Signal amplification and compression in ultra fast time modulated metamaterials due to a space-time resonant coupling

Florian Feppon – Habib Ammari

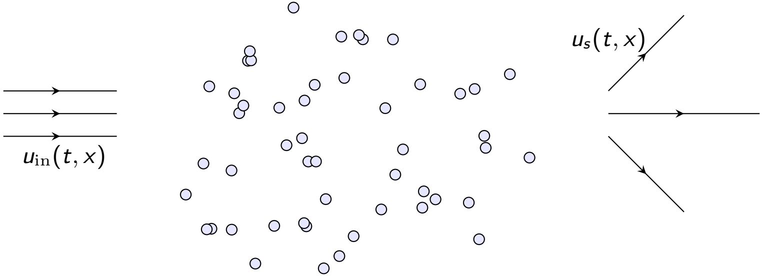
Applied Mathematics Seminar
Yale University (Online), March 30th 2022

Seminar for Applied Mathematics

ETH zürich

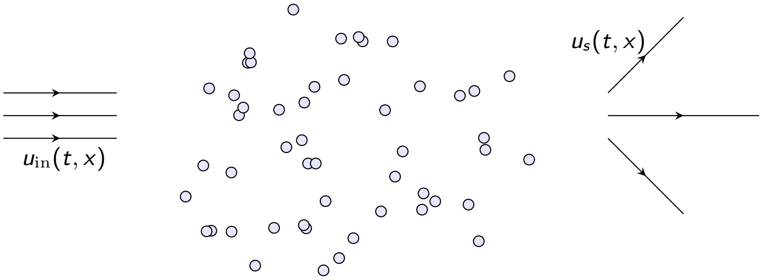
Motivation: time-modulated acoustic metamaterials

Acoustic scattering of an incident field u_{in} through a cloud of high-contrast obstacles:



Motivation: time-modulated acoustic metamaterials

Acoustic scattering of an incident field u_{in} through a cloud of high-contrast obstacles:



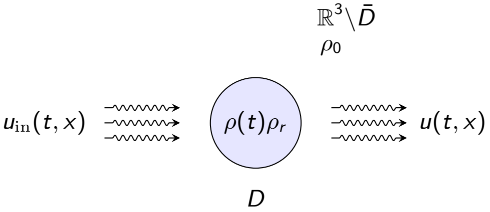
Our goal: understand the physics of the effective medium for time-modulated high-contrast obstacles.

Motivation: time-modulated acoustic metamaterials

Potential applications of high-contrast time-modulated metamaterials:

1. frequency conversion
2. signal amplification
3. spontaneous radiation
4. non-reciprocal propagation
5. spacetime cloaking

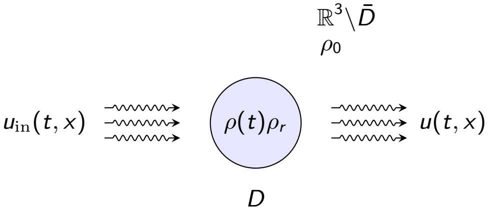
Motivation: time-modulated acoustic metamaterials



In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$

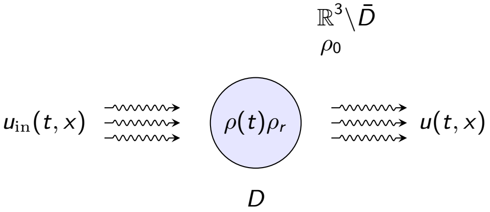
Motivation: time-modulated acoustic metamaterials



In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$

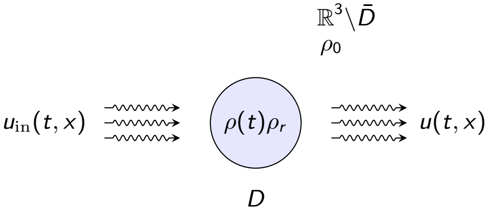
Motivation: time-modulated acoustic metamaterials



In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \frac{\rho_r}{\rho_0} \to 0$
- subwavelength setting $\omega \to 0$

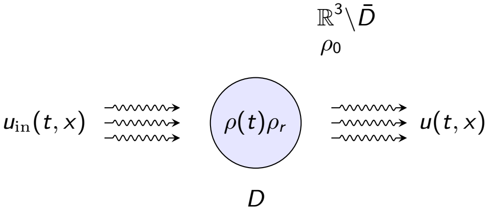
Motivation: time-modulated acoustic metamaterials



In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$
- subwavelength setting $\omega \to 0$

Motivation: time-modulated acoustic metamaterials

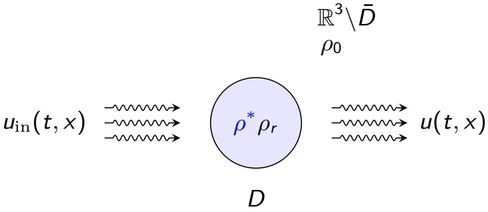


In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$
- subwavelength setting $\omega \to 0$

In most situations the fast modulation $\rho(t)$ is **averaged**.

Motivation: time-modulated acoustic metamaterials

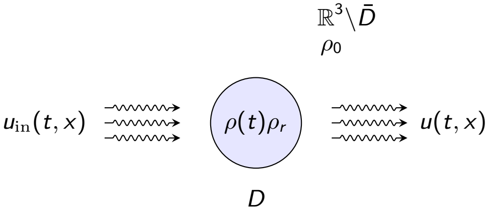


In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$
- subwavelength setting $\omega \to 0$

In most situations the fast modulation $\rho(t)$ is **averaged**. Everything happens as if we have a static material with some effective parameter ρ^*.

Motivation: time-modulated acoustic metamaterials

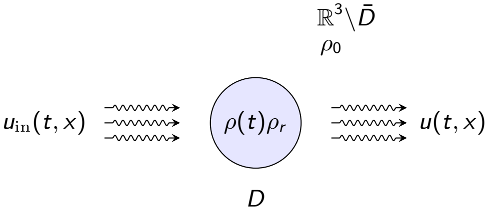


In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$
- subwavelength setting $\omega \to 0$

In most situations the fast modulation $\rho(t)$ is **averaged**. Everything happens as if we have a static material with some effective parameter ρ^*. The scattered field propagates with frequency ω.

Motivation: time-modulated acoustic metamaterials



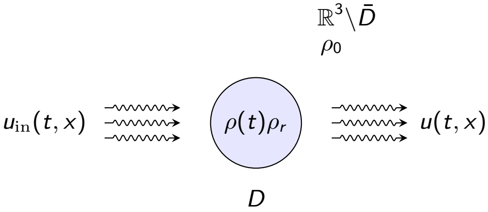
In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$
- subwavelength setting $\omega \to 0$

In most situations the fast modulation $\rho(t)$ is **averaged**. Everything happens as if we have a static material with some effective parameter ρ^*. The scattered field propagates with frequency ω.

However for an **exceptional tuning** of $\rho(t)$, a strong coupling arises.

Motivation: time-modulated acoustic metamaterials



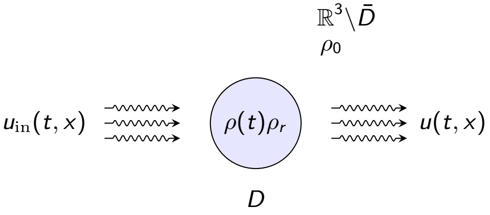
In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$
- subwavelength setting $\omega \to 0$

In most situations the fast modulation $\rho(t)$ is **averaged**. Everything happens as if we have a static material with some effective parameter ρ^*. The scattered field propagates with frequency ω.

However for an **exceptional tuning** of $\rho(t)$, a strong coupling arises. The scattered field contains **high frequency components**.

Motivation: time-modulated acoustic metamaterials



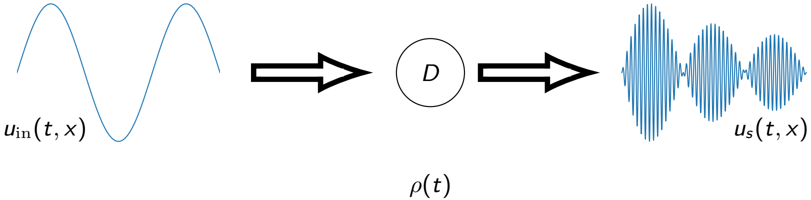
In our setting:

- modulation $\rho(t)$ of the physical parameter periodic, with **high** frequency $\Omega \gg \omega$
- high contrast $\delta := \dfrac{\rho_r}{\rho_0} \to 0$
- subwavelength setting $\omega \to 0$

In most situations the fast modulation $\rho(t)$ is **averaged**. Everything happens as if we have a static material with some effective parameter ρ^*. The scattered field propagates with frequency ω.

However for an **exceptional tuning** of $\rho(t)$, a strong coupling arises. The scattered field contains **high frequency components**. Outgoing modes **growing exponentially in time** may also arise.

Motivation: time-modulated acoustic metamaterials



Outline

1. The unmodulated case: point scatterer approximations and effective medium theory
2. Resonances in the static case and in the time-modulated case
3. Point scatterer approximation and effective medium in the time-modulated case.

Outline

1. The unmodulated case: point scatterer approximations and effective medium theory
2. Resonances in the static case and in the time-modulated case
3. Point scatterer approximation and effective medium in the time-modulated case.

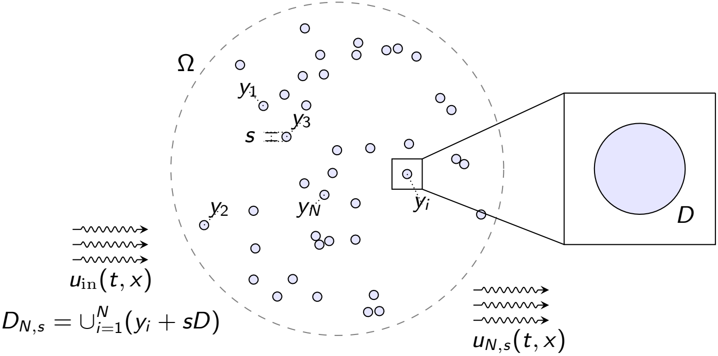
Outline

1. The unmodulated case: point scatterer approximations and effective medium theory
2. Resonances in the static case and in the time-modulated case
3. Point scatterer approximation and effective medium in the time-modulated case.

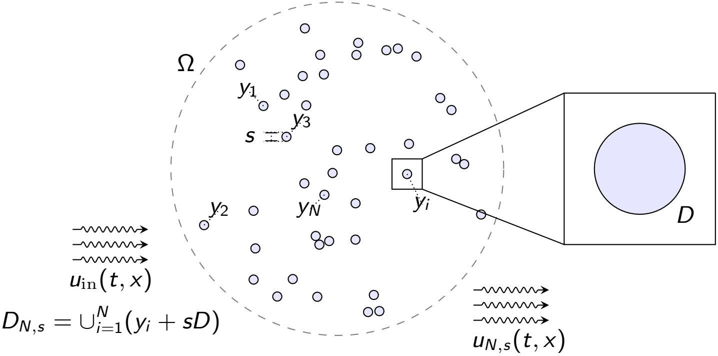
Outline

1. The unmodulated case: point scatterer approximations and effective medium theory
2. Resonances in the static case and in the time-modulated case
3. Point scatterer approximation and effective medium in the time-modulated case.

The static case



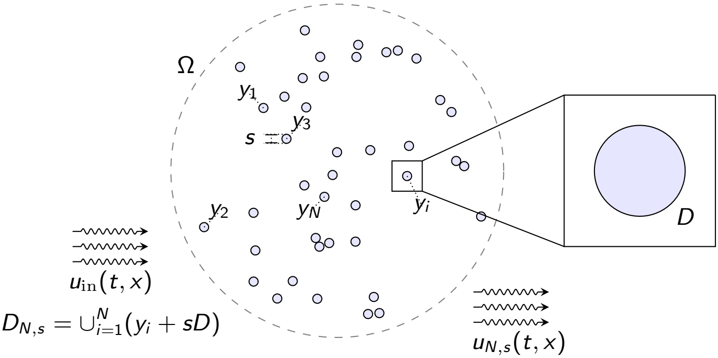
The static case



Material parameters:

- in D: density ρ_r, bulk modulus κ_r

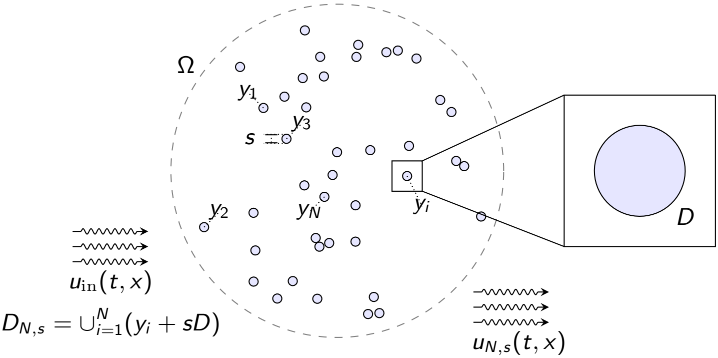
The static case



Material parameters:

- in D: density ρ_r, bulk modulus κ_r
- outside D: density ρ_0, bulk modulus κ_0.

The static case

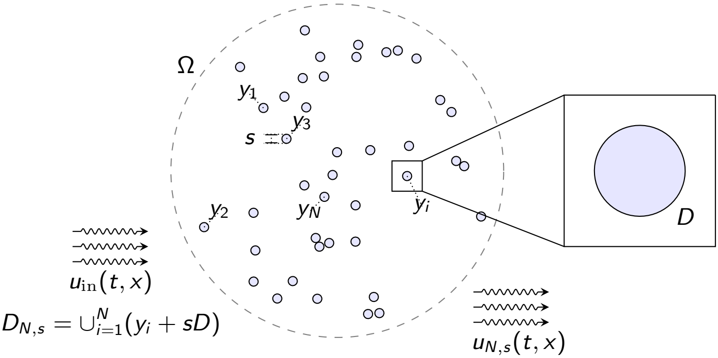


Material parameters:

- in D: density ρ_r, bulk modulus κ_r
- outside D: density ρ_0, bulk modulus κ_0.
- high-contrast regime:

$$\delta := \frac{\rho_r}{\rho_0} \to 0$$

The static case



Material parameters:

- in D: density ρ_r, bulk modulus κ_r
- outside D: density ρ_0, bulk modulus κ_0.
- high-contrast regime:

$$\delta := \frac{\rho_r}{\rho_0} \to 0$$

- Homogenization regime: $N \to +\infty$, $s \to 0$, centers $(y_i)_{1\le i\le N}$ randomly distributed in Ω according to a density $V(x)\mathrm{d}x$.

The static case

- The scattering problem for this high-contrast material:

$$
\begin{cases}
\dfrac{1}{\kappa_0}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_0}\Delta u = 0 \text{ in } \mathbb{R}\times\mathbb{R}^3\backslash\bar{D}_{N,s}, \\
\dfrac{1}{\kappa_r}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_r}\Delta u = 0 \text{ in } \mathbb{R}\times D_{N,s}, \\
\qquad \dfrac{1}{\rho_0}\left.\dfrac{\partial u}{\partial \boldsymbol{n}}\right|_+ = \dfrac{1}{\rho_r}\left.\dfrac{\partial u}{\partial \boldsymbol{n}}\right|_- \quad \text{on } \mathbb{R}\times\partial D_{N,s}, \quad 1\le i\le N, \\
\qquad u_{|+} = u_{|-} \text{ on } \mathbb{R}\times\partial D_{N,s}, \\
\quad u - u_{\text{in}} \text{ is outgoing.}
\end{cases}
$$

The static case

- The scattering problem for this high-contrast material:

$$\begin{cases} \dfrac{1}{\kappa_0}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_0}\Delta u = 0 \text{ in } \mathbb{R}\times\mathbb{R}^3\backslash \bar{D}_{N,s}, \\ \dfrac{1}{\kappa_r}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_r}\Delta u = 0 \text{ in } \mathbb{R}\times D_{N,s}, \\ \qquad \dfrac{1}{\rho_0}\dfrac{\partial u}{\partial \boldsymbol{n}}\bigg|_{+} = \dfrac{1}{\rho_r}\dfrac{\partial u}{\partial \boldsymbol{n}}\bigg|_{-} \quad \text{on } \mathbb{R}\times\partial D_{N,s}, \quad 1 \le i \le N, \\ \qquad u_{|+} = u_{|-} \text{ on } \mathbb{R}\times\partial D_{N,s}, \\ \qquad u - u_{\mathrm{in}} \text{ is outgoing.} \end{cases}$$

- Denote by v_0 and v_r the wave speeds in D and $\mathbb{R}^3\backslash D$:

$$v_0 := \sqrt{\frac{\kappa_0}{\rho_0}}, \qquad v_r = \sqrt{\frac{\kappa_r}{\rho_r}}.$$

The static case

- The scattering problem for this high-contrast material:

$$\begin{cases} \dfrac{1}{\kappa_0}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_0}\Delta u = 0 \text{ in } \mathbb{R}\times\mathbb{R}^3\backslash\bar{D}_{N,s}, \\ \dfrac{1}{\kappa_r}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_r}\Delta u = 0 \text{ in } \mathbb{R}\times D_{N,s}, \\ \qquad \dfrac{1}{\rho_0}\left.\dfrac{\partial u}{\partial \boldsymbol{n}}\right|_+ = \dfrac{1}{\rho_r}\left.\dfrac{\partial u}{\partial \boldsymbol{n}}\right|_- \quad \text{on } \mathbb{R}\times\partial D_{N,s}, \quad 1\leq i\leq N, \\ \qquad\qquad u_{|+} = u_{|-} \text{ on } \mathbb{R}\times\partial D_{N,s}, \\ \qquad u - u_{\text{in}} \text{ is outgoing.} \end{cases}$$

- Denote by v_0 and v_r the wave speeds in D and $\mathbb{R}^3\backslash D$:

$$v_0 := \sqrt{\frac{\kappa_0}{\rho_0}}, \qquad v_r = \sqrt{\frac{\kappa_r}{\rho_r}}.$$

- The incident field $u_{\text{in}}(t,x)$ is time-harmonic with frequency ω:

$$u_{\text{in}}(t,x) = e^{-\mathrm{i}\omega t}\hat{u}_{\text{in}}(x) \text{ with } \Delta\hat{u}_{\text{in}} + \frac{\omega^2}{v_0^2}\hat{u}_{\text{in}} = 0 \text{ in } \mathbb{R}^3.$$

The static case

- The scattering problem for this high-contrast material:

$$\begin{cases} \frac{1}{\kappa_0}\frac{\partial^2 u}{\partial t^2} - \frac{1}{\rho_0}\Delta u = 0 \text{ in } \mathbb{R}\times\mathbb{R}^3\backslash\bar{D}_{N,s}, \\ \frac{1}{\kappa_r}\frac{\partial^2 u}{\partial t^2} - \frac{1}{\rho_r}\Delta u = 0 \text{ in } \mathbb{R}\times D_{N,s}, \\ \qquad \frac{1}{\rho_0}\left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_+ = \frac{1}{\rho_r}\left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_- \quad \text{on } \mathbb{R}\times\partial D_{N,s}, \quad 1\le i\le N, \\ \qquad u_{|+} = u_{|-} \text{ on } \mathbb{R}\times\partial D_{N,s}, \\ \qquad u - u_{\mathrm{in}} \text{ is outgoing.} \end{cases}$$

- Denote by v_0 and v_r the wave speeds in D and $\mathbb{R}^3\backslash D$:

$$v_0 := \sqrt{\frac{\kappa_0}{\rho_0}}, \qquad v_r = \sqrt{\frac{\kappa_r}{\rho_r}}.$$

- The incident field $u_{\mathrm{in}}(t,x)$ is time-harmonic with frequency ω:

$$u_{\mathrm{in}}(t,x) = e^{-\mathrm{i}\omega t}\hat{u}_{\mathrm{in}}(x) \text{ with } \Delta\hat{u}_{\mathrm{in}} + \frac{\omega^2}{v_0^2}\hat{u}_{\mathrm{in}} = 0 \text{ in } \mathbb{R}^3.$$

- The total field is itself time-harmonic, $u(t,x) = e^{-\mathrm{i}\omega t}\hat{u}(x)$ and $\hat{u}$ satisfies the outgoing Sommerfeld Radiation condition $(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0})\hat{u} = O(|x|^{-2})$ as $|x| \to +\infty$.

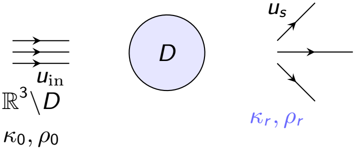
The static case

The key ingredient for understanding the effective physics of the medium is the point scatterer approximation.

The static case

The key ingredient for understanding the effective physics of the medium is the point scatterer approximation.

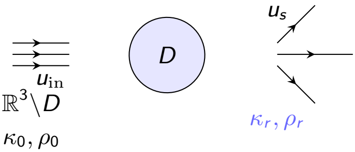
- Consider scattering against a single resonator D, centered at 0:



The static case

The key ingredient for understanding the effective physics of the medium is the point scatterer approximation.

- Consider scattering against a single resonator D, centered at 0:



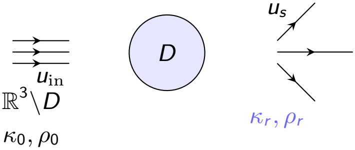
- In this rescaled situation, consider the subwavelength regime

$$\omega \to 0$$

The static case

The key ingredient for understanding the effective physics of the medium is the point scatterer approximation.

- Consider scattering against a single resonator D, centered at 0:



- In this rescaled situation, consider the subwavelength regime

$$\omega \to 0$$

- Two complex subwavelength resonant frequencies:

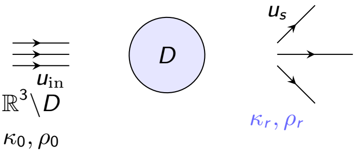
$$\omega^{\pm}(\delta) = v_r \delta^{\frac{1}{2}} \sqrt{\frac{\mathrm{cap}\,(D)}{|D|}} - \frac{\mathrm{i} v_r^2 \mathrm{cap}\,(D)^2}{8\pi v_0 |D|}\delta + O(\delta^{\frac{3}{2}}),$$

with negative imaginary part.

The static case

The key ingredient for understanding the effective physics of the medium is the point scatterer approximation.

- Consider scattering against a single resonator D, centered at 0:



- In this rescaled situation, consider the subwavelength regime

$$\omega \to 0$$

- Two complex subwavelength resonant frequencies:

$$\omega^{\pm}(\delta) = v_r \delta^{\frac{1}{2}} \sqrt{\frac{\mathrm{cap}\,(D)}{|D|}} - \frac{\mathrm{i} v_r^2 \mathrm{cap}\,(D)^2}{8\pi v_0 |D|} \delta + O(\delta^{\frac{3}{2}}),$$

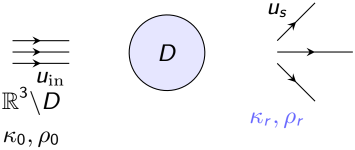
  with negative imaginary part.
- We have the point scatterer (far field) approximation

$$\hat{u}(x) - \hat{u}_{\text{in}}(x) \simeq \frac{\hat{u}_{\text{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega \mathrm{cap}\,(D)}{4\pi v_0}} \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(x) \text{ as } |x| \to +\infty,$$

The static case

The key ingredient for understanding the effective physics of the medium is the point scatterer approximation.

- Consider scattering against a single resonator D, centered at 0:



- In this rescaled situation, consider the subwavelength regime

$$\omega \to 0$$

- Two complex subwavelength resonant frequencies:

$$\omega^{\pm}(\delta) = v_r \delta^{\frac{1}{2}} \sqrt{\frac{\mathrm{cap}\,(D)}{|D|}} - \frac{\mathrm{i} v_r^2 \mathrm{cap}\,(D)^2}{8\pi v_0 |D|} \delta + O(\delta^{\frac{3}{2}}),$$

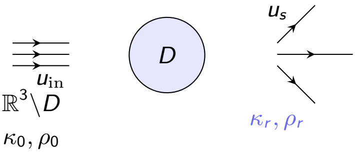
  with negative imaginary part.
- We have the point scatterer (far field) approximation

$$\hat{u}(x) - \hat{u}_{\rm in}(x) \simeq \frac{\hat{u}_{\rm in}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega \mathrm{cap}\,(D)}{4\pi v_0}} \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(x) \text{ as } |x| \to +\infty,$$

The static case

The key ingredient for understanding the effective physics of the medium is the point scatterer approximation.

- Consider scattering against a single resonator D, centered at 0:



- In this rescaled situation, consider the subwavelength regime

$$\omega \to 0$$

- Two complex subwavelength resonant frequencies:

$$\omega^{\pm}(\delta) = v_r \delta^{\frac{1}{2}} \sqrt{\frac{\mathrm{cap}\,(D)}{|D|}} - \frac{\mathrm{i} v_r^2 \mathrm{cap}\,(D)^2}{8\pi v_0 |D|} \delta + O(\delta^{\frac{3}{2}}),$$

  with negative imaginary part.
- We have the point scatterer (far field) approximation

$$\hat{u}(x) - \hat{u}_{\text{in}}(x) \simeq \frac{\hat{u}_{\text{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega \mathrm{cap}\,(D)}{4\pi v_0}} \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(x) \text{ as } |x| \to +\infty,$$

  where $\mathrm{cap}\,(D)$ is the capacity of D and

The static case

- We have the point scatterer (far field) approximation

$$\hat{u}(x) - \hat{u}_{\mathrm{in}}(x) \simeq \frac{\hat{u}_{\mathrm{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega \mathrm{cap}\,(D)}{4\pi v_0}} \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(x) \text{ as } |x| \to +\infty,$$

where $\mathrm{cap}\,(D)$ is the capacity of D and

$$\Gamma^{\frac{\omega}{v_0}}(x) = -\frac{e^{\mathrm{i}\frac{\omega}{v_0}|x|}}{4\pi|x|}. \text{ and } \omega_M := v_r \sqrt{\frac{\mathrm{cap}\,(D)}{|D|}} \delta^{\frac{1}{2}}.$$

The static case

- We have the point scatterer (far field) approximation

$$\hat{u}(x) - \hat{u}_{\mathrm{in}}(x) \simeq \frac{\hat{u}_{\mathrm{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega \mathrm{cap}\,(D)}{4\pi v_0}} \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(x) \text{ as } |x| \to +\infty,$$

where $\mathrm{cap}\,(D)$ is the capacity of D and

$$\Gamma^{\frac{\omega}{v_0}}(x) = -\frac{e^{\mathrm{i}\frac{\omega}{v_0}|x|}}{4\pi|x|}. \text{ and } \omega_M := v_r \sqrt{\frac{\mathrm{cap}\,(D)}{|D|}} \delta^{\frac{1}{2}}.$$

- Field enhancement of order $O(\delta^{-\frac{1}{2}})$ for ω close to ω_M.

The static case

- We have the point scatterer (far field) approximation

$$\hat{u}(x) - \hat{u}_{\mathrm{in}}(x) \simeq \frac{\hat{u}_{\mathrm{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega \mathrm{cap}\,(D)}{4\pi v_0}} \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(x) \text{ as } |x| \to +\infty,$$

where $\mathrm{cap}\,(D)$ is the capacity of D and

$$\Gamma^{\frac{\omega}{v_0}}(x) = -\frac{e^{\mathrm{i}\frac{\omega}{v_0}|x|}}{4\pi|x|}. \text{ and } \omega_M := v_r \sqrt{\frac{\mathrm{cap}\,(D)}{|D|}} \delta^{\frac{1}{2}}.$$

- Field enhancement of order $O(\delta^{-\frac{1}{2}})$ for ω close to ω_M.

The static case

- For a rescaled resonator sD:

$$\operatorname{cap}(sD) = s\operatorname{cap}(D), \qquad |sD| = s^3|D|$$

so ω_M rescales as

$$\omega_M := v_r\sqrt{\frac{\operatorname{cap}(D)}{|D|}}\frac{\delta^{\frac{1}{2}}}{s},$$

The static case

- For a rescaled resonator sD:

$$\mathrm{cap}\,(sD) = s\mathrm{cap}\,(D), \qquad |sD| = s^3|D|$$

so ω_M rescales as

$$\omega_M := v_r \sqrt{\frac{\mathrm{cap}\,(D)}{|D|}} \frac{\delta^{\frac{1}{2}}}{s},$$

The static case

- For a rescaled resonator sD:

$$\operatorname{cap}(sD) = s\operatorname{cap}(D), \qquad |sD| = s^3|D|$$

so ω_M rescales as

$$\omega_M := v_r \sqrt{\frac{\operatorname{cap}(D)}{|D|}}\frac{\delta^{\frac{1}{2}}}{s},$$

i.e. $\omega_M = O(1)$ if $s = O(\delta^{\frac{1}{2}})$.

The static case

- For a rescaled resonator sD:

$$\mathrm{cap}\,(sD) = s\mathrm{cap}\,(D), \qquad |sD| = s^3|D|$$

so ω_M rescales as

$$\omega_M := v_r\sqrt{\frac{\mathrm{cap}\,(D)}{|D|}}\frac{\delta^{\frac{1}{2}}}{s},$$

i.e. $\omega_M = O(1)$ if $s = O(\delta^{\frac{1}{2}})$.

- Foldy-Lax approximation:

$$\text{single scatterer: } \hat{u}(x) - \hat{u}_{\mathrm{in}}(x) \simeq \frac{\hat{u}_{\mathrm{in}}(0)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{\mathrm{i}\omega\mathrm{cap}\,(D)}{4\pi v_0}}\mathrm{cap}\,(D)\Gamma^{\frac{\omega}{v_0}}(x)$$

The static case

- For a rescaled resonator sD:

$$\operatorname{cap}(sD) = s\operatorname{cap}(D), \qquad |sD| = s^3|D|$$

so ω_M rescales as

$$\omega_M := v_r\sqrt{\frac{\operatorname{cap}(D)}{|D|}}\,\frac{\delta^{\frac{1}{2}}}{s},$$

i.e. $\omega_M = O(1)$ if $s = O(\delta^{\frac{1}{2}})$.

- Foldy-Lax approximation:

multiple scatterers: $\hat{u}_{N,s}(y_i) - \hat{u}_{\mathrm{in}}(y_i) \simeq \displaystyle\sum_{1\leq j\neq i\leq N} \frac{\hat{u}_{N,s}(y_j)}{\frac{\omega^2}{\omega_M^2} - 1 + \frac{i\omega s\operatorname{cap}(D)}{4\pi v_0}} s\operatorname{cap}(D)\Gamma^{\frac{\omega}{v_0}}(y_i - y_j)$

The static case

- For a rescaled resonator sD:

$$\mathrm{cap}\,(sD) = s\mathrm{cap}\,(D), \qquad |sD| = s^3|D|$$

so ω_M rescales as

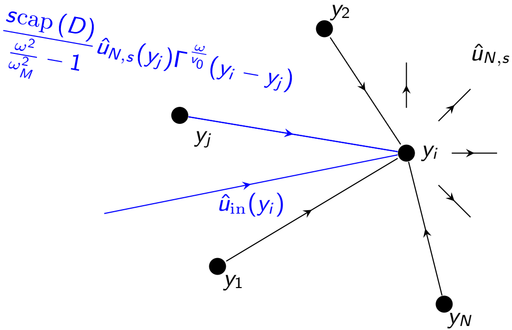
$$\omega_M := v_r\sqrt{\frac{\mathrm{cap}\,(D)}{|D|}}\frac{\delta^{\frac{1}{2}}}{s},$$

i.e. $\omega_M = O(1)$ if $s = O(\delta^{\frac{1}{2}})$.

- Foldy-Lax approximation:

$$\text{multiple scatterers: } \hat{u}_{N,s}(y_i) - \hat{u}_{\mathrm{in}}(y_i) \simeq \sum_{1\leq j\neq i\leq N} \frac{\hat{u}_{N,s}(y_j)}{\frac{\omega^2}{\omega_M^2} - 1} s\mathrm{cap}\,(D)\Gamma^{\frac{\omega}{v_0}}(y_i - y_j)$$

The static case



The static case

- Foldy-Lax approximation:

$$\hat{u}_{N,s}(y_i) - \hat{u}_{\mathrm{in}}(y_i) \simeq \sum_{1 \leq j \neq i \leq N} \frac{\hat{u}_{N,s}(y_j)}{\frac{\omega^2}{\omega_M^2} - 1} s \quad \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(y_i - y_j)$$

The static case

- Foldy-Lax approximation:

$$\hat{u}_{N,s}(y_i) - \hat{u}_{\mathrm{in}}(y_i) \simeq \frac{1}{N} \sum_{1 \le j \ne i \le N} \frac{\hat{u}_{N,s}(y_j)}{\frac{\omega^2}{\omega_M^2} - 1} sN \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}} (y_i - y_j)$$

The static case

- Foldy-Lax approximation:

$$\hat{u}_{N,s}(y_i) - \hat{u}_{\rm in}(y_i) \simeq \frac{1}{N} \sum_{1 \le j \neq i \le N} \frac{\hat{u}_{N,s}(y_j)}{\frac{\omega^2}{\omega_M^2} - 1} sN \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(y_i - y_j)$$

- Law of large numbers: convergence to the integral equation

$$\hat{u}(y) - \hat{u}_{\rm in}(y) = \int_\Omega \frac{\hat{u}(y')}{\frac{\omega^2}{\omega_M^2} - 1} sN \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(y - y') V(y') \mathrm{d}y', \qquad y \in \Omega.$$

The static case

- Foldy-Lax approximation:

$$\hat{u}_{N,s}(y_i) - \hat{u}_{\rm in}(y_i) \simeq \frac{1}{N} \sum_{1 \leq j \neq i \leq N} \frac{\hat{u}_{N,s}(y_j)}{\frac{\omega^2}{\omega_M^2} - 1} sN \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(y_i - y_j)$$

- Law of large numbers: convergence to the integral equation

$$\hat{u}(y) - \hat{u}_{\rm in}(y) = \int_\Omega \frac{\hat{u}(y')}{\frac{\omega^2}{\omega_M^2} - 1} sN \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(y - y') V(y') \mathrm{d}y', \qquad y \in \Omega.$$

- Effective medium:

$$\begin{cases} \Delta \hat{u} + \left(\frac{\omega^2}{v_0^2} - sN \frac{\mathrm{cap}\,(D)}{\frac{\omega^2}{\omega_M^2} - 1} V 1_\Omega \right) u = 0 \text{ in } \mathbb{R}^3, \\ \left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} \right) (\hat{u} - \hat{u}_{\rm in})(x) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases}$$

The static case

- Foldy-Lax approximation:

$$\hat{u}_{N,s}(y_i) - \hat{u}_{\mathrm{in}}(y_i) \simeq \frac{1}{N} \sum_{1\le j\neq i\le N} \frac{\hat{u}_{N,s}(y_j)}{\frac{\omega^2}{\omega_M^2} - 1} sN\mathrm{cap}\,(D)\Gamma^{\frac{\omega}{v_0}}(y_i - y_j)$$

- Law of large numbers: convergence to the integral equation

$$\hat{u}(y) - \hat{u}_{\mathrm{in}}(y) = \int_\Omega \frac{\hat{u}(y')}{\frac{\omega^2}{\omega_M^2} - 1} sN\mathrm{cap}\,(D)\Gamma^{\frac{\omega}{v_0}}(y - y')V(y')\mathrm{d}y', \qquad y \in \Omega.$$

- Effective medium:

$$\begin{cases} \Delta \hat{u} + \left(\frac{\omega^2}{v_0^2} - sN \frac{\mathrm{cap}\,(D)}{\frac{\omega^2}{\omega_M^2} - 1} V 1_\Omega \right) u = 0 \text{ in } \mathbb{R}^3, \\ \left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} \right)(\hat{u} - \hat{u}_{\mathrm{in}})(x) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases}$$

The static case

- Foldy-Lax approximation:

$$\hat{u}_{N,s}(y_i) - \hat{u}_{\rm in}(y_i) \simeq \frac{1}{N} \sum_{1 \le j \ne i \le N} \frac{\hat{u}_{N,s}(y_j)}{\frac{\omega^2}{\omega_M^2} - 1} sN\mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(y_i - y_j)$$

- Law of large numbers: convergence to the integral equation

$$\hat{u}(y) - \hat{u}_{\rm in}(y) = \int_\Omega \frac{\hat{u}(y')}{\frac{\omega^2}{\omega_M^2} - 1} sN\mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(y - y') V(y') \mathrm{d}y', \qquad y \in \Omega.$$

- Effective medium:

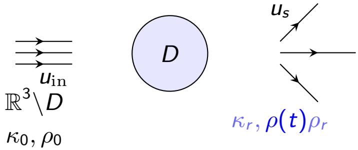
$$\begin{cases} \Delta \hat{u} + \left(\frac{\omega^2}{v_0^2} - sN \frac{\mathrm{cap}\,(D)}{\frac{\omega^2}{\omega_M^2} - 1} V 1_\Omega \right) u = 0 \text{ in } \mathbb{R}^3, \\ \left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} \right) (\hat{u} - \hat{u}_{\rm in})(x) = O(|x|^{-2}) \text{ as } |x| \to +\infty. \end{cases}$$

Dissipative when $\omega > \omega_M$, dispersive if $\omega < \omega_M$.

Outline

1. The unmodulated case: point scatterer approximations and effective medium theory
2. Resonances in the static case and in the time-modulated case
3. Point scatterer approximation and effective medium in the time-modulated case.

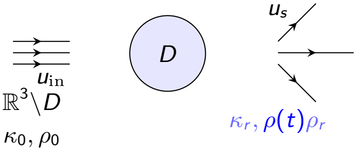
Consider scattering against a single time-modulated resonator D, centered at 0:



$$
\begin{cases}
\dfrac{1}{\kappa_0}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_0}\Delta u = 0 \text{ in } \mathbb{R} \times \mathbb{R}^3 \backslash \bar{D}, \\
\dfrac{1}{\kappa_r}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho(t)\rho_r}\Delta u = 0 \text{ in } \mathbb{R} \times D, \\
\dfrac{1}{\rho_0}\dfrac{\partial u}{\partial \boldsymbol{n}}\bigg|_+ = \dfrac{1}{\rho_r \rho(t)}\dfrac{\partial u}{\partial \boldsymbol{n}}\bigg|_- \text{ on } \mathbb{R} \times \partial D, \quad 1 \leq i \leq N, \\
u_{|+} = u_{|-} \text{ on } \mathbb{R} \times \partial D, \\
u - u_{\text{in}} \text{ is outgoing.}
\end{cases}
$$

Time-modulated metamaterials

Consider scattering against a single time-modulated resonator D, centered at 0:



$$
\begin{cases}
\dfrac{1}{\kappa_0}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho_0}\Delta u = 0 \text{ in } \mathbb{R}\times\mathbb{R}^3\backslash\bar{D}, \\
\dfrac{1}{\kappa_r}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{1}{\rho(t)\rho_r}\Delta u = 0 \text{ in } \mathbb{R}\times D, \\
\dfrac{1}{\rho_0}\dfrac{\partial u}{\partial \boldsymbol{n}}\bigg|_+ = \dfrac{1}{\rho_r\rho(t)}\dfrac{\partial u}{\partial \boldsymbol{n}}\bigg|_- \quad \text{on } \mathbb{R}\times\partial D, \quad 1\le i\le N, \\
u_{|+} = u_{|-} \text{ on } \mathbb{R}\times\partial D, \\
u - u_{\text{in}} \text{ is outgoing.}
\end{cases}
$$

ρ is periodic of period T: $\rho(t+T) = \rho(t)$ and

$$\Omega := \frac{2\pi}{T} = O(1).$$

Time-modulated metamaterials

We still assume that u_{in} is time harmonic:

$$u_{\mathrm{in}}(t, x) = e^{-\mathrm{i}\omega t}\hat{u}_{\mathrm{in}}(x),$$

and the regimes $\omega \to 0$, $\delta = \rho_r/\rho_0 \to 0$.

We still assume that u_{in} is time harmonic:

$$u_{\text{in}}(t,x) = e^{-\mathrm{i}\omega t}\hat{u}_{\text{in}}(x),$$

and the regimes $\omega \to 0$, $\delta = \rho_r/\rho_0 \to 0$.
Question: what does "$u - u_{\text{in}}$ **outgoing**" mean ?

We still assume that u_{in} is time harmonic:

$$u_{\mathrm{in}}(t,x) = e^{-\mathrm{i}\omega t}\hat{u}_{\mathrm{in}}(x),$$

and the regimes $\omega \to 0$, $\delta = \rho_r/\rho_0 \to 0$.
Question: what does "$u - u_{\mathrm{in}}$ **outgoing**" mean ?
Seek u in the form of

$$u(t,x) = e^{-\mathrm{i}\omega t}\hat{u}(t,x) \text{ with } t \mapsto u(t,x) \ T\text{–periodic}.$$

We still assume that u_{in} is time harmonic:

$$u_{\mathrm{in}}(t,x) = e^{-\mathrm{i}\omega t}\hat{u}_{\mathrm{in}}(x),$$

and the regimes $\omega \to 0$, $\delta = \rho_r/\rho_0 \to 0$.
Question: what does "$u - u_{\mathrm{in}}$ **outgoing**" mean ?
Seek u in the form of

$$u(t,x) = e^{-\mathrm{i}\omega t}\hat{u}(t,x) \text{ with } t \mapsto u(t,x) \ T\text{–periodic.}$$

This can be justified with the Bloch transform.

Time-modulated metamaterials

$\hat{u}$ satisfies then

$$\begin{cases} \dfrac{1}{\kappa_0}\left(-\mathrm{i}\omega + \partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho_0}\Delta \hat{u}(t,x) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \backslash \bar{D}, \\ \dfrac{1}{\kappa_r}\left(-\mathrm{i}\omega + \partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho(t)\rho_r}\Delta \hat{u}(t,x) = 0, & (t,x) \in \mathbb{R} \times D, \\ \dfrac{1}{\rho_0} \left.\dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+ = \dfrac{1}{\rho_r \rho(t)} \left.\dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+, & (t,x) \in \mathbb{R} \times \partial D, \\ \hat{u}_{|+}(t,x) = \hat{u}_{|-}(t,x), & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto \hat{u}(t,x) \text{ is } T\text{–periodic}, \\ e^{-\mathrm{i}\omega t}(\hat{u}(t,x) - \hat{u}_{\mathrm{in}}(t,x)) \text{ is outgoing.} \end{cases}$$

$\hat{u}$ satisfies then

$$\begin{cases} \dfrac{1}{\kappa_0}\left(-\mathrm{i}\omega+\partial_t\right)^2\hat{u}(t,x)-\dfrac{1}{\rho_0}\Delta\hat{u}(t,x)=0, & (t,x)\in\mathbb{R}\times\mathbb{R}^3\backslash\bar{D}, \\ \dfrac{1}{\kappa_r}\left(-\mathrm{i}\omega+\partial_t\right)^2\hat{u}(t,x)-\dfrac{1}{\rho(t)\rho_r}\Delta\hat{u}(t,x)=0, & (t,x)\in\mathbb{R}\times D, \\ \dfrac{1}{\rho_0}\left.\dfrac{\partial\hat{u}(t,x)}{\partial\boldsymbol{n}}\right|_+=\dfrac{1}{\rho_r\rho(t)}\left.\dfrac{\partial\hat{u}(t,x)}{\partial\boldsymbol{n}}\right|_+, & (t,x)\in\mathbb{R}\times\partial D, \\ \hat{u}_{|+}(t,x)=\hat{u}_{|-}(t,x), & (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto\hat{u}(t,x)\text{ is }T\text{–periodic}, \\ e^{-\mathrm{i}\omega t}(\hat{u}(t,x)-\hat{u}_{\mathrm{in}}(t,x))\text{ is outgoing.} \end{cases}$$

We can show that the “right” outgoing radiation condition for $\hat{u}$ is

$$\left(\partial_{|x|}-\frac{\mathrm{i}\omega}{v_0}+\frac{1}{v_0}\partial_t\right)\hat{u}(t,x)=O(|x|^{-2}).$$

Time-modulated metamaterials

$\hat{u}$ satisfies then

$$\begin{cases} \dfrac{1}{\kappa_0}\left(-\mathrm{i}\omega + \partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho_0}\Delta \hat{u}(t,x) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \backslash \bar{D}, \\ \dfrac{1}{\kappa_r}\left(-\mathrm{i}\omega + \partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho(t)\rho_r}\Delta \hat{u}(t,x) = 0, & (t,x) \in \mathbb{R} \times D, \\ \dfrac{1}{\rho_0} \left.\dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+ = \dfrac{1}{\rho_r \rho(t)} \left.\dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+, & (t,x) \in \mathbb{R} \times \partial D, \\ \hat{u}_{|+}(t,x) = \hat{u}_{|-}(t,x), & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto \hat{u}(t,x) \text{ is } T\text{-periodic}, \\ e^{-\mathrm{i}\omega t}(\hat{u}(t,x) - \hat{u}_{\mathrm{in}}(t,x)) \text{ is outgoing.} \end{cases}$$

We can show that the "right" outgoing radiation condition for $\hat{u}$ is

$$\left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} + \frac{1}{v_0}\partial_t\right)\hat{u}(t,x) = O(|x|^{-2}).$$

Equivalent to say that all Fourier modes are outgoing.

$\hat{u}$ satisfies then

$$\begin{cases} \dfrac{1}{\kappa_0}\left(-\mathrm{i}\omega+\partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho_0}\Delta \hat{u}(t,x) = 0, & (t,x)\in \mathbb{R}\times\mathbb{R}^3\backslash\bar{D}, \\ \dfrac{1}{\kappa_r}\left(-\mathrm{i}\omega+\partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho(t)\rho_r}\Delta \hat{u}(t,x) = 0, & (t,x)\in \mathbb{R}\times D, \\ \delta \left.\dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+ = \dfrac{1}{\rho(t)} \left.\dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+, & (t,x)\in \mathbb{R}\times\partial D, \\ \hat{u}_{|+}(t,x) = \hat{u}_{|-}(t,x), & (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto \hat{u}(t,x) \text{ is } T\text{–periodic}, & \\ e^{-\mathrm{i}\omega t}(\hat{u}(t,x)-\hat{u}_{\mathrm{in}}(t,x)) \text{ is outgoing}. & \end{cases}$$

We can show that the “right” outgoing radiation condition for $\hat{u}$ is

$$\left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} + \frac{1}{v_0}\partial_t\right)\hat{u}(t,x) = O(|x|^{-2}).$$

Equivalent to say that all Fourier modes are outgoing.

To analyze resonances, it is useful to introduce the Dirichlet-to-Neumann operator. For a T−periodic Dirichlet datum $f(t,x)$, consider the solution $w_f(t,x)$ to

$$\begin{cases} \frac{1}{v_0^2}\left(-\mathrm{i}\omega + \partial_t\right)^2 w_f(t,x) - \Delta w_f(t,x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^3\backslash\bar{D}, \\ w_f(t,x) = f(t,x), & (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto w_f(t,x) \text{ is } T\text{−periodic} & \\ \left(\partial_{|x|} - \frac{\mathrm{i}\omega}{v_0} + \frac{1}{v_0}\partial_t\right) w_f(t,x) = O(|x|^{-2}) \text{ as } |x|\to+\infty. & \end{cases}$$

To analyze resonances, it is useful to introduce the Dirichlet-to-Neumann operator. For a T–periodic Dirichlet datum $f(t,x)$, consider the solution $w_f(t,x)$ to

$$\begin{cases} \dfrac{1}{v_0^2}\left(-\mathrm{i}\omega + \partial_t\right)^2 w_f(t,x) - \Delta w_f(t,x) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \backslash \bar{D}, \\ w_f(t,x) = f(t,x), & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto w_f(t,x) \text{ is } T\text{–periodic} & \\ \left(\partial_{|x|} - \dfrac{\mathrm{i}\omega}{v_0} + \dfrac{1}{v_0}\partial_t\right) w_f(t,x) = O(|x|^{-2}) \text{ as } |x| \to +\infty. & \end{cases}$$

The Dirichlet-to-Neumann map is the operator $\mathcal{T}^\omega$ defined by

$$\mathcal{T}^\omega[f] = \frac{\partial w_f}{\partial \boldsymbol{n}} \text{ on } \partial D.$$

Time-modulated metamaterials

To analyze resonances, it is useful to introduce the Dirichlet-to-Neumann operator. For a T−periodic Dirichlet datum $f(t,x)$, consider the solution $w_f(t,x)$ to

$$\begin{cases} \dfrac{1}{v_0^2}\left(-\mathrm{i}\omega+\partial_t\right)^2 w_f(t,x) - \Delta w_f(t,x) = 0, & (t,x)\in\mathbb{R}\times\mathbb{R}^3\backslash\bar{D}, \\ w_f(t,x) = f(t,x), & (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto w_f(t,x) \text{ is } T\text{−periodic} & \\ \left(\partial_{|x|} - \dfrac{\mathrm{i}\omega}{v_0} + \dfrac{1}{v_0}\partial_t\right) w_f(t,x) = O(|x|^{-2}) \text{ as } |x|\to+\infty. & \end{cases}$$

The Dirichlet-to-Neumann map is the operator $\mathcal{T}^\omega$ defined by

$$\mathcal{T}^\omega[f] = \frac{\partial w_f}{\partial \boldsymbol{n}} \text{ on } \partial D.$$

Static case retrieved by removing the ∂_t term.

The scattering problem can be rephrased in D only thanks to $\mathcal{T}^{\omega}$:

$$\frac{1}{\rho(t)} \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_{-}$$

Time-modulated metamaterials

The scattering problem can be rephrased in D only thanks to $\mathcal{T}^{\omega}$:

$$\frac{1}{\rho(t)}\left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_{-}=\delta\left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_{+}$$

Time-modulated metamaterials

The scattering problem can be rephrased in D only thanks to $\mathcal{T}^\omega$:

$$\frac{1}{\rho(t)} \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_- = \delta \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+ = \delta \left.\frac{\partial (\hat{u} - \hat{u}_{\mathrm{in}})}{\partial \boldsymbol{n}}\right|_+ + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}}$$

Time-modulated metamaterials

The scattering problem can be rephrased in D only thanks to $\mathcal{T}^\omega$:

$$\frac{1}{\rho(t)} \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_{-} = \delta \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_{+} = \delta \left.\frac{\partial(\hat{u} - \hat{u}_{\mathrm{in}})}{\partial \boldsymbol{n}}\right|_{+} + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} = \delta \mathcal{T}^\omega[(\hat{u} - \hat{u}_{\mathrm{in}})] + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}}.$$

The scattering problem can be rephrased in D only thanks to $\mathcal{T}^{\omega}$:

$$\frac{1}{\rho(t)}\left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_{-} = \delta \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_{+} = \delta \left.\frac{\partial(\hat{u}-\hat{u}_{\mathrm{in}})}{\partial \boldsymbol{n}}\right|_{+} + \delta\frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} = \delta\mathcal{T}^{\omega}[(\hat{u}-\hat{u}_{\mathrm{in}})] + \delta\frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}}.$$

Therefore $\hat{u}$ satisfies:

$$\begin{cases} \dfrac{1}{v_r^2}\left(-\mathrm{i}\omega+\partial_t\right)^2\hat{u}(t,x) - \dfrac{1}{\rho(t)}\Delta\hat{u}(t,x) = 0, & (t,x)\in\mathbb{R}\times D, \\ \dfrac{1}{\rho(t)}\dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}} - \delta\mathcal{T}^{\omega}[\hat{u}(t,x)] = \delta\left(\dfrac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} - \mathcal{T}^{\omega}[\hat{u}_{\mathrm{in}}]\right), & (t,x)\in\mathbb{R}\times\partial D, \\ t\mapsto \hat{u}(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

The scattering problem can be rephrased in D only thanks to $\mathcal{T}^\omega$:

$$\frac{1}{\rho(t)} \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_- = \delta \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+ = \delta \left.\frac{\partial (\hat{u} - \hat{u}_{\mathrm{in}})}{\partial \boldsymbol{n}}\right|_+ + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} = \delta \mathcal{T}^\omega[(\hat{u} - \hat{u}_{\mathrm{in}})] + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}}.$$

Therefore $\hat{u}$ satisfies:

$$\begin{cases} \dfrac{1}{v_r^2} \left(-\mathrm{i}\omega + \partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho(t)} \Delta \hat{u}(t,x) = 0, & (t,x) \in \mathbb{R} \times D, \\ \dfrac{1}{\rho(t)} \dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}} - \delta \mathcal{T}^\omega[\hat{u}(t,x)] = \delta \left(\dfrac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} - \mathcal{T}^\omega[\hat{u}_{\mathrm{in}}] \right), & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto \hat{u}(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

Resonances: complex values $\omega(\delta)$ for which the above admits a non-zero solution with $\hat{u}_{\mathrm{in}} = 0$.

The scattering problem can be rephrased in D only thanks to $\mathcal{T}^\omega$:

$$\frac{1}{\rho(t)} \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_- = \delta \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+ = \delta \left.\frac{\partial(\hat{u}-\hat{u}_{\mathrm{in}})}{\partial \boldsymbol{n}}\right|_+ + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} = \delta \mathcal{T}^\omega[(\hat{u}-\hat{u}_{\mathrm{in}})] + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}}.$$

Therefore $\hat{u}$ satisfies:

$$\begin{cases} \dfrac{1}{v_r^2}\left(-\mathrm{i}\omega + \partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho(t)}\Delta \hat{u}(t,x) = 0, & (t,x) \in \mathbb{R}\times D, \\ \dfrac{1}{\rho(t)}\dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}} - \delta \mathcal{T}^\omega[\hat{u}(t,x)] = \delta\left(\dfrac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} - \mathcal{T}^\omega[\hat{u}_{\mathrm{in}}]\right), & (t,x)\in \mathbb{R}\times\partial D, \\ t \mapsto \hat{u}(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

Resonances: complex values $\omega(\delta)$ for which the above admits a non-zero solution with $\hat{u}_{\mathrm{in}} = 0$.
Equivalently,

$$A(\omega,\delta)\hat{u} = F(\hat{u}_{\mathrm{in}}) \Rightarrow \hat{u} = A(\omega,\delta)^{-1}F(\hat{u}_{\mathrm{in}}).$$

The scattering problem can be rephrased in D only thanks to $\mathcal{T}^\omega$:

$$\frac{1}{\rho(t)} \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_- = \delta \left.\frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}}\right|_+ = \delta \left.\frac{\partial (\hat{u} - \hat{u}_{\mathrm{in}})}{\partial \boldsymbol{n}}\right|_+ + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} = \delta \mathcal{T}^\omega[(\hat{u} - \hat{u}_{\mathrm{in}})] + \delta \frac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}}.$$

Therefore $\hat{u}$ satisfies:

$$\begin{cases} \dfrac{1}{v_r^2}\left(-\mathrm{i}\omega + \partial_t\right)^2 \hat{u}(t,x) - \dfrac{1}{\rho(t)}\Delta \hat{u}(t,x) = 0, & (t,x) \in \mathbb{R} \times D, \\ \dfrac{1}{\rho(t)} \dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}} - \delta \mathcal{T}^\omega[\hat{u}(t,x)] = \delta \left(\dfrac{\partial \hat{u}_{\mathrm{in}}}{\partial \boldsymbol{n}} - \mathcal{T}^\omega[\hat{u}_{\mathrm{in}}] \right), & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto \hat{u}(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

Resonances: complex values $\omega(\delta)$ for which the above admits a non-zero solution with $\hat{u}_{\mathrm{in}} = 0$.

Equivalently,

$$A(\omega,\delta)\hat{u} = F(\hat{u}_{\mathrm{in}}) \Rightarrow \hat{u} = A(\omega,\delta)^{-1} F(\hat{u}_{\mathrm{in}}).$$

Resonances are the poles of $A(\omega,\delta)$, yield amplification for $\omega \simeq \omega(\delta)$.

Set $\delta = 0$ and $\omega = 0$:

$$\begin{cases} \frac{1}{v_r^2}\partial_t^2 \hat{u} - \frac{1}{\rho(t)}\Delta \hat{u} = 0, & (t,x) \in \mathbb{R} \times D \\ \frac{1}{\rho(t)} \frac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}} = 0, & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto \hat{u}(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

Set $\delta = 0$ and $\omega = 0$:

$$\begin{cases} \dfrac{1}{v_r^2}\partial_t^2 \hat{u} - \dfrac{1}{\rho(t)}\Delta \hat{u} = 0, & (t,x) \in \mathbb{R} \times D \\ \dfrac{1}{\rho(t)} \dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}} = 0, & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto \hat{u}(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

Separation of variables shows that $\hat{u}(t,x) = p_m(t)\phi_l(x)$ for $p_m(t)$ and $\phi_l(x)$ solutions to the eigenvalue problems

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} p_m(t) = \dfrac{\mu_m}{\rho(t)} p_m(t), \\ p_m \text{ is } T\text{–periodic.} \end{cases} \text{ and } \begin{cases} -\Delta\phi_l = \lambda_l \phi_l \text{ in } D, \\ \dfrac{\partial \phi_l}{\partial \boldsymbol{n}} = 0 \text{ on } \partial D, \end{cases} \qquad l \in \mathbb{N}.$$

Time-modulated metamaterials

Set $\delta = 0$ and $\omega = 0$:

$$\begin{cases} \dfrac{1}{v_r^2}\partial_t^2 \hat{u} - \dfrac{1}{\rho(t)}\Delta \hat{u} = 0, & (t,x) \in \mathbb{R} \times D \\ \dfrac{1}{\rho(t)} \dfrac{\partial \hat{u}(t,x)}{\partial \boldsymbol{n}} = 0, & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto \hat{u}(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

Separation of variables shows that $\hat{u}(t,x) = p_m(t)\phi_l(x)$ for $p_m(t)$ and $\phi_l(x)$ solutions to the eigenvalue problems

$$\begin{cases} -\dfrac{\mathrm{d}^2}{\mathrm{d}t^2} p_m(t) = \dfrac{\mu_m}{\rho(t)} p_m(t), \\ p_m \text{ is } T\text{–periodic.} \end{cases} \quad \text{and} \quad \begin{cases} -\Delta \phi_l = \lambda_l \phi_l \text{ in } D, \\ \dfrac{\partial \phi_l}{\partial \boldsymbol{n}} = 0 \text{ on } \partial D, \end{cases} \qquad l \in \mathbb{N}.$$

at the condition that the Sturm-Liouville and Neumann eigenvalue coincide!

$$\frac{\mu_m}{v_r^2} = \lambda_l.$$

Time-modulated metamaterials

Set

$$\Lambda := \{(l, m) \in \mathbb{N} \times \mathbb{N} \mid \frac{\mu_m}{v_r^2} = \lambda_l\}.$$

- In general $\Lambda = \{(0,0)\}$ associated to $\mu_0/v_r^2 = 0 = \lambda_0$ and constant p_0, ϕ_0.

Time-modulated metamaterials

Set

$$\Lambda := \{(l, m) \in \mathbb{N} \times \mathbb{N} \,|\, \frac{\mu_m}{v_r^2} = \lambda_l\}.$$

- In general $\Lambda = \{(0,0)\}$ associated to $\mu_0/v_r^2 = 0 = \lambda_0$ and constant p_0, ϕ_0.
- If $\rho(t)$ is well tuned, then it is possible that $\Lambda \neq \{(0,0)\}$.

Time-modulated metamaterials

Set

$$\Lambda := \{(l, m) \in \mathbb{N} \times \mathbb{N} \mid \frac{\mu_m}{v_r^2} = \lambda_l\}.$$

- In general $\Lambda = \{(0,0)\}$ associated to $\mu_0/v_r^2 = 0 = \lambda_0$ and constant p_0, ϕ_0.
- If $\rho(t)$ is well tuned, then it is possible that $\Lambda \neq \{(0,0)\}$.
- If $\Lambda = \{(0,0)\}$: situation analogous to the static case, the modulation $\rho(t)$ is averaged.

Time-modulated metamaterials

Set

$$\Lambda := \{(l,m) \in \mathbb{N}\times\mathbb{N} \,|\, \frac{\mu_m}{v_r^2} = \lambda_l\}.$$

- In general $\Lambda = \{(0,0)\}$ associated to $\mu_0/v_r^2 = 0 = \lambda_0$ and constant p_0, ϕ_0.
- If $\rho(t)$ is well tuned, then it is possible that $\Lambda \neq \{(0,0)\}$.
- If $\Lambda = \{(0,0)\}$: situation analogous to the static case, the modulation $\rho(t)$ is averaged.
- Assume $\Lambda = \{(0,0),(l,m)\}$ for some $(l,m) \neq (0,0)$. Then, by perturbation, one can construct **two** oscillating modes

$$v(t,x) \simeq \alpha_{0,0}(\omega,\delta) + \alpha_{l,m}(\omega,\delta) p_m(t)\phi_l(x)$$

Time-modulated metamaterials

Set

$$\Lambda := \{(l, m) \in \mathbb{N} \times \mathbb{N} \mid \frac{\mu_m}{v_r^2} = \lambda_l\}.$$

- In general $\Lambda = \{(0, 0)\}$ associated to $\mu_0/v_r^2 = 0 = \lambda_0$ and constant p_0, ϕ_0.
- If $\rho(t)$ is well tuned, then it is possible that $\Lambda \neq \{(0, 0)\}$.
- If $\Lambda = \{(0, 0)\}$: situation analogous to the static case, the modulation $\rho(t)$ is averaged.
- Assume $\Lambda = \{(0, 0), (l, m)\}$ for some $(l, m) \neq (0, 0)$. Then, by perturbation, one can construct **two** oscillating modes

$$v(t, x) \simeq \alpha_{0,0}(\omega, \delta) + \alpha_{l,m}(\omega, \delta) p_m(t) \phi_l(x)$$

Time-modulated metamaterials

Set

$$\Lambda := \{(l,m) \in \mathbb{N} \times \mathbb{N} \,|\, \frac{\mu_m}{v_r^2} = \lambda_l\}.$$

- In general $\Lambda = \{(0,0)\}$ associated to $\mu_0/v_r^2 = 0 = \lambda_0$ and constant p_0, ϕ_0.
- If $\rho(t)$ is well tuned, then it is possible that $\Lambda \neq \{(0,0)\}$.
- If $\Lambda = \{(0,0)\}$: situation analogous to the static case, the modulation $\rho(t)$ is averaged.
- Assume $\Lambda = \{(0,0),(l,m)\}$ for some $(l,m) \neq (0,0)$. Then, by perturbation, one can construct **two** oscillating modes

$$v(t,x) \simeq \alpha_{0,0}(\omega,\delta) + \alpha_{l,m}(\omega,\delta) p_m(t) \phi_l(x)$$

such that

$$\begin{cases} \frac{1}{v_r^2}\left(-\mathrm{i}\omega + \partial_t\right)^2 v(t,x) - \frac{1}{\rho(t)}\Delta v(t,x) = 0, & (t,x) \in \mathbb{R} \times D, \\ \frac{1}{\rho(t)} \frac{\partial v(t,x)}{\partial \boldsymbol{n}} - \delta \mathcal{T}^{\omega}[v(t,x)] = 0, & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto v(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

Time-modulated metamaterials

Set

$$\Lambda := \{(l, m) \in \mathbb{N} \times \mathbb{N} \mid \frac{\mu_m}{v_r^2} = \lambda_l\}.$$

- In general $\Lambda = \{(0,0)\}$ associated to $\mu_0/v_r^2 = 0 = \lambda_0$ and constant p_0, ϕ_0.
- If $\rho(t)$ is well tuned, then it is possible that $\Lambda \neq \{(0,0)\}$.
- If $\Lambda = \{(0,0)\}$: situation analogous to the static case, the modulation $\rho(t)$ is averaged.
- Assume $\Lambda = \{(0,0),(l,m)\}$ for some $(l,m) \neq (0,0)$. Then, by perturbation, one can construct **two** oscillating modes

$$v(t,x) \simeq \alpha_{0,0}(\omega,\delta) + \alpha_{l,m}(\omega,\delta) p_m(t) \phi_l(x)$$

such that

$$\begin{cases} \dfrac{1}{v_r^2}\left(-\mathrm{i}\omega + \partial_t\right)^2 v(t,x) - \dfrac{1}{\rho(t)} \Delta v(t,x) = 0, & (t,x) \in \mathbb{R} \times D, \\ \dfrac{1}{\rho(t)} \dfrac{\partial v(t,x)}{\partial \boldsymbol{n}} - \delta \mathcal{T}^{\omega}[v(t,x)] = 0, & (t,x) \in \mathbb{R} \times \partial D, \\ t \mapsto v(t,x) \text{ is } T\text{–periodic.} \end{cases}$$

for some complex resonant frequencies $\omega \equiv \omega_i^{\pm}(\delta)$, $i = 1,2$, satisfying $\omega_i^{\pm}(\delta) \to 0$ as $\delta \to 0$.

Proposition 1

The leading asymptotic of the subwavelength resonances $\omega_i^{\pm}(\delta)$ *satisfy, to the first order:*

$$\omega_i^{\pm}(\delta) \sim \pm v_r \delta^{\frac{1}{2}} \lambda_i^{\frac{1}{2}}, \qquad 1 \leq i \leq \#\Lambda,$$

where $(\lambda_i)_{1 \leq i \leq \#\Lambda}$ *are the (complex) eigenvalues of the generalized eigenvalue problem*

$$T\boldsymbol{a}_i + \lambda_i G\boldsymbol{a}_i = 0, \qquad G = \mathrm{diag}(\gamma_m)_{(m,l)\in\Lambda}.$$

Proposition 1

The leading asymptotic of the subwavelength resonances $\omega_i^{\pm}(\delta)$ *satisfy, to the first order:*

$$\omega_i^{\pm}(\delta) \sim \pm v_r \delta^{\frac{1}{2}} \lambda_i^{\frac{1}{2}}, \qquad 1 \leq i \leq \#\Lambda,$$

where $(\lambda_i)_{1 \leq i \leq \#\Lambda}$ *are the (complex) eigenvalues of the generalized eigenvalue problem*

$$T\boldsymbol{a}_i + \lambda_i G \boldsymbol{a}_i = 0, \qquad G = \mathrm{diag}(\gamma_m)_{(m,l)\in\Lambda}.$$

$T \equiv (T_{ml,m'l'})_{(m,l),(m',l')\in\Lambda\times\Lambda}$ *is the (real) matrix*

$$T_{ml,m'l'} := \int_0^T \int_{\partial D} \mathcal{T}^0[p_m \phi_l] p_{m'}(t) \phi_{l'}(x) \mathrm{d}t \mathrm{d}\sigma(x).$$

Proposition 1

The leading asymptotic of the subwavelength resonances $\omega_i^{\pm}(\delta)$ satisfy, to the first order:

$$\omega_i^{\pm}(\delta) \sim \pm v_r \delta^{\frac{1}{2}} \lambda_i^{\frac{1}{2}}, \qquad 1 \leq i \leq \#\Lambda,$$

where $(\lambda_i)_{1 \leq i \leq \#\Lambda}$ are the (complex) eigenvalues of the generalized eigenvalue problem

$$T\boldsymbol{a}_i + \lambda_i G \boldsymbol{a}_i = 0, \qquad G = \mathrm{diag}(\gamma_m)_{(m,l)\in\Lambda}.$$

$T \equiv (T_{ml,m'l'})_{(m,l),(m',l')\in\Lambda\times\Lambda}$ is the (real) matrix

$$T_{ml,m'l'} := \int_0^T \int_{\partial D} \mathcal{T}^0[p_m \phi_l] p_{m'}(t) \phi_{l'}(x) \mathrm{d}t \mathrm{d}\sigma(x).$$

and $(\gamma_m)_{1 \leq m \leq \#\Lambda}$ are the numbers

$$\gamma_m := \int_0^T \left(-2 \frac{\mathrm{d}}{\mathrm{d}t} p_m^1 p_m + |p_m(t)|^2 \right) \mathrm{d}t$$

Proposition 1

The leading asymptotic of the subwavelength resonances $\omega_i^{\pm}(\delta)$ satisfy, to the first order:

$$\omega_i^{\pm}(\delta) \sim \pm v_r \delta^{\frac{1}{2}} \lambda_i^{\frac{1}{2}}, \qquad 1 \leq i \leq \#\Lambda,$$

where $(\lambda_i)_{1 \leq i \leq \#\Lambda}$ are the (complex) eigenvalues of the generalized eigenvalue problem

$$T\boldsymbol{a}_i + \lambda_i G \boldsymbol{a}_i = 0, \qquad G = \mathrm{diag}(\gamma_m)_{(m,l)\in\Lambda}.$$

$T \equiv (T_{ml,m'l'})_{(m,l),(m',l')\in\Lambda\times\Lambda}$ is the (real) matrix

$$T_{ml,m'l'} := \int_0^T \int_{\partial D} \mathcal{T}^0[p_m \phi_l] p_{m'}(t) \phi_{l'}(x) \mathrm{d}t \mathrm{d}\sigma(x).$$

and $(\gamma_m)_{1 \leq m \leq \#\Lambda}$ are the numbers

$$\gamma_m := \int_0^T \left(-2 \frac{\mathrm{d}}{\mathrm{d}t} p_m^1 p_m + |p_m(t)|^2 \right) \mathrm{d}t$$

where p_m^1 is the unique solution to the ODE

$$\begin{cases} -\dfrac{\mathrm{d}^2 p_m^1}{\mathrm{d}t^2} - \dfrac{\mu_m}{\rho(t)} p_m^1 = -2 \dfrac{\mathrm{d}p_m}{\mathrm{d}t}, \\ p_m^1 \text{ is } T\text{–periodic}, \\ \displaystyle\int_0^T \frac{1}{\rho(t)} p_m^1(t) p_m(t) \mathrm{d}t = 0. \end{cases}$$

- T and $(\gamma_m)_{(m,l)\in\Lambda}$ have no distinguished signs if $\Lambda \neq \{(0,0)\}$.

- T and $(\gamma_m)_{(m,l)\in\Lambda}$ have no distinguished signs if $\Lambda \neq \{(0,0)\}$.
- As a consequence, the eigenvalues λ_i are in general complex, and one of the resonant frequencies, say $\omega_i^+(\delta)$, **has positive imaginary part** .

- T and $(\gamma_m)_{(m,l)\in\Lambda}$ have no distinguished signs if $\Lambda \neq \{(0,0)\}$.
- As a consequence, the eigenvalues λ_i are in general complex, and one of the resonant frequencies, say $\omega_i^+(\delta)$, **has positive imaginary part** .
- This means that $e^{-\mathrm{i}\omega_i^+(\delta)t}v(t,x)$ is an **outgoing, and exponentially growing solution to the scattering problem.**

In any case, the scattered field contains high frequency components.

Proposition 2

Assume $\Lambda = \{(0,0),(l,m)\}$. *We have the following asymptotic expansion for* $\hat{u}(t,x)$ *inside* D*:*

$$\hat{u}(t,x) = \frac{\hat{u}_{\mathrm{in}}(0)}{g(\omega,\delta)}\left[-\left(\frac{\omega^2}{v_r^2\delta}\gamma_m + T_{ml,ml}\right)1_D + \left(\int_0^T p_m(t)\mathrm{d}t\int_{\partial D}\frac{\partial \Phi}{\partial \boldsymbol{n}}\phi_l \mathrm{d}\sigma\right) p_m(t)\phi_l(x)\right] + O(\delta^{\frac{1}{2}}). \quad (1)$$

In any case, the scattered field contains high frequency components.

Proposition 2

Assume $\Lambda = \{(0,0),(l,m)\}$. We have the following asymptotic expansion for $\hat{u}(t,x)$ inside D:

$$\hat{u}(t,x) = \frac{\hat{u}_{\mathrm{in}}(0)}{g(\omega,\delta)} \left[- \left(\frac{\omega^2}{v_r^2 \delta} \gamma_m + T_{ml,ml} \right) 1_D + \left(\int_0^T p_m(t) \mathrm{d}t \int_{\partial D} \frac{\partial \Phi}{\partial \boldsymbol{n}} \phi_l \mathrm{d}\sigma \right) p_m(t) \phi_l(x) \right] + O(\delta^{\frac{1}{2}}). \quad (1)$$

where $g(\omega,\delta)$ is vanishing at the complex frequency $\omega = \omega_i^{\pm}(\delta)$

In any case, the scattered field contains high frequency components.

Proposition 2

Assume $\Lambda = \{(0,0),(l,m)\}$. We have the following asymptotic expansion for $\hat{u}(t,x)$ inside D:

$$\hat{u}(t,x) = \frac{\hat{u}_{\mathrm{in}}(0)}{g(\omega,\delta)}\left[-\left(\frac{\omega^2}{v_r^2\delta}\gamma_m + T_{ml,ml}\right)1_D + \left(\int_0^T p_m(t)\mathrm{d}t\int_{\partial D}\frac{\partial\Phi}{\partial\boldsymbol{n}}\phi_l\mathrm{d}\sigma\right)p_m(t)\phi_l(x)\right] + O(\delta^{\frac{1}{2}}). \quad (1)$$

where $g(\omega,\delta)$ is vanishing at the complex frequency $\omega = \omega_l^{\pm}(\delta)$ and Φ is the solution to the exterior problem

$$\begin{cases} \Delta\Phi = 0 \ in\ \mathbb{R}^3\backslash D \\ \Phi = 1 \ on\ \partial D \\ \Phi(x) = O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases}$$

In any case, the scattered field contains high frequency components.

Proposition 2

Assume $\Lambda = \{(0,0),(l,m)\}$. We have the following asymptotic expansion for $\hat{u}(t,x)$ inside D:

$$\hat{u}(t,x) = \frac{\hat{u}_{\mathrm{in}}(0)}{g(\omega,\delta)} \left[- \left(\frac{\omega^2}{v_r^2 \delta} \gamma_m + T_{ml,ml} \right) 1_D + \left(\int_0^T p_m(t) \mathrm{d}t \int_{\partial D} \frac{\partial \Phi}{\partial \boldsymbol{n}} \phi_l \mathrm{d}\sigma \right) p_m(t) \phi_l(x) \right] + O(\delta^{\frac{1}{2}}). \quad (1)$$

where $g(\omega,\delta)$ is vanishing at the complex frequency $\omega = \omega_i^{\pm}(\delta)$ and Φ is the solution to the exterior problem

$$\begin{cases} \Delta \Phi = 0 \ \ in \ \mathbb{R}^3 \backslash D \\ \Phi = 1 \ \ on \ \partial D \\ \Phi(x) = O(|x|^{-1}) \ \text{as} \ |x| \to +\infty. \end{cases}$$

In the "worst" case, $1/g(\omega,\delta) = O(1)$, so that the high frequency effect due to $p_m(t)\phi_l(x)$ is visible at first order.

In any case, the scattered field contains high frequency components.

Proposition 2

Assume $\Lambda = \{(0,0),(l,m)\}$. We have the following asymptotic expansion for $\hat{u}(t,x)$ inside D:

$$\hat{u}(t,x) = \frac{\hat{u}_{\mathrm{in}}(0)}{g(\omega,\delta)}\left[-\left(\frac{\omega^2}{v_r^2\delta}\gamma_m + T_{ml,ml}\right)1_D + \left(\int_0^T p_m(t)\mathrm{d}t\int_{\partial D}\frac{\partial\Phi}{\partial\boldsymbol{n}}\phi_l\mathrm{d}\sigma\right)p_m(t)\phi_l(x)\right] + O(\delta^{\frac{1}{2}}). \quad (1)$$

where $g(\omega,\delta)$ is vanishing at the complex frequency $\omega = \omega_i^{\pm}(\delta)$ and Φ is the solution to the exterior problem

$$\begin{cases} \Delta\Phi = 0 \ \mathit{in}\ \mathbb{R}^3\backslash D \\ \Phi = 1 \ \mathit{on}\ \partial D \\ \Phi(x) = O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases}$$

In the "worst" case, $1/g(\omega,\delta) = O(1)$, so that the high frequency effect due to $p_m(t)\phi_l(x)$ is visible at first order.
If the imaginary part of $\omega_i^{\pm}(\delta)$ is of order $O(\delta)$, then we have a strong amplification for ω close to $\Re(\omega_i^{\pm}(\delta))$.

Outline

1. The unmodulated case: point scatterer approximations and effective medium theory
2. Resonances in the static case and in the time-modulated case
3. Point scatterer approximation and effective medium in the time-modulated case.

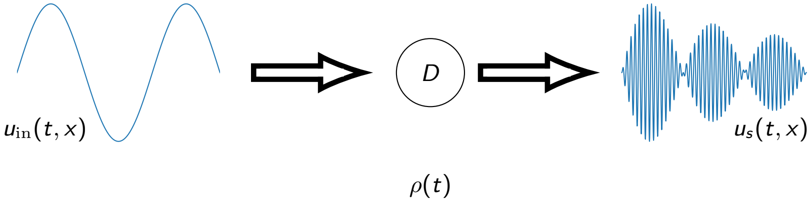
Proposition 3

The scattered field generated by the time modulated resonator satisfies the following far field expansion:

$$\hat{u}(x) - \hat{u}_{\mathrm{in}}(x) \simeq \frac{\hat{u}_{\mathrm{in}}(0)}{g(\omega,\delta)} \left[\left(\frac{\omega^2}{v_r^2 \delta} \gamma_m + T_{ml,ml} \right) \mathrm{cap}\,(D) + \left(\int_0^T p_m(t) \mathrm{d}t \int_{\partial D} \frac{\partial \Phi}{\partial \boldsymbol{n}} \phi_l \mathrm{d}\sigma \right) F_{ml} \left(t - \frac{|x|}{v_0}, \frac{x}{|x|} \right) \right] \Gamma^{\frac{\omega_0}{v_0}}(x) + \hat{u}_{\mathrm{in}}(0) \mathrm{cap}\,(D) \Gamma^{\frac{\omega}{v_0}}(x).$$

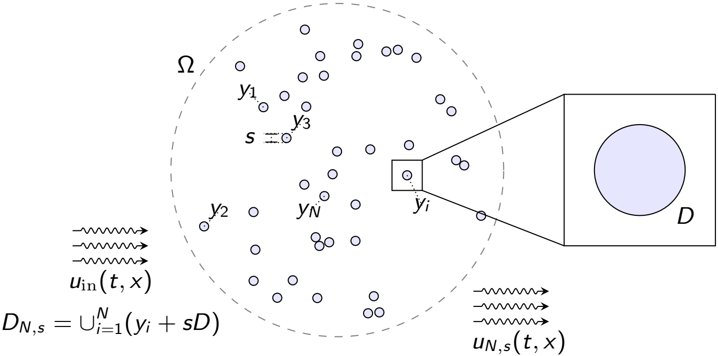
for function $F_{ml}(t,x)$ which is T–periodic in the variable t.

Effective medium theory



Effective medium theory

Effective medium theory in a heterogeneous medium.



Effective medium theory

Scalings:

- $D \to sD$
- $T \to sT, \qquad \rho \to \rho(\cdot/s)$. Fast modulation with large frequency $2\pi/(sT) \to +\infty$!
- The Neumann and Sturm-Liouville eigenvalues scale as

$$\mu_m \to \frac{\mu_m}{s^2} \text{ and } \lambda_l \to \frac{\lambda_l}{s^2},$$

  so that it is possible to keep a constant set

$$\Lambda = \left\{ (m,l) \in \mathbb{N} \times \mathbb{N} \,|\, \frac{\lambda_l}{s^2} = \frac{\mu_m}{s^2 v_r^2} \right\} = \left\{ (m,l) \in \mathbb{N} \times \mathbb{N} \,|\, \lambda_l = \frac{\mu_m}{v_r^2} \right\}$$

- The resonant frequencies scales again as

$$\omega_i^{\pm}(\delta) \sim \lambda_i^{\frac{1}{2}} v_r \frac{\delta^{\frac{1}{2}}}{s},$$

  which shows that for $s = O(\delta^{\frac{1}{2}})$, ω can be of order one.

Effective medium theory

We find then the following effective homogenized equation for the scattering of wave in the fast temporal medium:

$$u(t,y) - sN \int_\Omega K_{\omega,\delta}\left(t - \frac{|y-y'|}{v_0}, \frac{y-y'}{|y-y'|}\right) \Gamma^{\frac{\omega_0}{v_0}}(y-y')V(y')\hat{u}(t,y')\mathrm{d}y' = \hat{u}_{\rm in}(y), \quad y \in \Omega.$$

where

$$K_{\omega,\delta}(t,y) := [A(s\omega,\delta) + B(s\omega,\delta)F_{ml}(t,y)]$$

for some coefficients $A(s\omega,\delta)$ and $B(s\omega,\delta)$.

Effective medium theory

We find then the following effective homogenized equation for the scattering of wave in the fast temporal medium:

$$u(t,y) - sN \int_\Omega K_{\omega,\delta}\left(t - \frac{|y-y'|}{v_0}, \frac{y-y'}{|y-y'|}\right) \Gamma^{\frac{\omega_0}{v_0}}(y-y') V(y') \hat{u}(t,y') \mathrm{d}y' = \hat{u}_{\mathrm{in}}(y), \quad y \in \Omega.$$

where

$$K_{\omega,\delta}(t,y) := [A(s\omega,\delta) + B(s\omega,\delta) F_{ml}(t,y)]$$

for some coefficients $A(s\omega,\delta)$ and $B(s\omega,\delta)$.
Using Fourier series, this is equivalent to a cascade of Helmholtz equation for each of the Fourier modes with a frequency dependent refractive index.

Preprint

The full details soon available in a preprint

Preprint

The full details soon available in a preprint

Related works:

Feppon and Ammari, *Analysis of a Monte-Carlo Nystrom method* (2022)

Feppon and Ammari, *Modal decompositions and point scatterer approximations near the Minnaert resonance frequencies* (2022)

Feppon and Ammari, *Homogenization of sound-absorbing and high-contrast acoustic metamaterials in subcritical regimes* (2021).

The full details soon available in a preprint

Related works:

Feppon and Ammari, *Analysis of a Monte-Carlo Nystrom method* (2022)

Feppon and Ammari, *Modal decompositions and point scatterer approximations near the Minnaert resonance frequencies* (2022)

Feppon and Ammari, *Homogenization of sound-absorbing and high-contrast acoustic metamaterials in subcritical regimes* (2021).

Thank you for your attention.