Homogenization of sound-absorbing resonant, and temporal acoustic metamaterials

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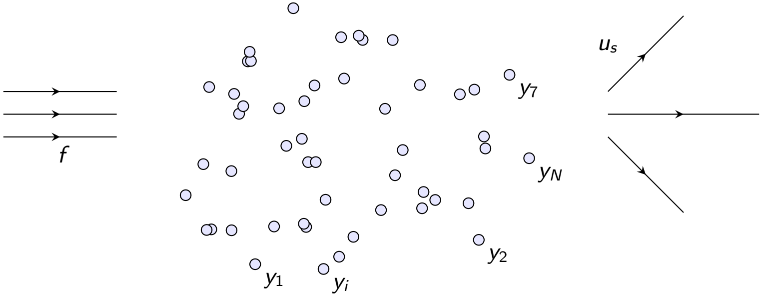
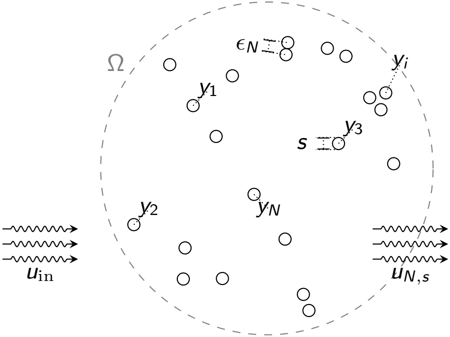
Seminar for Applied Mathematics

ETH zürich

Motivation: acoustic metamaterials

Acoustic scattering of an incident field f through N obstacles $(y_i + sD_i)_{1\leq i\leq N}$ located at $(y_i)_{1\leq i\leq N}$:





D_i

Figure: Setting of the homogenization problem.

We assume there are N packets of obstacles of size s filling a bounded domain Ω.

$$D_{N,s} = \cup_{i=1}^{N}(y_i + sD_i)$$

Assumption 1

$(y_i)_{1\leq i\leq N}$ are distributed randomly and independently according to $\rho \mathrm{d}x$ with $\rho \in L^\infty(\Omega)$ supported in $\Omega \subset \mathbb{R}^3$. In particular, $\rho \geqslant 0$ and $\int_\Omega \rho \mathrm{d}x = 1$, and

$\sum_{i=1}^{N} \delta_{y_i} \to \rho \mathrm{d}x$ as $N \to +\infty$, in the sense of distributions.

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Assumption 2

The packets of resonators are identical.

$$D_i = D \qquad \forall 1 \leq i \leq N.$$

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Sound-absorbing obstacles:

$$\begin{cases} \Delta u_{N,s} + k^2 u_{N,s} = 0 \text{ in } \mathbb{R}^3 \backslash D_{N,s}, \\ u_{N,s} = 0 \text{ on } \partial D_{N,s}, \\ \left(\frac{\partial}{\partial |x|} - \mathrm{i}k \right) (u_{N,s}(x) - u_{\mathrm{in}}(x)) = O(|x|^{-2}) \text{ as } |x| \to +\infty, \end{cases}$$

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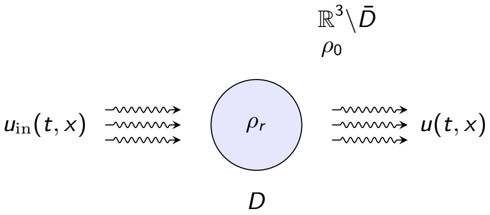
High-contrast obstacles:



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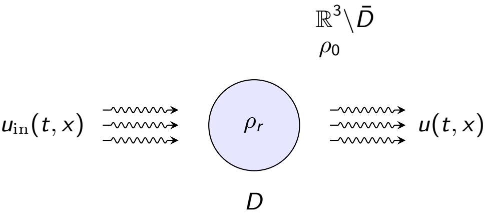

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$$\begin{cases} \mathrm{div}\left(\dfrac{1}{\rho_r}\nabla u_{N,s}\right) + \dfrac{\omega^2}{\kappa_r} u_{N,s} = 0 \text{ in } D_{N,s}, \\ \mathrm{div}\left(\dfrac{1}{\rho}\nabla u_{N,s}\right) + \dfrac{\omega^2}{\kappa} u_{N,s} = 0 \text{ in } \mathbb{R}^3\backslash D_{N,s}, \\ u_{N,s}|_+ - u_{N,s}|_- = 0 \text{ on } \partial D_{N,s}, \\ \dfrac{1}{\rho_r}\dfrac{\partial u_{N,s}}{\partial n}\Big|_- = \dfrac{1}{\rho}\dfrac{\partial u_{N,s}}{\partial n}\Big|_+ \text{ on } \partial D_{N,s}, \\ \left(\dfrac{\partial}{\partial|x|} - \mathrm{i}k\right)(u_{N,s} - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty, \end{cases}$$

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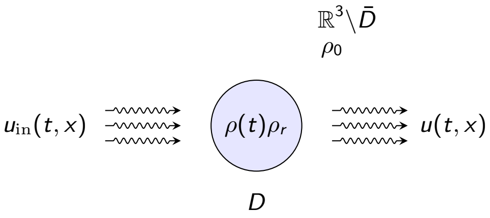

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$$\begin{cases} \dfrac{1}{\kappa_0}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_0}\Delta u = 0 \text{ in } \mathbb{R} \times \mathbb{R}^3 \backslash \bar{D}, \\ \dfrac{1}{\kappa_r}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho(t)\rho_r}\Delta u = 0 \text{ in } \mathbb{R} \times D, \\ \dfrac{1}{\rho_0}\left.\dfrac{\partial u}{\partial \boldsymbol{n}}\right|_+ = \dfrac{1}{\rho_r \rho(t)}\left.\dfrac{\partial u}{\partial \boldsymbol{n}}\right|_- \text{ on } \mathbb{R} \times \partial D, \quad 1 \leq i \leq N, \\ u_{|+} = u_{|-} \text{ on } \mathbb{R} \times \partial D, \\ u - u_{\text{in}} \text{ is outgoing,} \end{cases}$$

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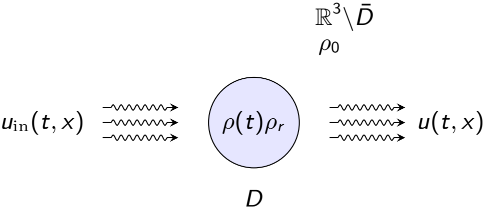

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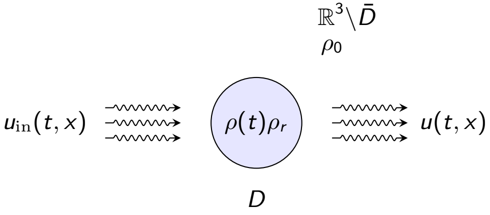

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Subwavelength and fast-modulation regimes: Ω is fixed while $\omega \to 0$.

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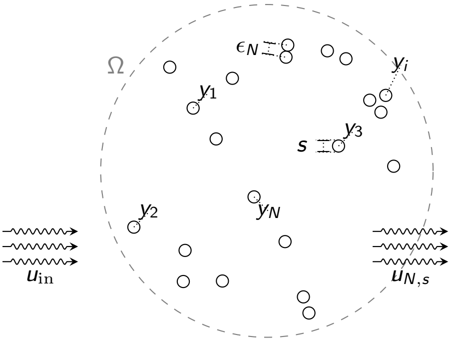

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Figure: Setting of the homogenization problem.

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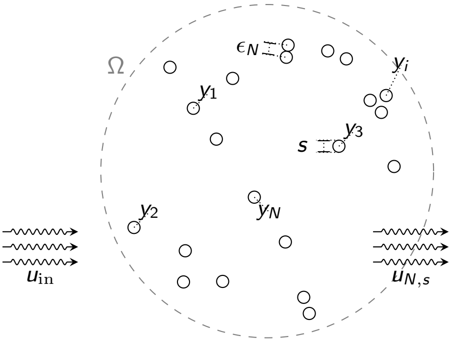

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Asymptotic analysis performed with $s \to 0$, $N \to +\infty$, $\delta \to 0$.

Outline

1. Sound-absorbing metamaterials
2. High-contrast metamaterials
3. Time-modulated high-contrast metamaterials.

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Proposition 1

Denote by u the solution to the Lippmann-Schwinger equation

$$\begin{cases} \Delta u + (k^2 - sN\mathrm{cap}\,(D)\rho 1_\Omega)u = 0 \ \ in\ \mathbb{R}^3, \\ \left(\dfrac{\partial}{\partial|x|} - \mathrm{i}k\right)(u - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases}$$

Assume $sN = O(1)$. There exists an event $\mathcal{H}_{N_0}$ which holds with large probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that when $\mathcal{H}_{N_0}$ is realized, the function u is an approximation of the total wave field $u_{N,s}$ with the following error estimates:

1. *on any ball $B(0,r)$ containing the obstacles, $\Omega \subset B(0,r)$ and for any $N \geqslant N_0$:*

$$\mathbb{E}[||u_{N,s} - u||^2_{L^2(B(0,r))}|\mathcal{H}_{N_0}]^{\frac{1}{2}} \leq csN \max((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}});$$

2. *on any bounded open subset $A \subset \mathbb{R}^3 \backslash \Omega$ away from the obstacles and for any $N \geqslant N_0$:*

$$\mathbb{E}[||\nabla u_{N,s} - \nabla u||^2_{L^2(A)}|\mathcal{H}_{N_0}]^{\frac{1}{2}} \leq csN \max((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}).$$

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2. For $sN \to \Lambda$ with $\Lambda > 0$, the effective medium is dissipative, $u_{N,s} \to u$, the solution to the Helmholtz equation with "strange term"

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3. For $sN \to +\infty$, we expect that the obstacles "solidify" in a single sound-hard obstacle Ω, and that $u_{N,s} \to u$ where u is the solution to the problem

$$\begin{cases} \Delta u + k^2 u = 0 \text{ in } \mathbb{R}^3, \\ u = 0 \text{ on } \Omega, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases}$$

However this would require a significantly different analysis.

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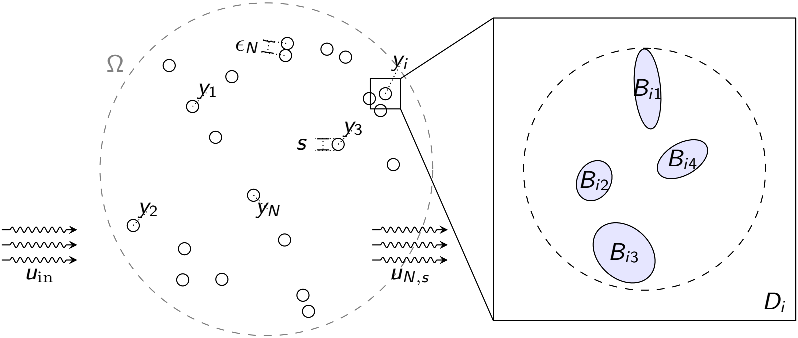
High-contrast metamaterials



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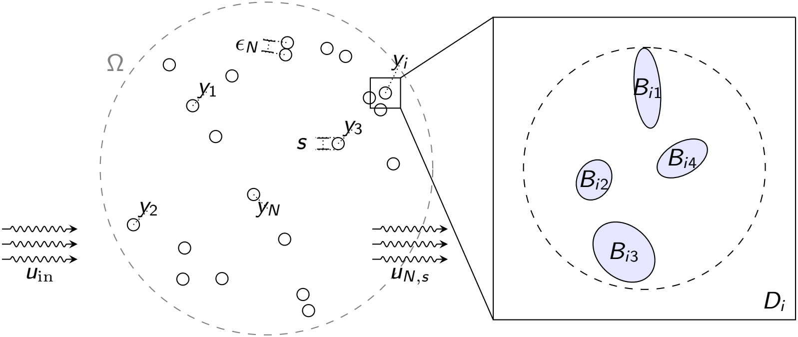
High-contrast metamaterials

High-contrast metamaterials feature resonances. Denote by $(\boldsymbol{a}_k)_{1\le k\le K}$ and $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_K$ the eigenvectors and eigenvalues of the generalized eigenvalue problem

$$C\boldsymbol{a}_j = \lambda_j V\boldsymbol{a}_j \text{ with } C := \left(-\int_{\partial B_i} \mathcal{S}_D^{-1}[1_{\partial B_j}]\mathrm{d}\sigma\right)_{1\le i,j\le K} \text{ and } V := \mathrm{diag}(|B_i|)_{1\le i\le K}, \quad (1)$$



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- The metamaterial constituted of N identical packets of K connected resonators $sD=\cup_{i=1}^{K}sB_i$ admits K resonant frequencies

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$$s_i(\delta):=\frac{\delta^{\frac{1}{2}}}{\omega}\lambda_i^{\frac{1}{2}}v_r, \qquad 1\leq i\leq K.$$

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$$s_i(\delta):=\frac{\delta^{\frac{1}{2}}}{\omega}\lambda_i^{\frac{1}{2}}v_r, \qquad 1\leq i\leq K.$$

- As $s\to s_i(\delta)$, the relevant "critical quantity" is

$$sNQ(s,\delta) \text{ with } Q(s,\delta):=\sum_{i=1}^{K}\frac{\lambda_i}{\frac{s^2}{s_i(\delta)^2}-1}(\boldsymbol{a}_i^T V 1)^2,$$

where $1=(1)_{1\leq i\leq K}$ is the vector of ones.

Proposition 2

Assume $sNQ(s,\delta) = O(1)$ and denote by u the solution to the following Lippmann-Schwinger equation:

$$\begin{cases} \left(\Delta + k^2 - sNQ(s,\delta)\rho 1_\Omega\right) u = 0, \\ \left(\frac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{\mathrm{in}}) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases} \tag{2}$$

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  - If $\Lambda < 0$ (s is slightly smaller than the resonant size $s_i(\delta)$, but not too close), then the effective medium is dispersive .
  - If $sNQ(s,\delta) \to +\infty$, we expect that the medium solidifies as for sound-absorbing obstacles. If $sNQ(s,\delta) \to -\infty$, then the medium becomes highly dispersive. The analysis of these cases remain opened.

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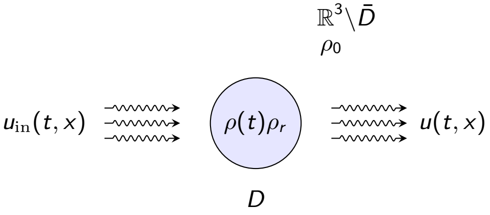
Time-modulated acoustic high-contrast metamaterials



In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$

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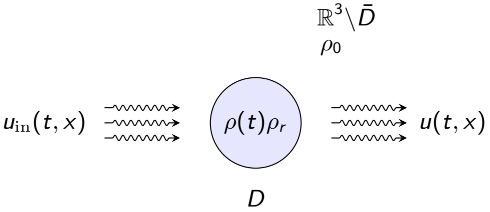

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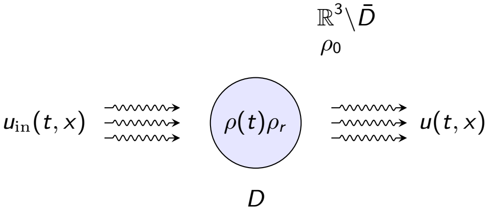

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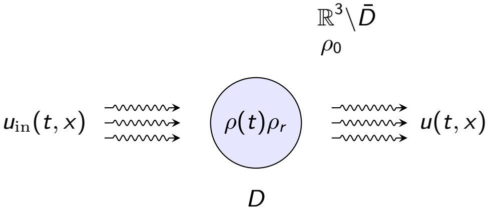

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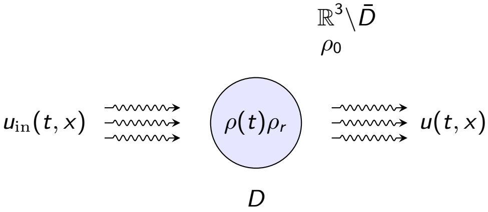

In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$
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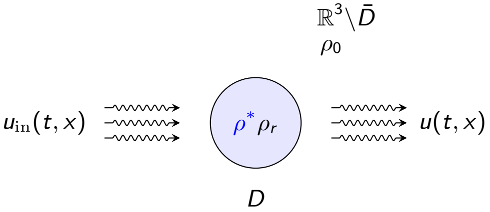

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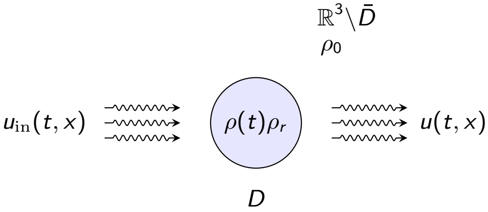

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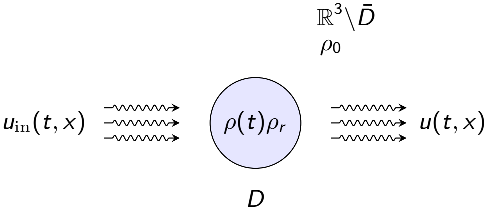

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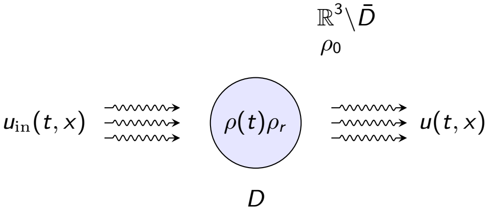

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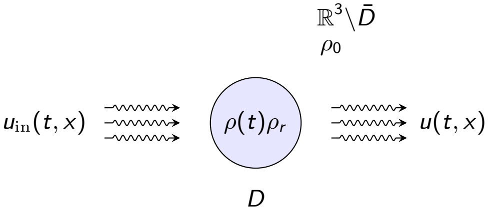

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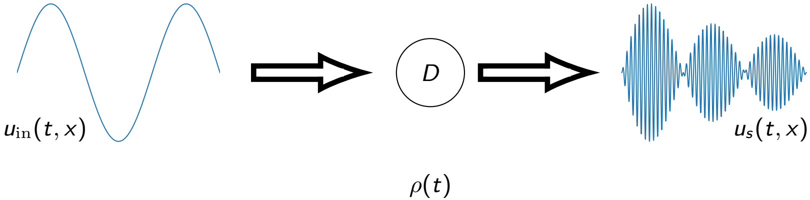
$u_{\mathrm{in}}(t, x)$
D
$u_s(t, x)$
$\rho(t)$

Time-modulated acoustic high-contrast metamaterials

Consider the limit equation in D when $\delta \to 0$:

$$\begin{cases} \dfrac{1}{v_r^2}\partial_t^2 \hat{u} - \dfrac{1}{\rho(t)}\Delta \hat{u} = 0, & (t,x) \in \mathbb{R} \times D \\ \dfrac{1}{\rho(t)} \dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}} = 0, & (t,x) \in \mathbb{R} \times \partial D, \\ & t \mapsto \hat{u}(t,x) \text{ is } T\text{−periodic.} \end{cases}$$

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$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} p_m(t) = \dfrac{\mu_m}{\rho(t)} p_m(t), \\ p_m \text{ is } T\text{–periodic.} \end{cases} \quad \text{and} \quad \begin{cases} -\Delta \phi_l = \lambda_l \phi_l \text{ in } D, \\ \dfrac{\partial \phi_l}{\partial \boldsymbol{n}} = 0 \text{ on } \partial D, \end{cases} \qquad l \in \mathbb{N}.$$

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at the condition that the Sturm-Liouville and Neumann eigenvalue coincide!

$$\frac{\mu_m}{v_r^2} = \lambda_l.$$

Set

$$\Lambda := \{(l, m) \in \mathbb{N} \times \mathbb{N} \mid \frac{\mu_m}{v_r^2} = \lambda_l\}.$$

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- If $\Lambda = \{(0,0)\}$: situation analogous to the static case, the modulation $\rho(t)$ is averaged.
- Assume $\Lambda = \{(0,0),(l,m)\}$ for some $(l,m) \neq (0,0)$. Then, one can construct two oscillating modes satisfying

$$v(t,x) \simeq \alpha_{0,0} + \alpha_{l,m} p_m(t)\phi_l(x) \text{ in } D$$

Proposition 3

There exist $2\#\Lambda$ subwavelength resonances $\omega_i^{\pm}(\delta)$ whose leading asymptotic satisfy, to the first order:

$$\omega_i^{\pm}(\delta) \sim \pm v_r \delta^{\frac{1}{2}} \lambda_i^{\frac{1}{2}}, \qquad 1 \leq i \leq \#\Lambda,$$

where $(\lambda_i)_{1\leq i\leq \#\Lambda}$ are the (complex) eigenvalues of a generalized eigenvalue problem

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T and $(\gamma_m)_{m,l\in\Lambda}$ have no distinguished signes, hence the eigenvalues λ_i are in general **complex** and one of them has **positive** imaginary part, leading to ougoing exponentially increasing modes.

Proposition 4

The scattered field generated by the time modulated resonator admits the following far field expansion as $|x| \to +\infty$ *and* $\omega = O(\delta^{\frac{1}{2}})$*:*

$$\hat{u}(x) - \hat{u}_{\mathrm{in}}(x) = \hat{u}_{\mathrm{in}}(0)\left(A\left(\frac{\omega^2}{v_r^2\delta}\right) + B\left(\frac{\omega^2}{v_r^2\delta}\right) G_{ml}\left(t - \frac{|x|}{v_0}, \frac{x}{|x|}\right)\right)(1 + O(\delta^{\frac{1}{2}}) + O(|x|^{-1}))\Gamma^{\frac{\omega}{v_0}}(x),$$

for function $G_{ml}(t,x)$ *which is* T*–periodic in the variable* t *and constant coefficients* $A(\omega^2/v_r^2\delta)$ *and* $B(\omega^2/v_r^2\delta)$*.*

Effective medium theory

Scalings:

- $D \to sD$
- $T \to sT, \qquad \rho \to \rho(\cdot/s)$. Fast modulation with large frequency $2\pi/(sT) \to +\infty$!
- The Neumann and Sturm-Liouville eigenvalues scale as

$$\mu_m \to \frac{\mu_m}{s^2} \text{ and } \lambda_l \to \frac{\lambda_l}{s^2},$$

  so that it is possible to keep a constant set

$$\Lambda = \left\{ (m,l) \in \mathbb{N} \times \mathbb{N} \,|\, \frac{\lambda_l}{s^2} = \frac{\mu_m}{s^2 v_r^2} \right\} = \left\{ (m,l) \in \mathbb{N} \times \mathbb{N} \,|\, \lambda_l = \frac{\mu_m}{v_r^2} \right\}$$

- The resonant frequencies scales again as

$$\omega_i^{\pm}(\delta) \sim \lambda_i^{\frac{1}{2}} v_r \frac{\delta^{\frac{1}{2}}}{s},$$

  which shows that for $s = O(\delta^{\frac{1}{2}})$, ω can be of order one.

Effective medium theory

We find then the following effective homogenized equation for the scattering of wave in the fast temporal medium:

$$u_{\mathrm{eff}}(t,y) - sN \int_{\Omega} K_{\omega,\delta}\left(t - \frac{|y-y'|}{v_0}, \frac{y-y'}{|y-y'|}\right) \Gamma^{\frac{\omega_0}{v_0}}(y-y') V(y') \hat{u}_{\mathrm{eff}}(t,y') \mathrm{d}y' \\ = \hat{u}_{\mathrm{in}}(y), \quad y \in \Omega.$$

where

$$K_{\omega,\delta}(t,y) := \left[A(s^2\omega^2/v_r^2\delta) + B(s^2\omega^2/v_r^2\delta) G_{ml}(t,y) \right].$$

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Using Fourier series, this is equivalent to a cascade of Helmholtz equation for each of the Fourier modes with a frequency dependent refractive index.

Preprint

The full details are available in the preprints

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feppon:time, **feppon:time** (**feppon:time**)

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Thank you for your attention.