Homogenization of a system of subwavelength resonators

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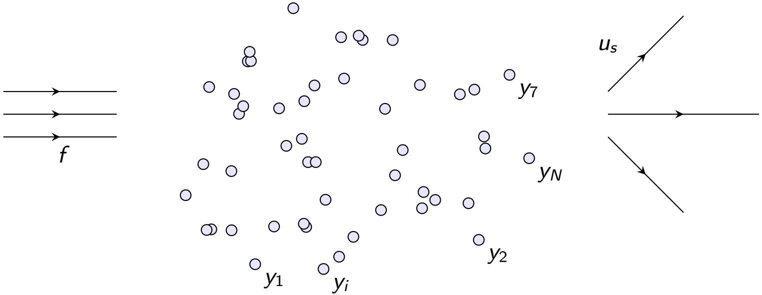
ETH zürich

Acoustic scattering of an incident field f through N packets of obstacles $(y_i + sD_i)_{1 \leq i \leq N}$ located at $(y_i)_{1 \leq i \leq N}$:



Motivation: acoustic metamaterials

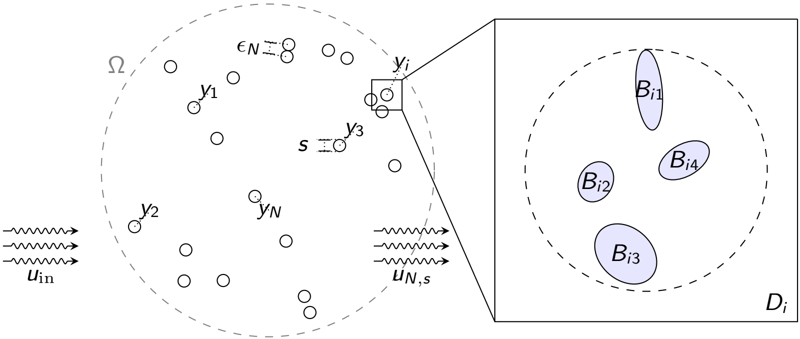


Figure: Setting of the homogenization problem.

We assume there are N packets of obstacles of size s filling a bounded domain Ω.

$$D_{N,s} = \cup_{i=1}^{N}(y_i + sD_i)$$

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Sound-absorbing obstacles:

$$\begin{cases} \Delta u_{N,s} + k^2 u_{N,s} = 0 \text{ in } \mathbb{R}^3 \backslash D_{N,s}, \\ u_{N,s} = 0 \text{ on } \partial D_{N,s}, \\ \left(\frac{\partial}{\partial |x|} - \mathrm{i}k \right) (u_{N,s}(x) - u_{\mathrm{in}}(x)) = O(|x|^{-2}) \text{ as } |x| \to +\infty, \end{cases}$$

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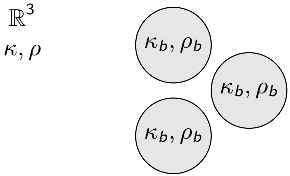
High-contrast obstacles:



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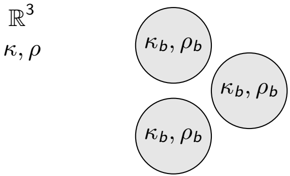
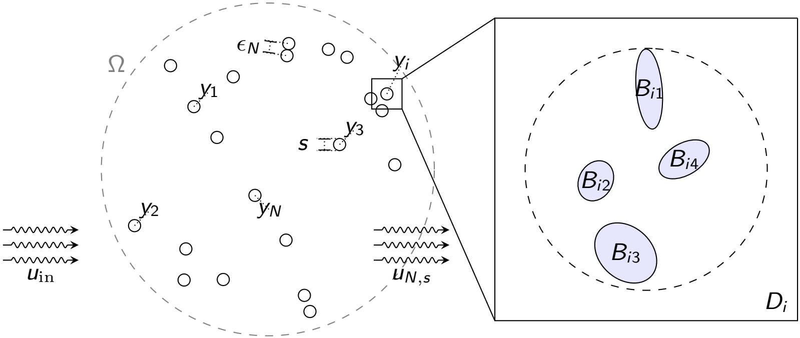


$$\delta := \frac{\rho_b}{\rho} \to 0.$$

$$\begin{cases} \operatorname{div}\left(\frac{1}{\rho_b}\nabla u_{N,s}\right) + \frac{\omega^2}{\kappa_b} u_{N,s} = 0 \text{ in } D_{N,s}, \\ \operatorname{div}\left(\frac{1}{\rho}\nabla u_{N,s}\right) + \frac{\omega^2}{\kappa} u_{N,s} = 0 \text{ in } \mathbb{R}^3 \backslash D_{N,s}, \\ u_{N,s}|_+ - u_{N,s}|_- = 0 \text{ on } \partial D_{N,s}, \\ \frac{1}{\rho_b} \frac{\partial u_{N,s}}{\partial n}\Big|_- = \frac{1}{\rho} \frac{\partial u_{N,s}}{\partial n}\Big|_+ \quad \text{on } \partial D_{N,s}, \\ \left(\frac{\partial}{\partial |x|} - \mathrm{i}k\right)(u_{N,s} - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty, \end{cases}$$



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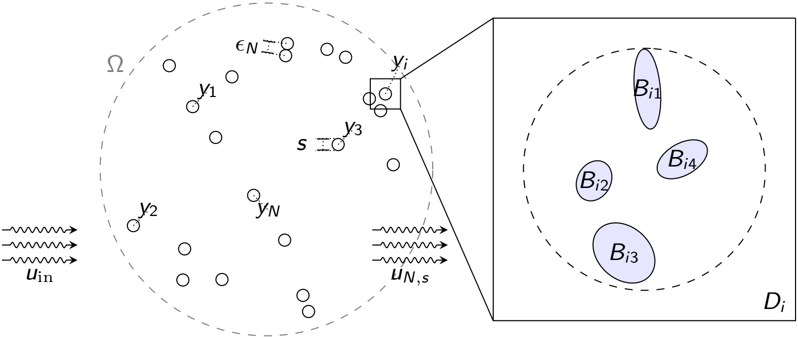

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The asymptotic analysis is performed with $s \to 0$, $N \to +\infty$, $\delta \to 0$.

Outline

1. Exposition of the results for sound-absorbing materials
2. Exposition of the results for high-contrast metamaterials
3. Main ingredients of the proof: layer potentials and convergence of a Foldy-Lax system.

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Our contribution: randomly distributed centers, quantitative error bounds in $L^2(B(0,R))$ for any $R > 0$ even close to the obstacles.

Assumption 1

$(y_i)_{1\leq i\leq N}$ are distributed randomly and independently according to $\rho \mathrm{d}x$ with $\rho \in L^\infty(\Omega)$ supported in $\Omega \subset \mathbb{R}^3$. In particular, $\rho \geqslant 0$ and $\int_\Omega \rho \mathrm{d}x = 1$, and

$\sum_{i=1}^{N} \delta_{y_i} \to \rho \mathrm{d}x$ as $N \to +\infty$, in the sense of distributions.

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Assumption 2

The packets of resonators are identical and constituted of K single components $(B_l)_{1\leq l\leq K}$:

$$D_i = D := \bigcup_{l=1}^K B_l, \qquad \forall 1 \leq i \leq N.$$

For sound-absorbing metamaterials, we assume further the subcritical regime $sN = O(1)$:

Assumption 3

There exists a constant $c > 0$ such that the parameters s and N satisfy

$$sN \leq c.$$

Proposition 1

Assume assumptions 1, 2 and 3 and denote by u the solution to the Lippmann-Schwinger equation

$$\begin{cases} \Delta u + (k^2 - sN\mathrm{cap}(D)\rho 1_\Omega)u = 0 \ in\ \mathbb{R}^3, \\ \left(\frac{\partial}{\partial|x|} - \mathrm{i}k\right)(u - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \tag{1}$$

There exists an event $\mathcal{H}_{N_0}$ which holds with large probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that when $\mathcal{H}_{N_0}$ is realized, the function u is an approximation of the total wave field $u_{N,s}$ with the following error estimates:

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1. *on any ball* $B(0,r)$ *containing the obstacles,* $\Omega \subset B(0,r)$ *and for any* $N \geqslant N_0$*:*

$$\mathbb{E}[||u_{N,s} - u||^2_{L^2(B(0,r))}|\mathcal{H}_{N_0}]^{\frac{1}{2}} \leq csN \max((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}); \tag{2}$$

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2. *on any bounded open subset $A \subset \mathbb{R}^3 \backslash \Omega$ away from the obstacles and for any $N \geqslant N_0$:*

$$\mathbb{E}[||\nabla u_{N,s} - \nabla u||^2_{L^2(A)}|\mathcal{H}_{N_0}]^{\frac{1}{2}} \leq csN \max((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}). \tag{3}$$

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The relative error is of order $O(\max((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}))$ because the scattered fields $u_{N,s} - u_{\mathrm{in}}$ and $u - u_{\mathrm{in}}$ are of order $O(sN)$.

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Sound-absorbing metamaterials

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2. For $sN \to \Lambda$ with $\Lambda > 0$, the effective medium is dissipative, $u_{N,s} \to u$, the solution to the Helmholtz equation with "strange term"

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3. For $sN \to +\infty$, we expect that the obstacles "solidify" in a single sound-hard obstacle Ω, and that $u_{N,s} \to u$ where u is the solution to the problem

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } \mathbb{R}^3, \\ u = 0 \text{ on } \Omega, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases}$$

However this would require a significantly different analysis.

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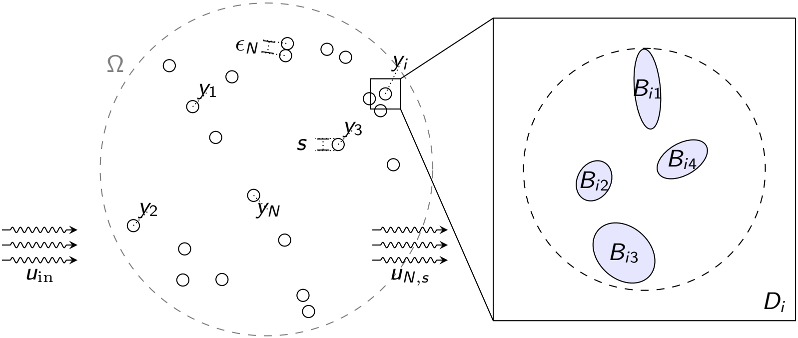
High-contrast metamaterials feature resonances. Denote by $(\boldsymbol{a}_k)_{1\leq k\leq K}$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_K$ the eigenvectors and eigenvalues of the generalized eigenvalue problem

$$C\boldsymbol{a}_j = \lambda_j V \boldsymbol{a}_j \text{ with } C := \left(-\int_{\partial B_i} \mathcal{S}_D^{-1}[1_{\partial B_j}]\mathrm{d}\sigma\right)_{1\leq i,j\leq K} \text{ and } V := \mathrm{diag}(|B_i|)_{1\leq i\leq K}, \quad (4)$$



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- The metamaterial constituted of N identical packets of K connected resonators $sD = \cup_{i=1}^{K} sB_i$ admits K resonant frequencies

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- Since in our analysis ω is fixed but s is variable, it is equivalent to say that there is K resonant sizes

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$$s_i(\delta):=\frac{\delta^{\frac{1}{2}}}{\omega}\lambda_i^{\frac{1}{2}}v_b, \qquad 1\leq i\leq K.$$

- As $s\to s_i(\delta)$, the relevant "critical quantity" is

$$sNQ(s,\delta) \text{ with } Q(s,\delta):=\sum_{i=1}^K\frac{\lambda_i}{\frac{s^2}{s_i(\delta)^2}-1}(\boldsymbol{a}_i^T V1)^2,$$

where $1=(1)_{1\leq i\leq K}$ is the vector of ones.

High-contrast metamaterials

Related previous works:

- Ammari and Zhang, *Effective medium theory for acoustic waves in bubbly fluids near minnaert resonant frequency* (2017). Single resonator $K = 1$, centers $(y_i)_{1\leq i\leq N}$ satisfying technical assumptions, case $sNQ(s,\delta) \to \Lambda$ for $\Lambda \in \mathbb{R}$, estimates in a small region away from the obstacles.

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Our contributions:

- identical packets of multiple resonators (K arbitrary) and identification of the role of $Q(s, \delta)$
- $s \to s_i(\delta)$ for a resonance of monopole type ($Q(s, \delta) \to +\infty$ as $s \to s_i(\delta) \Leftrightarrow \boldsymbol{a}_i^T V 1 \neq 0$), in the "subcritical regime" $sNQ(s, \delta) = O(1)$

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Our contributions:

- identical packets of multiple resonators (K arbitrary) and identification of the role of $Q(s, \delta)$
- $s \to s_i(\delta)$ for a resonance of monopole type ($Q(s, \delta) \to +\infty$ as $s \to s_i(\delta) \Leftrightarrow \boldsymbol{a}_i^T V 1 \neq 0$), in the "subcritical regime" $sNQ(s, \delta) = O(1)$
- quantitative estimates in any ball $B(0, R)$ with $R > 0$.

High-contrast metamaterials

High-contrast metamaterials feature resonances. Denote by $(\boldsymbol{a}_k)_{1\leq k\leq K}$ and $0<\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_K$ the eigenvectors and eigenvalues of the generalized eigenvalue problem

$$C\boldsymbol{a}_j=\lambda_j V\boldsymbol{a}_j \text{ with } C:=\left(-\int_{\partial B_i}\mathcal{S}_D^{-1}[1_{\partial B_j}]\mathrm{d}\sigma\right)_{1\leq i,j\leq K} \text{ and } V:=\mathrm{diag}(|B_i|)_{1\leq i\leq K}, \quad (4)$$

- The metamaterial constituted of N identical packets of K connected resonators $sD=\cup_{i=1}^K sB_i$ admits K resonant frequencies

$$\omega_i(\delta,s)=\frac{\delta^{\frac{1}{2}}}{s}\lambda_i^{\frac{1}{2}}v_b \text{ with } v_b:=\sqrt{\frac{\rho_b}{\kappa_b}},$$

- Since in our analysis ω is fixed but s is variable, it is equivalent to say that there is K resonant sizes

$$s_i(\delta):=\frac{\delta^{\frac{1}{2}}}{\omega}\lambda_i^{\frac{1}{2}}v_b, \qquad 1\leq i\leq K.$$

- As $s\to s_i(\delta)$, the relevant "critical quantity" is

$$sNQ(s,\delta) \text{ with } Q(s,\delta):=\sum_{i=1}^K\frac{\lambda_i}{\frac{s^2}{s_i(\delta)^2}-1}(\boldsymbol{a}_i^T V1)^2,$$

where $1=(1)_{1\leq i\leq K}$ is the vector of ones.

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For high-contrast metamaterials, we assume the following subcritical regime

Assumption 4

$\exists 1 \leq i \leq K, \quad s \sim s_i(\delta)$ with $\boldsymbol{a}_i^T V 1 \neq 0$, and there exists $c > 0$ independent of s, δ and N such that

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The contrast parameter is strictly smaller than N^{-2}*:*

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Proposition 2

Assume assumptions 1,2 and 4 and denote by u the solution to the following Lippmann-Schwinger equation:

$$\begin{cases} \left(\Delta + k^2 - sNQ(s,\delta)\rho 1_\Omega\right) u = 0, \\ \left(\frac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \tag{6}$$

There exists an event $\mathcal{H}_{N_0}$ which holds with large probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that when $\mathcal{H}_{N_0}$ is realized, u is an approximation of the solution field $u_{N,s}$ with the following error estimates:

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  - If $sNQ(s,\delta) \to +\infty$, we expect that the medium solidifies as for sound-absorbing obstacles. If $sNQ(s,\delta) \to -\infty$, then the medium becomes highly dispersive. This case remains opened.

Outline

1. Exposition of the results for sound-absorbing materials
2. Exposition of the results for high-contrast metamaterials
3. Main ingredients of the proof: layer potentials and convergence of a Foldy-Lax system.

Sketch of the derivation

We outline the proof for sound-absorbing metamaterials.
The main steps of the derivation are:

1. the scattered field of a single resonator sD centered at 0 can be written

$$u_s(x) \simeq -s\,\mathrm{cap}\,(D) u_{\mathrm{in}}(0)(1 + O(\omega) + O(|x|^{-1}))\Gamma^k(x)$$

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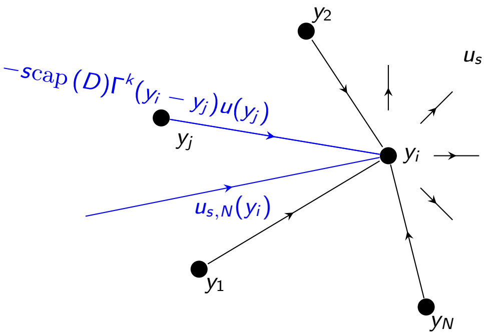
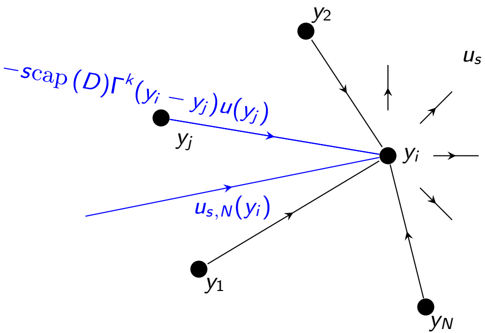


The Foldy-Lax approximation:



The Foldy-Lax approximation:

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Proposition 1

Assume assumptions 1, 2 and 3 and denote by u the solution to the Lippmann-Schwinger equation

$$\begin{cases} \Delta u + (k^2 - sN\mathrm{cap}\,(D)\rho 1_\Omega)u = 0 \ \mathit{in}\ \mathbb{R}^3, \\ \left(\frac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \quad (1)$$

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1. *on any ball $B(0,r)$ containing the obstacles, $\Omega \subset B(0,r)$ and for any $N \geqslant N_0$:*

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2. *on any bounded open subset $A \subset \mathbb{R}^3 \backslash \Omega$ away from the obstacles and for any $N \geqslant N_0$:*

$$\mathbb{E}[||\nabla u_{N,s} - \nabla u||^2_{L^2(A)}|\mathcal{H}_{N_0}]^{\frac{1}{2}} \leq csN \max((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}). \quad (3)$$

The relative error is of order $O(\max((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}))$ because the scattered fields $u_{N,s} - u_{\mathrm{in}}$ and $u - u_{\mathrm{in}}$ are of order $O(sN)$.

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2. the Foldy-Lax approximation assumes that the scattered field at a point y_i is obtained by summing the contributions of the total wave field experienced by the other resonators:

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   Justification with layer potentials

3. Assuming that $(y_i)_{i\in\mathbb{N}}$ is randomly and independently distributed according to $\rho\mathrm{d}x$, we expect the convergence of $(u_{s,N}(y_i)_{1\leq i\leq N})$ to the values $(u(y_i))_{1\leq i\leq N}$ where u is the solution to the integral equation:

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4. This equation an integral formulation of the claimed Lippmann-Schwinger equation.

Sketch of the derivation

We outline the proof for sound-absorbing metamaterials.
The main steps of the derivation are:

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where $D_{N,s} = \cup_{i=1}^{N}(y_i + sD_i)$ and

$$\mathcal{S}^k_{D_{N,s}}[\phi](x) := \int_{\partial D_{N,s}} \Gamma^k(x-y)\phi(y)\mathrm{d}y \text{ for } \phi \in L^2(D_{N,s}), \quad \Gamma^k(x-y) = -\frac{e^{\mathrm{i}k\pi|x-y|}}{4\pi|x-y|}.$$

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- There is some analiticity with respect to s:

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  for some holomorphic operator $\mathcal{S}^k_{\mathcal{D}}(s)$ on $L^2(D_1)\times\cdots\times L^2(D_N)$.

The operator $\mathcal{S}_{\mathcal{D}}^k(s)$ is given by

$$\mathcal{S}_{\mathcal{D}}^k(s) := s\mathcal{S}_{\mathcal{D},0} + s^2\mathcal{S}_{\mathcal{D},1}^k + \sum_{p=2}^{+\infty} s^{p+1}\mathcal{S}_{\mathcal{D},p}^k,$$

for some operators $\mathcal{S}_{\mathcal{D},p}^k$ which decay geometrically in the operator norm:

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The variable η_N is the ratio between the size s and the minimum distance ϵ_N between the centers (we can assume $\eta_N < 1$):

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Proposition 1

Assume assumptions 1, 2 and 3 and denote by u the solution to the Lippmann-Schwinger equation

$$\begin{cases} \Delta u + (k^2 - sN\mathrm{cap}\,(D)\rho 1_\Omega)u = 0 \ \mathit{in}\ \mathbb{R}^3, \\ \left(\dfrac{\partial}{\partial|x|} - \mathrm{i}k\right)(u - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \tag{1}$$

There exists an event $\mathcal{H}_{N_0}$ which holds with large probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that when $\mathcal{H}_{N_0}$ is realized, the function u is an approximation of the total wave field $u_{N,s}$ with the following error estimates:

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- Computing the inverse of $\mathcal{S}_{\mathcal{D}}^{k}(s)$, we obtain

$$(\mathcal{S}_{D_{N,s}}^{k})^{-1}[u_{\mathrm{in}}] \simeq -\frac{s^{-1}}{\mathrm{cap}\,(D)} \sum_{i=1}^{N} z_i^N \mathcal{S}_D^{-1}[1_{\partial D}] \circ \boldsymbol{\tau}_{y_i,s}^{-1},$$

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- This system is called the “Foldy-Lax approximation” of the scattering problem. Indeed, the scattered field has the following point-wise behavior away from the obstacles:

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[1] Feppon and Ammari, *Analysis of a Monte-Carlo Nystrom method* (2022)

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This is the origin of the error rate $O(N^{-\frac{1}{2}})$ in the convergence result.

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Sketch of the derivation

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Thank you for your attention.