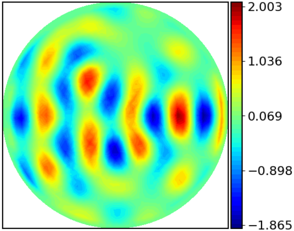
(b) $N = 1,000$



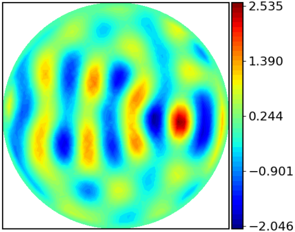
(c) $N = 5,000$



(d) $N = 10,000$



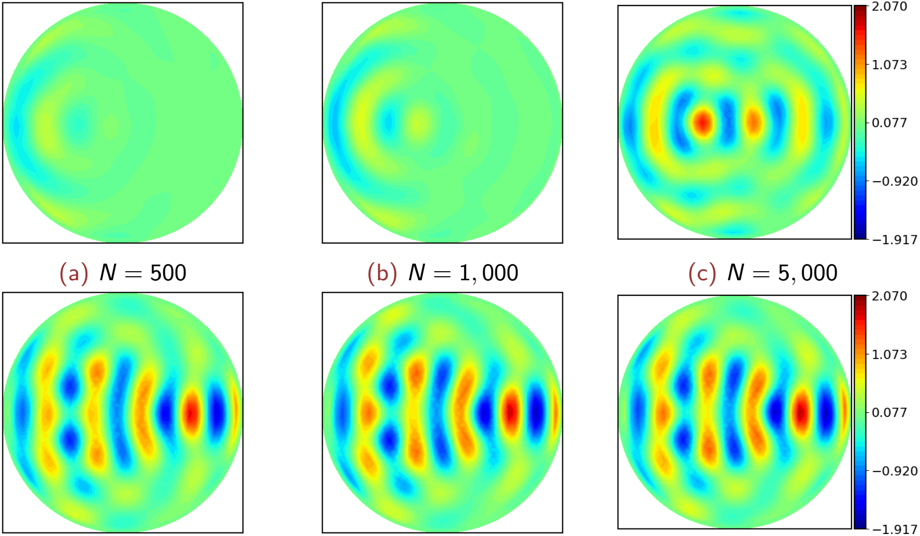
(e) $N = 20,000$



(f) $N = 40,000$

Figure: Monte-Carlo solutions $(z_i^p)_{1 \leq i \leq N}$

Numerical 2D example



(a) $N = 500$ (b) $N = 1,000$ (c) $N = 5,000$

(d) $N = 10,000$ (e) $N = 20,000$ (f) $N = 40,000$

Figure: Averaged field $\mathbb{E}[(\hat{z}_i^p)]$ at the vertices of the mesh $\mathcal{T}$.

Numerical 2D example

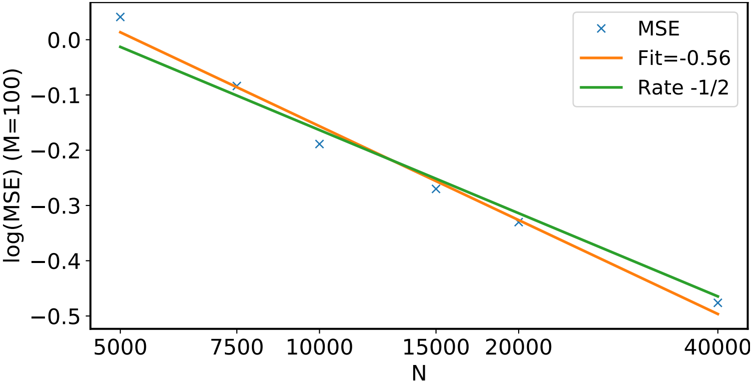


Figure: Mean-square error MSE.

Outline

1. Well-posedness and convergence results
2. Numerical illustration in 1D and 2D
3. Sketch of the proofs: random operator theory

Sketch of the proof

$$z_{i,N} + \frac{1}{N}\sum_{j=1}^{N} z_{j,N} k(y_i, y_j) = f(y_i), \quad 1 \le i \le N, \tag{1}$$

- (1) is equivalent to finding a function $z_N \in L^2(\Omega)$ such that

$$z_N(y) + \frac{1}{N}\sum_{j=1}^{N} k(y, y_j) z_N(y_j) = f(y), \quad \forall y \in \Omega. \tag{a}$$

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- (a) rewrites

$$\left(\mathrm{I} + \frac{1}{N}\sum_{i=1}^{N} A_i\right) z_N = f \text{ with } \begin{array}{rcccl} A_i & : & L^2(\Omega,\mathbb{C}) & \to & L^2(\Omega,\mathbb{C}) \\ & & z & \mapsto & k(\cdot, y_i) z(y_i). \end{array}$$

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- A_i are independent realizations of the random operator

$$\begin{array}{rcccl} A & : & \Omega \times L^2(\Omega, \mathbb{C}) & \to & L^2(\Omega, \mathbb{C}) \\ & & (y, z) & \mapsto & k(\cdot, y) z(y). \end{array} \tag{0.1}$$

Sketch of the proof

(a) rewrites

$$\left(\mathrm{I} + \frac{1}{N}\sum_{i=1}^{N} A_i\right) z_N = f$$

Proposition

Let $(A_i)_{i\in\mathbb{N}}$ be a family of independent realizations of a given bounded random operator $A : L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$. Then as $N \to +\infty$,

$$\frac{1}{N}\sum_{i=1}^{N} A_i \longrightarrow \mathbb{E}[A],$$

where the convergence holds at the rate $O(N^{-\frac{1}{2}})$ in the following mean-square sense:

$$\mathbb{E}\left[\left|\left|\left|\frac{1}{N}\sum_{i=1}^{N} A_i - \mathbb{E}[A]\right|\right|\right|^2\right]^{\frac{1}{2}} \leq \frac{\mathbb{E}[|||A - \mathbb{E}[A]|||^2]^{\frac{1}{2}}}{\sqrt{N}} \text{ for any } N \in \mathbb{N}.$$

Sketch of the proof

For the random operator

$$\begin{array}{rcl} A \;:\; \Omega \times L^2(\Omega, \mathbb{C}) & \rightarrow & L^2(\Omega, \mathbb{C}) \\ (y, z) & \mapsto & k(\cdot, y) z(y), \end{array}$$

the expectation $\mathbb{E}[A]$ is given by

$$\mathbb{E}[A] \,:\, z \mapsto \int_\Omega k(\cdot, y) z(y) \rho(y) \mathrm{d}y.$$

Sketch of the proof

Proposition

Let A be a bounded random operator and $(A_i)_{i\in\mathbb{N}}$ be a sequence of independent realizations of A. Then for any $\epsilon > 0$ sufficiently small, with probability one, any $\lambda \in B(-1, \epsilon)$ belongs to the resolvent set of $\frac{1}{N}\sum_{i=1}^{N} A_i$ for N large enough:

$$\left(\lambda \mathrm{I} - \frac{1}{N}\sum_{i=1}^{N} A_i\right)^{-1} \to (\lambda \mathrm{I} - \mathbb{E}[A])^{-1}$$

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- In particular $(\lambda = -1)$, $\mathrm{I} + \frac{1}{N}\sum_{i=1}^{N} A_i$ is invertible for N large enough.

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- In particular ($\lambda = -1$), $\mathrm{I} + \frac{1}{N}\sum_{i=1}^{N} A_i$ is invertible for N large enough.
- The convergence holds at rate $O(N^{-1/2})$ in the operator norm $|||\cdot|||$ of $L^2(\Omega)$; it yields

$$\mathbb{E}[||z_N - z||^2_{L^2(\Omega)}|\mathcal{H}_{N_0}]^{\frac{1}{2}} \leq CN^{-\frac{1}{2}}.$$

Result 2: convergence of the Nystrom interpolant

$$z_{i,N} + \frac{1}{N}\sum_{j=1}^{N} z_{j,N} k(y_i, y_j) = f(y_i), \qquad 1 \leq i \leq N, \tag{1}$$

$$z(y) + \int_\Omega z(y')k(y,y')\rho(y')\mathrm{d}y' = f(y), \qquad y \in \Omega. \tag{2}$$

Let $z_N(y)$ be the Nystrom interpolant

$$z_N(y) := f(y) - \frac{1}{N}\sum_{i=1}^{N} k(\cdot, y_i) z_{N,i}, \qquad y \in \Omega.$$

Proposition

There exists an event $\mathcal{H}_{N_0}$ satisfying $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that

1. *(1) is invertible for $N \geqslant N_0$ when $\mathcal{H}_{N_0}$ is realized*
2. *z_N converges to z at rate $O(N^{-\frac{1}{2}})$ in a mean-square sense:*

$$\mathbb{E}[||z_N - z||^2_{L^2(\Omega)}|\mathcal{H}_{N_0}]^{\frac{1}{2}} \leq CN^{-\frac{1}{2}}.$$

From operators to matrices

$$z_{i,N} + \frac{1}{N}\sum_{j=1}^{N} z_{j,N} k(y_i, y_j) = f(y_i), \qquad 1 \leq i \leq N, \tag{1}$$

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- The well-conditionning of the linear system (1)

From operators to matrices

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It remains to obtain

- The well-conditionning of the linear system (1)
- The point-wise convergence $z_{i,N} \to z(y_i)$.

Result 3: point-wise convergence

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Proposition

For the same event $\mathcal{H}_{N_0}$ satisfying $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$:

1. *(1) is invertible for $N \geqslant N_0$ when $\mathcal{H}_{N_0}$ is realized*
2. *the vector $(z_{N,i})_{1\leq i\leq N}$ converges to the point-wise values $(z(y_i))_{1\leq i\leq N}$ at rate $O(N^{-\frac{1}{2}})$ in a mean-square sense:*

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} |z_{N,i} - z(y_i)|^2 \,\middle|\, \mathcal{H}_{N_0}\right]^{\frac{1}{2}} \leq CN^{-\frac{1}{2}}.$$

Result 1: well-conditioning

$$z_{i,N} + \frac{1}{N}\sum_{j=1}^{N} z_{j,N} k(y_i, y_j) = f(y_i), \quad 1 \leq i \leq N \quad \Longleftrightarrow (\mathrm{I} + A_N) z_N = F \qquad (1)$$

with $z_N = (z_{i,N})_{1\leq i\leq N}$ and $F = (f(y_i))_{1\leq i\leq N}$ and where $(A_N)_{1\leq i,j\leq N}$ is the random matrix defined by

$$A_{N,ij} = \begin{cases} \frac{1}{N} k(y_i, y_j) \text{ if } i \neq j, \\ 0 \text{ if } i = j. \end{cases}$$

Proposition

Assume (i), (ii) and (iii). Then with probability one, there exists $N_0 \in \mathbb{N}$ such that the matrix $\mathrm{I} + A_N$ is invertible for any $N \geqslant N_0$, and there exists a constant $C > 0$ independent of N such that

$$\forall N \geqslant N_0, |||(\mathrm{I} + A_N)^{-1}|||_2 \leq C$$

where $|||\cdot|||_2$ is the operator norm ($|||A|||_2 := \sup_{||x||_2=1} ||Ax||_2$).

So (1) is well-posed if the continuous problem is well-posed.

Result 1: well-conditioning

- We know that $B(-1, \epsilon)$ belongs to the resolvent set of $(\mathrm{I} + A_N)$.

[3] Bandtlow, *Estimates for norms of resolvents and an application to the perturbation of spectra* (2004)

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$$|||(\mathrm{I}+A_N)^{-1}|||_2 \leq \frac{1}{d(-1,\sigma(A_N))}\exp\left(\frac{1}{2}\frac{\mathrm{Tr}(\overline{A_N^T}A_N)}{d(-1,\sigma(A_N))}+\frac{1}{2}\right).$$

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  We obtain the well conditioning of the matrix $\mathrm{I} + A_N$.
- Since the vector $v_N := (v_{N,i})_{1 \leq i \leq N}$ defined by $v_{N,i} := z_{N,i} - z(y_i)$ satisfies

$$(\mathrm{I} + A_N)v_N = -r_N$$

  with $\mathbb{E}[|r_N|_2^2] = O(N^{-1/2})$, we obtain the point-wise bound.

[3] Bandtlow, *Estimates for norms of resolvents and an application to the perturbation of spectra* (2004)

Preprint

The full details have been submitted in the preprint

Feppon F. and Ammari H., *Analysis of a Monte-Carlo Nystrom Method*. Submitted. (2021).

Thank you for your attention.