High order topological asymptotics: reconciling layer potentials and compound asymptotic expansions

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ETH zürich

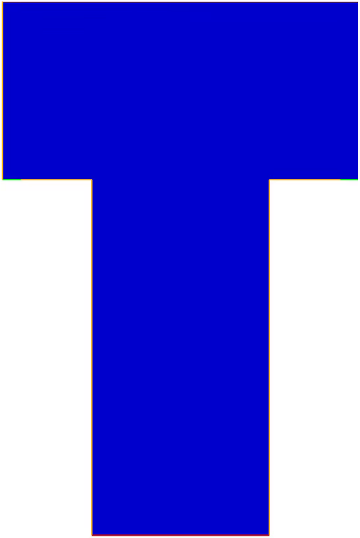
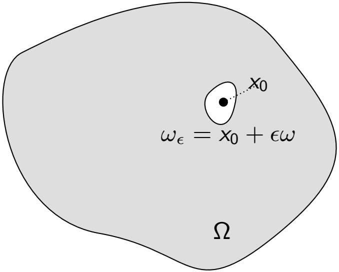


Figure: Topology optimization based on the topological derivative with an algorithm from S. Amstutz. Movie from C. Dapogny

Motivation: topological asymptotics

Topological derivative: the key ingredient is an asymptotic of the perturbation of a PDE problem with respect to the nucleation of a hole.



$$\begin{cases} -\Delta u_\epsilon = f \text{ in } \Omega, \\ u_\epsilon = 0 \text{ on } \partial\omega_\epsilon, \\ u_\epsilon = 0 \text{ on } \partial\Omega, \end{cases}$$

Reconciling compound asymptotic expansions and layer potentials

Two main ways for obtaining asymptotics in the literature:

- **Compound asymptotic expansions:** write a formal two-scale ansatz, e.g.

$$u_{\epsilon,N}(x) = u(x) + \sum_{p=0}^{N} \epsilon^p v_p\left(\frac{x - x_0}{\epsilon}\right) + \sum_{p=1}^{N} \epsilon^p w_p(x)$$

  then variational estimates on $u_{\epsilon,N} - u_\epsilon$;
- **Layer potential methods**: write an integral representation

$$u_\epsilon(x) = \int_{\partial\omega_\epsilon} G_\epsilon(x,y)\phi_\epsilon(y)\mathrm{d}\sigma(y)$$

  for an unknown potential ϕ_ϵ and perform asymptotic expansions with respect to ϵ.

Reconciling compound asymptotic expansions and layer potentials

Two main ways for obtaining asymptotics of solutions to PDEs in the literature:

- **Compound asymptotic expansions:** write a formal two-scale ansatz, e.g. Kozlov, Maz'ya, and Movchan (1999), Maz'ya, Nazarov, and Plamenevskij (2000), Samet, Amstutz, and Masmoudi (2003), Guillaume and Idris (2002), Pommier and Samet (2004), Faria and Novotny (2009), Samet, Amstutz, and Masmoudi (2003), Auroux, Jaafar-Belaid, and Rjaibi (2010), Hintermüller, Laurain, and Novotny (2012), Hassine and Khelifi (2016), Novotny and Sokołowski (2013)
- **Layer potential methods**: Ammari et al. (2002), Ammari et al. (2012), Ammari et al. (2002), Capdeboscq and Vogelius (2003), Cristoforis (2008), Maz'ya, Movchan, Nieves, et al. (2013).

Reconciling compound asymptotic expansions and layer potentials

Strengths and weaknesses:

- **Compound asymptotic expansions:**
  - Clear physical interpretation of the ansatz and its terms defined in terms of **exterior problems**
  - Variational estimates with H^1 norm.
  - The ansatz has to be proposed **a priori**. Sometimes hard to find.
- **Layer potential methods**:
  - Explicit dependence w.r.t. the small parameter ϵ fully ellucidated. Analiticity becomes clear.
  - The asymptotic of ϕ_ϵ involve **Neumann series**, hard to interpret physically and tedious to manipulate.

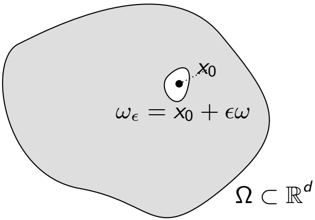
Reconciling compound asymptotic expansions and layer potentials

The goal of this presentation: introduce a mixed **systematic** procedure for computing **full asymptotic expansions** of the solution w.r.t the nucleation of a hole:

1. find an **explicit integral representation** of the solution with layer potentials
2. with a change of variable, write a first asymptotic expansion to identify the **correct form of the ansatz**
3. use the method of compound asymptotic expansions to **identify the terms of the ansatz** and prove **variational error estimates**.

The perforated Dirichlet problem

Works for the Dirichlet perforated problem:



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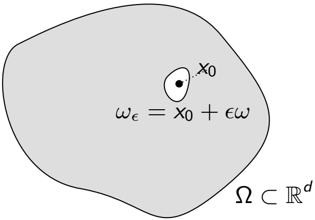

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$u_\epsilon \to u$ where $\begin{cases} -\Delta u = f \text{ in } \Omega, \\ u_\epsilon = 0 \text{ on } \partial\Omega, \end{cases}$

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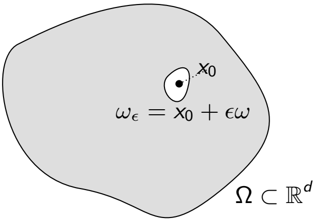

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Proposition 1

Assume $d \geqslant 3$. There exist functions $(v_p)_{p\geqslant 0}$ and $(w_p)_{p\geqslant d-2}$ such that the following ansatz holds for the solution u_ϵ:

$$u_\epsilon(x) = u(x) + \sum_{p=0}^{+\infty} \epsilon^p v_p\left(\frac{x - x_0}{\epsilon}\right) + \sum_{p=d-2}^{+\infty} \epsilon^p w_p(x), \qquad x \in \Omega\backslash\omega_\epsilon. \tag{1}$$

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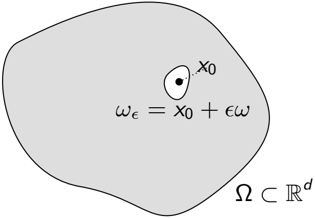

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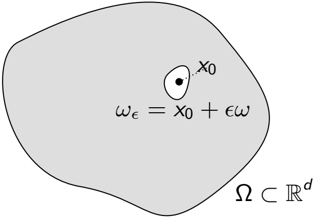

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Assume $d = 2$. There exist functions $(v_{p,q})_{p\geqslant 0, 0\leq q\leq p}$ and $(w_{p,q})_{p\geqslant 1, 0\leq q\leq p}$, R_Ω, Φ and constants Φ^∞, $(v^\infty_{p,q})_{p\geqslant 0,0\leq q\leq p})$ such that the following ansatz holds for the solution u_ϵ:

$$u_\epsilon(x) = u(x) + \left(\frac{1}{a_\epsilon}\sum_{p=0}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}v^\infty_{p,q}\right)\left(\Phi\left(\frac{x-x_0}{\epsilon}\right) + R_\Omega(x,x_0) - R_\Omega(x_0,x_0)\right)$$

$$+ \sum_{p=0}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}v_{p,q}\left(\frac{x-x_0}{\epsilon}\right) + \sum_{p=1}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}w_{p,q}(x).$$

where $a_\epsilon = \frac{1}{2\pi}\log\epsilon - \Phi^\infty + R_\Omega(x_0,x_0)$.

The perforated Dirichlet problem

Idea of the proof: use the single layer potential representation

$$u_\epsilon(x) = u(x) - \mathcal{S}_{\Omega,\omega_\epsilon}[\mathcal{S}^{-1}_{\Omega,\omega_\epsilon}[u|_{\partial\omega_\epsilon}]](x), \qquad x \in \Omega_\epsilon$$

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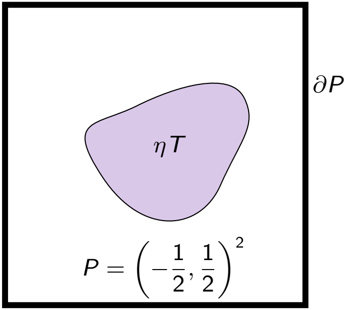
where

$$\forall \phi \in H^{-\frac{1}{2}}(\partial\omega_\epsilon), \quad \forall x \in \Omega, \quad \mathcal{S}_{\Omega,\omega_\epsilon}[\phi](x) := \int_{\partial\omega_\epsilon} G_\Omega(x,y)\phi(y)\mathrm{d}\sigma(y),$$

for G_Ω the Dirichlet Green function on Ω.

A singular problem with periodicity conditions



$$\begin{cases} -\Delta \mathcal{X}_\eta = 1 \text{ in } P \backslash (\eta T), \\ \mathcal{X}_\eta = 0 \text{ on } \partial(\eta T), \\ \mathcal{X}_\eta \text{ is } P-\text{periodic}, \end{cases}$$

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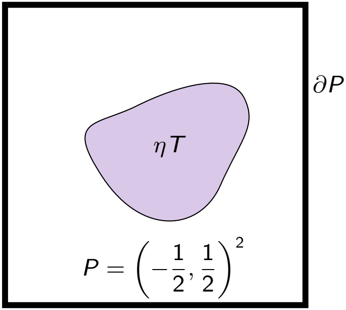

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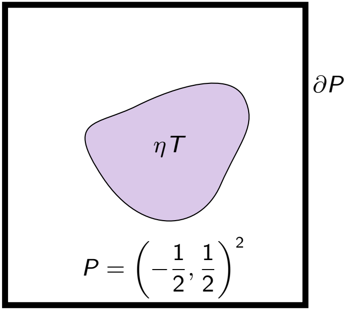

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- Leading order asymptotic identified by Allaire (1991) and then quantitative estimate by Jing (2020):

$$\mathcal{X}_\eta = \Phi(\cdot/\eta) + O(1) \text{ where } \begin{cases} -\Delta \Phi = 0 \text{ in } \mathbb{R}^2 \backslash \overline{\omega}, \\ \Phi = 0 \text{ on } \partial\omega, \\ \Phi(x) \sim \frac{1}{2\pi} \log|x| \text{ as } |x| \to +\infty, \end{cases}$$

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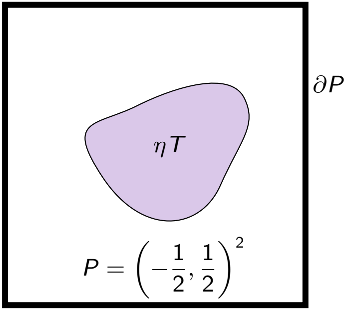

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- What would be the correct ansatz for high order topological asymptotic expansions?

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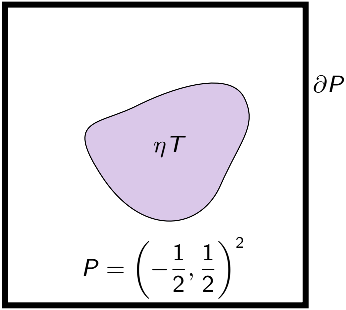

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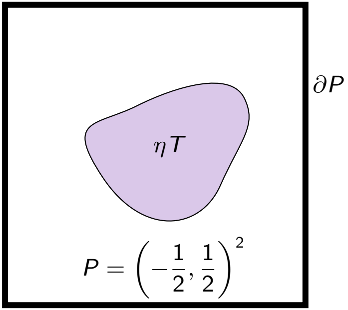

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- What would be the correct ansatz for high order topological asymptotic expansions? Logarithmic terms or not ?

The periodic Green function

Definition 1

There exists a unique function $G_\#$, defined up to an additive constant, such that

$$\begin{cases} \Delta G_\# = \delta_0 - 1 \text{ in } P, \\ \quad G_\# \text{ is } P\text{–periodic.} \end{cases}$$

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We have e.g.

$$G_\#(x) = -\sum_{\boldsymbol{\xi} \in \mathbb{Z}^d \setminus \{0\}} \frac{e^{2\mathrm{i}\pi \boldsymbol{\xi} \cdot x}}{4\pi^2 |\boldsymbol{\xi}|^2}, \qquad x \in P \setminus \{0\}.$$

The periodic Green function

Proposition 2

The periodic Green kernel $G_\#$ is given by

$$G_\#(x) = \Gamma(x) + R_\#(x), \qquad x \in P,$$

where $\Gamma(x)$ is the fundamental solution to the Laplace problem:

$$\Gamma(x) = \frac{1}{2\pi} \log |x|,$$

and where $R_\# \in H^1(P)$ is the unique solution, up to a constant, to the difference problem

$$\begin{cases} -\Delta R_\# = 1 \ \mathit{in} \ P, \\ R_\# + \Gamma \ \mathit{is} \ P\text{–}\mathit{periodic}, \\ \dfrac{\partial R_\#}{\partial n}\boldsymbol{n} + \dfrac{\partial \Gamma}{\partial n}\boldsymbol{n} \ \mathit{is} \ P\text{–}\mathit{periodic}. \end{cases}$$

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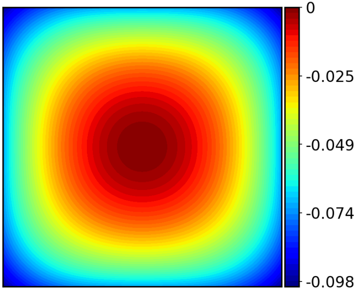
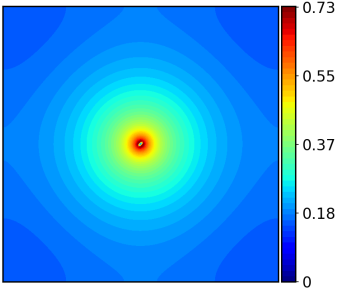
For our application, we choose the constant to be set such that

$$R_\#(0) = 0.$$

The periodic Green function



(a) $R_{\#}$



(b) $G_{\#}$

Figure: Periodic Green function $G_{\#} = \Gamma + R_{\#}$ in $P = (-\frac{1}{2}, \frac{1}{2})^2$.

Periodic single layer potential

Definition 2

For a given real $\kappa \in \mathbb{R}$, we define $\mathcal{S}^{\kappa}_{\#,\eta T}$ to be the single layer potential defined by

$$\mathcal{S}^{\kappa}_{\#,\eta T}[\phi](x) := \int_{\partial(\eta T)} G_{\#}(x-y)\phi(y)\mathrm{d}\sigma(y) + \eta^{2-d}\left(-\frac{\log \eta}{2\pi} + \kappa\right)\int_{\partial(\eta T)} \phi \mathrm{d}\sigma, \qquad (1)$$

for any $\phi \in H^{-\frac{1}{2}}(\partial(\eta T))$ and $x \in P$.

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Proposition 3

$\mathcal{S}^{\kappa}_{\#,\eta T}$ satisfies $\Delta \mathcal{S}^{\kappa}_{\#,\eta T} = 0$ in $P \backslash (\eta T)$ and in ηT, and the jump relations

$$[\![\mathcal{S}^{\kappa}_{\#,\eta T}[\phi]]\!] = 0, \qquad \left[\!\!\left[\frac{\partial \mathcal{S}^{\kappa}_{\#,\eta T}[\phi]}{\partial \boldsymbol{n}}\right]\!\!\right] = \phi.$$

Lemma 3

The kernel of $\mathcal{S}^{\kappa}_{\#,\eta}$ is either trivial or is the space :

$$\mathrm{Ker}(\mathcal{S}^{\kappa}_{\#,\eta T}) \subset \mathrm{span}\left(\frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}}\right).$$

Moreover, $\mathcal{X}_\eta$ has the following single layer potential representation when this kernel is not trivial:

$$\mathcal{X}_\eta(x) = \frac{1}{-1+\eta^d |T|} \mathcal{S}^{\kappa}_{\#,\eta T}\left[\frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}}\right](x), \qquad x \in P \backslash (\eta T).$$

Characterization of $\mathcal{X}_\eta$

We characterize κ and $\frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}}$ by the implicit function theorem!

Characterization of $\mathcal{X}_\eta$

We characterize κ and $\dfrac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}}$ by the implicit function theorem! We solve

$$\mathcal{S}^{\kappa}_{\#,\eta T}[\phi] = 0 \text{ on } \partial(\eta T).$$

Characterization of $\mathcal{X}_\eta$

Let $\boldsymbol{\tau}_\eta$ be the rescaling function

$$\boldsymbol{\tau}_\eta(t) := \eta t \text{ for any } t \in \ \partial T,$$

and P_η the rescaling operator

$$\mathcal{P}_\eta[\phi] := \phi \circ \boldsymbol{\tau}_\eta \text{ for any } \phi \in H^s(\partial(\eta T)), \text{ for any } s \in \mathbb{R}.$$

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Proposition 4

The following factorization holds:

$$\mathcal{S}^\kappa_{\#,\eta T} = \eta \mathcal{P}_\eta^{-1} \mathcal{S}^\kappa_T(\eta) \mathcal{P}_\eta,$$

where $\mathcal{S}^\kappa_T(\eta) : H^{-\frac{1}{2}}(\partial T) \to H^{\frac{1}{2}}(\partial T)$ *is given by*

$$\mathcal{S}^\kappa_T(\eta)[\phi](t) = \mathcal{S}_T[\phi](t) + \kappa \int_{\partial T} \phi \mathrm{d}\sigma + \int_{\partial T} R_\#(\eta(t - t'))\phi(t')\mathrm{d}\sigma(t'), \qquad t \in \partial T.$$

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No $\log \eta$ anymore after the rescaling!

Characterization of $\mathcal{X}_\eta$

We have

$$S_T^\kappa(\eta)[\phi](t) = \mathcal{S}_T[\phi](t) + \kappa \int_{\partial T} \phi \mathrm{d}\sigma + O(\eta),$$

where

$$\mathcal{S}_T[\phi](t) = \int_{\partial T} \Gamma(t - t')\phi(t')\mathrm{d}\sigma(t').$$

Proposition 5

1. *There exists a unique solution Φ to the problem*

$$\begin{cases} -\Delta\Phi = 0 \ \mathit{in} \ \mathbb{R}^2 \setminus \overline{T}, \\ \Phi = 0 \ \mathit{on} \ \partial T, \\ \Phi(x) \sim \dfrac{1}{2\pi} \log|x| \ \textit{as} \ |x| \to +\infty, \end{cases} \tag{2}$$

satisfying $\Phi - \Gamma \in \mathcal{D}^{1,2}(\mathbb{R}^2 \setminus \overline{T})$.

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   satisfying $\Phi - \Gamma \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{T})$.

2. *There exists a constant* Φ^∞ *such that:*

$$\Phi(x) = \mathcal{S}_T\left[\left.\frac{\partial\Phi}{\partial \boldsymbol{n}}\right|_+\right](x) + \Phi^\infty, \qquad x \in \mathbb{R}^2\backslash\overline{T}. \tag{3}$$

   Consequently, we have the asymptotic expansion

$$\Phi(x) = \frac{1}{2\pi}\log|x| + \Phi^\infty + O(|x|^{-1}). \tag{4}$$

The capacity in dimension 2

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   satisfying $\Phi - \Gamma \in \mathcal{D}^{1,2}(\mathbb{R}^2 \backslash \overline{T})$*.*

2. *There exists a constant* Φ^∞ *such that:*

$$\Phi(x) = \mathcal{S}_T \left[\left. \frac{\partial \Phi}{\partial \boldsymbol{n}} \right|_+ \right](x) + \Phi^\infty, \qquad x \in \mathbb{R}^2 \backslash \overline{T}. \tag{3}$$

   Consequently, we have the asymptotic expansion

$$\Phi(x) = \frac{1}{2\pi} \log|x| + \Phi^\infty + O(|x|^{-1}). \tag{4}$$

3. *Independently of* T*, the normal flux of* Φ *is equal to one:*

$$\int_{\partial T} \left. \frac{\partial \Phi}{\partial \boldsymbol{n}} \right|_+ \mathrm{d}\sigma = 1. \tag{5}$$

Analiticity of $\partial \mathcal{X}_\eta / \partial n$

We have

$$S_T^\kappa(\eta)[\phi](t) = \mathcal{S}_T[\phi](t) + \kappa \int_{\partial T} \phi \mathrm{d}\sigma + O(\eta),$$

where

$$\mathcal{S}_T[\phi](t) = \int_{\partial T} \Gamma(t - t')\phi(t')\mathrm{d}\sigma(t').$$

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Set $\phi = \dfrac{\partial \Phi}{\partial \boldsymbol{n}}$ and $\kappa = \Phi^{\infty}$.

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Set $\phi = \dfrac{\partial \Phi}{\partial \boldsymbol{n}}$ and $\kappa = \Phi^\infty$.

Then $\mathcal{S}_T^\kappa(\eta)[\phi] = O(\eta)$.

Analiticity of $\partial \mathcal{X}_\eta / \partial n$

Proposition 6

There exists a real analytic function $\eta \mapsto \kappa_\eta$ such that the operator $\mathcal{S}^{\kappa_\eta}_{\#,\eta T}$ has a non-trivial kernel, given by

$$\mathrm{Ker}(\mathcal{S}^{\kappa_\eta}_{\#,\eta T}) = \mathrm{span}\left(\frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}}\right).$$

Moreover, κ_η and $\frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}}$ admit the following series representations:

$$\kappa_\eta = \Phi^\infty + \sum_{p \geqslant 2} \eta^p c_p \text{ and } \frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}} = (1 - \eta^2 |T|)\left(-\eta^{-1} \phi^* \circ \boldsymbol{\tau}_\eta^{-1} + \sum_{p \geqslant 1} \eta^p \phi_p \circ \boldsymbol{\tau}_\eta^{-1}\right),$$

for some constants $(c_p)_{p \geqslant 2}$ and functions $(\phi_p)_{p \geqslant 1}$ of $L^2(\partial T)$ satisfying

$$\int_{\partial T} \phi_p \mathrm{d}\sigma = 0 \text{ for all } p \geqslant 1.$$

Full asymptotic expansion for $\mathcal{X}_\eta$

Coming back to the representation

$$\mathcal{X}_\eta(x) = \frac{1}{-1+\eta^d|T|}\mathcal{S}^{\kappa}_{\#,\eta T}\left[\frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}}\right](x), \qquad x \in P\backslash(\eta T),$$

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we obtain a surprising result:

Proposition 7

There exist functions $(v_p)_{p\geqslant 2}$ and $(w_p)_{p\geqslant 0}$ such that the following ansatz holds:

$$\mathcal{X}_\eta(x) = \Phi\left(x/\eta\right) + \sum_{p=2}^{+\infty}\eta^p v_p(x/\eta) + \sum_{p=0}^{+\infty}\eta^p w_p(x),$$

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2. *$v_p \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash T)$ is the solution to an exterior Dirichlet problem in $\mathbb{R}^d\backslash\bar{T}$ satisfying $v_p(x) = O(|x|^{-1})$ as $|x| \to +\infty$, for $p \geqslant 2$ (namely, satisfying additionally $v_p^\infty = 0$);*

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This is 2D but no logarithms !

Proposition 8

There exist functions $(v_p)_{p\geqslant 2}$ and $(w_p)_{p\geqslant 0}$ such that the following ansatz holds:

$$\mathcal{X}_\eta(x) = \Phi\,(x/\eta) + \sum_{p=2}^{+\infty} \eta^p v_p(x/\eta) + \sum_{p=0}^{+\infty} \eta^p w_p(x),$$

Proof.

Recall $\mathcal{S}^{\kappa_\eta}_{\eta,\eta T} = \eta \mathcal{P}_\eta^{-1} \mathcal{S}^{\kappa_\eta}_{T} \mathcal{P}_\eta$.

$$\mathcal{X}_\eta(x) = \frac{1}{-1+\eta^d|T|}\mathcal{S}^{\kappa_\eta}_{\#,\eta T}\left[\frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}}\right](x) = \frac{1}{-1+\eta^d|T|}\mathcal{S}^{\kappa_\eta}_{T}(\eta)\left[\frac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}} \circ \boldsymbol{\tau}_\eta\right](x/\eta)$$

$$= \frac{1}{-1+\eta^d|T|}\left(\mathcal{S}_T[\phi_\eta](x/\eta) + \kappa_\eta \int_{\partial T} \phi_\eta \mathrm{d}\sigma + \int_{\partial T} R_\#(x-\eta t')\phi_\eta(t')\mathrm{d}\sigma(t')\right)$$

with $\phi_\eta := \dfrac{\partial \mathcal{X}_\eta}{\partial \boldsymbol{n}} \circ \boldsymbol{\tau}_\eta$. □

Proposition 9

The functions $(v_p)_{p\geqslant 2}$ and $(w_p)_{p\geqslant 0}$ are uniquely characterized as the solutions to the following recursive systems of exterior and interior problems:

$$\left\{ -\Delta w_p = \begin{cases} 1 \text{ if } p = 0, \\ 0 \text{ if } p \geqslant 1, \end{cases} \quad \text{in } P, \right.$$

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Full asymptotic expansion for $\mathcal{X}_\eta$

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and

$$\begin{cases} -\Delta v_p = 0 \ in \ \mathbb{R}^d \setminus T, \\ v_p(t) = -w_p(0) - \displaystyle\sum_{k=1}^{p} \frac{1}{k!} \nabla^k w_{p-k}(0) \cdot t^k \text{ for } t \in \partial T, \quad p \geqslant 2, \\ v_p(x) = O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases}$$

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- Here, $w_0(0) = 0$, $w_1(0) = 0$, and $w_p(0)$ is determined by the condition $v_p(x) = O(|x|^{-1})$ as $|x| \to +\infty$ for $p \geqslant 2$

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- We have more explicitly:

$$v_p^{(k)}(x) := \frac{(-1)^{k-d+2}}{(k-d+2)!} \nabla^{k-d+2}\Gamma(x) \cdot \int_{\partial T} \left[\!\left[\frac{\partial v_p}{\partial \boldsymbol{n}} \right]\!\right] t^{k-d+2} \mathrm{d}\sigma(t), \quad k \geqslant d-1, \tag{6}$$

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Proposition 10

For any $N \in \mathbb{N}$, *let* $\mathcal{X}_\eta^N$ *be the truncated ansatz at rank* N*:*

$$\mathcal{X}_\eta^N(x) := \Phi(x/\eta) + \sum_{p=2}^{N} \eta^p v_p(x/\eta) + \sum_{p=0}^{N} \eta^p w_p,$$

We have the following variational estimates:

$$||\mathcal{X}_\eta - \mathcal{X}_\eta^0||_{L^2(P\setminus(\eta T))} + ||\nabla \mathcal{X}_\eta - \nabla \mathcal{X}_\eta^0||_{L^2(P\setminus(\eta T))} \leq C_N \eta,$$

and

$$|\log \eta|^{-\frac{1}{2}} ||\mathcal{X}_\eta - \mathcal{X}_\eta^N||_{L^2(P\setminus(\eta T))} + ||\nabla \mathcal{X}_\eta - \nabla \mathcal{X}_\eta^N||_{L^2(P\setminus(\eta T))} \leq C_N \eta^{N+1} \qquad \text{for any } N \geqslant 1.$$

Preprint

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Thank you for your attention.