Lecture 4: physical models in mechanical and aeronautic engineering, numerical solutions, formulation of optimal design problems

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ETH zürich

Outline

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1. The membrane's equation, reminders on FEM and weak formulations
2. Linear elasticity
3. Fluid flows
4. Heat diffusion
5. Coupled physics:
   5.1 convective heat transfer
   5.2 fluid-structure interactions
   5.3 thermoelasticity

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In industrial applications, $J(\Omega)$, $G_i(\Omega)$ or $H_j(\Omega)$ involve the solution u_Ω defined with respect to a PDE model posed on Ω .

The membrane's equation

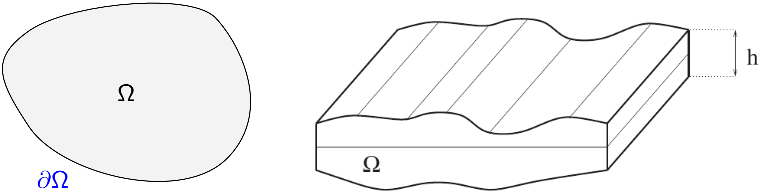

Figure: A membrane with variable height $h(x)$. Figure from Allaire 2004.

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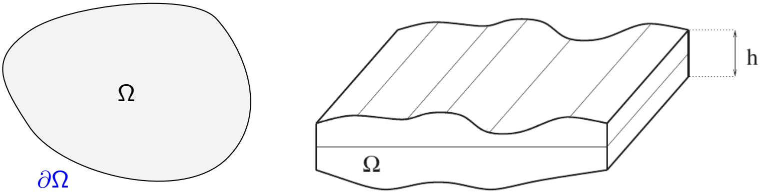

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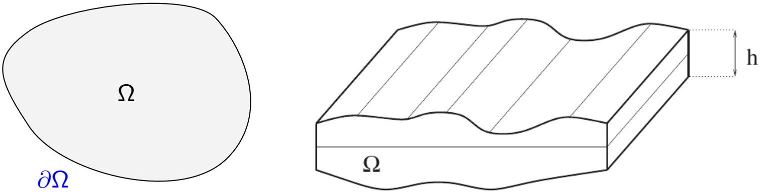

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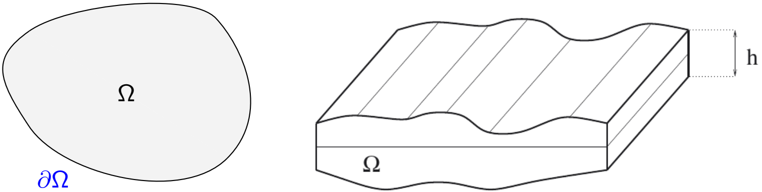

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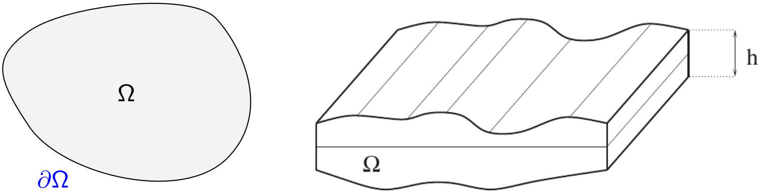

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- $f(x)$ is a distribution of forces applied on the membrane (e.g. a pressure distribution)
- the membrane is clamped on the boundary of Ω whence $u = 0$ on $\partial\Omega$

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- In numerical practice we work with the weak formulation:
$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \int_\Omega A(x)\nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x \text{ for any } v \in H_0^1(\Omega)$$

Reminders on weak formulations

Definition 1

Let $\Omega \subset \mathbb{R}^d$ a smooth bounded domain. We denote by $H^1(\Omega)$ the Sobolev space

$$H^1(\Omega) = \{v \mid v \in L^2(\Omega) \text{ and } \nabla v \in L^2(\Omega, \mathbb{R}^d)\}.$$

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We denote by $||v||_{H^1(\Omega)} := ||v||_{L^2(\Omega)} + ||\nabla v||_{L^2(\Omega)}$ the norm on $H^1(\Omega)$, which is a Hilbert space for the inner product

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Proposition 1

There exists a constant $C > 0$ such that

$$||\phi||_{L^2(\partial\Omega)} \leq C||\phi||_{H^1(\Omega)} \text{ for any } \phi \in \mathcal{C}^\infty(\bar{\Omega}).$$

Hence the application $\gamma : \phi \mapsto \phi_{|\partial\Omega}$ extends continuously to an application of $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$. $\gamma(v)$ is called the trace of v and is still denoted by $v_{|\partial\Omega}$.

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Proposition 2 (Green formula)

Let $u, v \in H^1(\Omega)$. Then

$$\int_\Omega \frac{\partial u}{\partial x_i} \mathrm{d}x = \int_{\partial\Omega} u n_i \mathrm{d}\sigma$$

where $\boldsymbol{n} = (n_1, \ldots, n_d)$ is the outward *normal to Ω.*

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We find therefore, assuming enough regularity

$$\int_\Omega -\mathrm{div}(A\nabla u) v \mathrm{d}x = \int_\Omega A\nabla u \cdot \nabla v \mathrm{d}x - \int_{\partial\Omega} v A\nabla u \cdot n \mathrm{d}\sigma = \int_\Omega A\nabla u \cdot \nabla v \mathrm{d}x$$

if $v = 0$ on $\partial\Omega$.

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Proposition 3 (Lax Milgram's theorem)

Let V be a Hilbert space and $a : V \times V \to \mathbb{R}$ be a coercive symmetric bilinear form:

- $a(\lambda u + v, w) = \lambda a(u, w) + a(v, w)$ *and* $a(u, v) = a(v, u)$ *for any* $u, v, w \in V$, $\lambda \in \mathbb{R}$,

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- *$|a(u, v)| \leq C||u||_V||v||_V$ for any $u, v \in V$ with $C > 0$,*

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Then for any $b \in V'$, there exists a unique $u \in V$ solving the variational formulation

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$$\textit{Find } u \in V \textit{ such that } \forall v \in V,\ a(u, v) = b(v).$$

Here, $V = H_0^1(\Omega)$, $a(u, v) = \displaystyle\int_\Omega A(x)\nabla u \cdot \nabla v \mathrm{d}x$, $b(v) = \displaystyle\int_\Omega f v \mathrm{d}x$.

Proposition 4 (Poincaré inequality)

Assume Ω is a smooth bounded domain. There exists $C > 0$ such that

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Proof.

Assume the result is wrong. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that $\int_\Omega |\nabla u_n|^2 \mathrm{d}x \leq \frac{1}{n}\int_\Omega |u_n|^2 \mathrm{d}x$. Up to a rescaling, we may assume that $\int_\Omega |u_n|^2 \mathrm{d}x = 1$. Then $||\nabla u_n||_{L^2(\Omega)} \to 0$ while $||u_n||_{H^1(\Omega)}$ remains bounded. By the Rellich theorem, we may assume, up to extracting a subsequence, that $u_n \to u$ strongly in $L^2(\Omega)$ for some $u \in H^1(\Omega)$. Since $||\nabla u_n||_{L^2(\Omega)} \to 0$, the convergence is in fact strong in $H^1(\Omega)$ and we obtain that $\nabla u = 0$, hence u is constant on Ω. Since $u_n \in H_0^1(\Omega)$, we find that $u = 0$ on Ω, which is not possible because $||u_n||_{L^2(\Omega)} = 1$ implies $||u||_{L^2(\Omega)} = 1$. □

Corollary:

- if $-\mathrm{div}(A(x)\nabla\cdot)$ is uniformly elliptic, i.e.

$$\exists C > 0\,,\ A_{ij}(x)\boldsymbol{\xi}_i\boldsymbol{\xi}_j \geqslant C|\boldsymbol{\xi}|^2 \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d$$

then

$$a(u,u) = \int_\Omega A(x)\nabla u \cdot \nabla u \mathrm{d}x \geqslant C||\nabla u||^2_{L^2(\Omega)} \geqslant C||u||_{H^1(\Omega)}$$

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- Then the weak formulation

$$\text{Find } u \in H^1_0(\Omega) \text{ such that } \forall v \in H^1_0(\Omega),\; a(u,v) = b(v),$$

  admits a unique solution $u \in H^1_0(\Omega)$.

Reminders on the finite element method

The finite element (or Galerkin approximation) method amount to approximate the variational formulation

$$\text{Find } u \in V \text{ such that } \forall v \in V,\ a(u,v) = b(v),$$

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$$\text{Find } u_h \in \ \text{ such that } \forall v \in V_h,\ a(u_h,v_h) = b(v_h), \tag{1}$$

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Proof.

Let $(\phi_i)_{1 \le i \le N_h}$ a basis of V_h. Then if $u_h = \sum_{j=1}^{N_h} u_i \phi_i$, eq. (1) is equivalent to

$$\text{find } (u_i)_{1 \le i \le N_h} \text{ such that } a\left(\sum_{j=1}^{N_h} u_j \phi_j, \phi_i\right) = b(\phi_i),$$

that is $a(\phi_i, \phi_j) u_j = b(\phi_i), \quad 1 \le i \le N.$ □

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- The matrix $A_h = (a(\phi_i, \phi_j))_{1 \leq i,j \leq N_h}$ is called the rigidity matrix.

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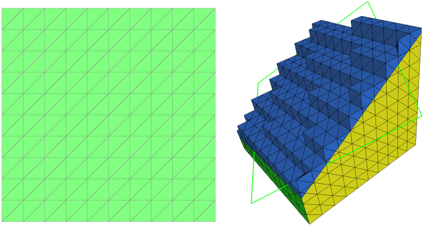
- When using the finite element method, it is common to numerically approximate the domain Ω by a triangular or tetrahedral mesh $\mathcal{T}_h = \cup_{i=1}^{N} T_i$ with N triangles/tetrahedra T_i with size h



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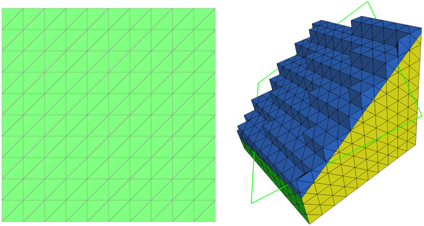

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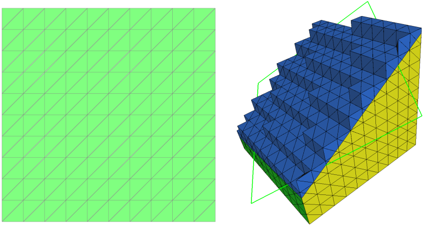

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- Typically, h is the maximum element size (maximum edge size, or maximum height of a tetrahedron)
- It is common to use $\mathbb{P}_k(\mathcal{T}_h)$-Lagrange finite elements for the space V_h:

$$\mathbb{P}_k(\mathcal{T}_h) = \{v \text{ continuous on } \mathcal{T}_h \,|\, v_{T_i} \text{ is a polynomial of degree less than } k\}.$$

Reminders on the finite element method

- Under reasonable assumptions on the mesh $\mathcal{T}_h$ it is possible to prove that the variational approximation is convergent:

$$\lim_{h \to 0} ||u - u_h||_{H^1(\Omega)} = 0.$$

Moreover, **if** $u \in H_{k+1}(\Omega)$, then we have the error estimate

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- there is no need to use higher order finite element if the solution is only in $H^1(\Omega)$.

Reminders on the finite element method

- The equation
$$\begin{cases} -\mathrm{div}(A\nabla u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$
is called the **strong** formulation of the model.
- In numerical practice we work with the weak formulation:
$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \int_\Omega A(x)\nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x \text{ for any } v \in H_0^1(\Omega)$$
- Regularity estimates show that if $f \in H^k(\Omega)$, then $u \in H^{k+2}(\Omega)$.

Numerical solution in FreeFEM:

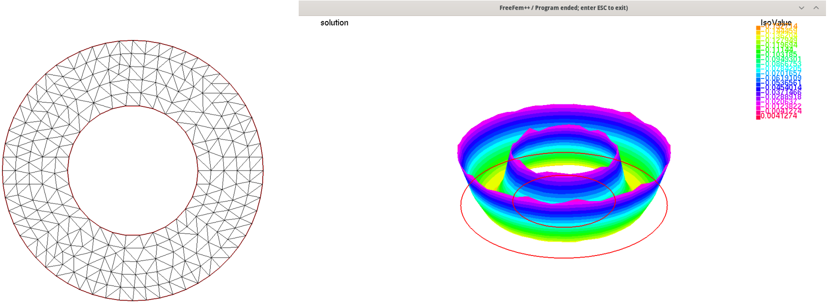

Figure: Numerical solution of the membrane problem with uniform $A = I$ and $f = -1$ on an annulus domain Ω.

Formulation of a shape optimization problem

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- an **objective function** $J(\Omega, h, u)$ corresponding to a measure of the performance,

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- an **objective function** $J(\Omega, h, u)$ corresponding to a measure of the performance,
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- a design variable and an **admissible set** : thickness $h(x)$ of the membrane, shape of the domain Ω

Formulation of a shape optimization problem

A common objective function: the compliance $b(u)$:

$$J(u) := \int_\Omega fu \mathrm{d}x = \int_\Omega A\nabla u \cdot \nabla u \mathrm{d}x = \int_\Omega h(x)|\nabla u|^2 \mathrm{d}x.$$

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- Minimum and maximum thickness if the design variable is h:

$$\forall x \in \Omega, h_{\mathrm{min}} \leq h(x) \leq h_{\mathrm{max(x)}}$$

In that case, we need to deal with a **point-wise** constraint.

Formulation of a shape optimization problem

For instance, an optimal design problem of the membrane reads

$$\min_{\Omega} \quad J(u_\Omega) := \int_\Omega f u \mathrm{d}x$$
$$s.t. \left\{ \mathrm{Vol}(\Omega) := \int_\Omega \mathrm{dx} = \mathtt{Vol}_0. \right.$$

and $u_\Omega \in H_0^1(\Omega)$ is solution to $a(u_\Omega, v) = b(v)$ for all $v \in H_0^1(\Omega)$.

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For other physical models posed on bounded domains, the methodology is similar:

- determining the weak formulation of the PDE,
- being sure of the well-posedness, regularity of the solution,
- devising a numerical method to solve the **forward problem**.
- formulating an optimal design problem: modelling of the performance through an objective function, and of the specification constraints.

Outline

Linear elasticity

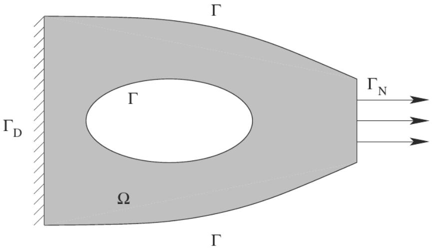

Figure: A cantilever beam subjected to traction forces on Γ_N and zero displacement on Γ_D. Figure from Allaire, 2004.

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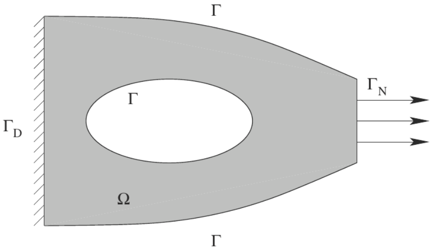

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$\sigma_s(\boldsymbol{u}) := Ae(\boldsymbol{u}) \in \mathbb{R}^{d \times d}$ is the strain or **solid stress tensor**. The Hooke's law states that

$$Ae(\boldsymbol{u}) = 2\mu e(\boldsymbol{u}) + \lambda \mathrm{Tr}(e(\boldsymbol{u}))I \text{ with } \boldsymbol{e}(\boldsymbol{u}) := \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}.$$

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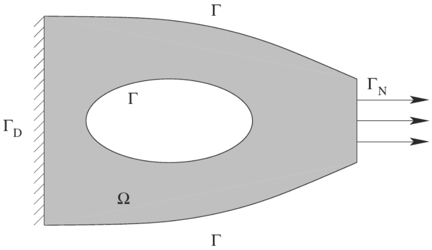

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μ and λ are the Lamé coefficients, related to the Young modulus and Poisson ratio from the formula:

$$\lambda = \frac{\nu E}{(1+\nu)(1-(d-1)\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

Linear elasticity

The weak formulation of

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is find $\boldsymbol{u} \in H_0^1(\Omega, \mathbb{R}^d)$ such that for any $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^d)$,

$$\int_\Omega Ae(\boldsymbol{u}) : e(\boldsymbol{v})\mathrm{d}x = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\mathrm{d}x + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}\mathrm{d}\sigma,$$

where

$$H_0^1(\Omega, \mathbb{R}^d) = \{\boldsymbol{v} \in H^1(\Omega, \mathbb{R}^d) \,|\, \boldsymbol{v} = 0 \text{ on } \Gamma_D\}.$$

The compliance minimization problem reads

$$\min_{\Omega \subset D} \quad J(\Omega, \boldsymbol{u}(\Omega)) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \mathrm{d}x + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v} \mathrm{d}\sigma = \int_\Omega Ae(\boldsymbol{u}) : e(\boldsymbol{u}) \mathrm{d}x$$

$$s.t. \quad \mathrm{Vol}(\Omega) := \int_\Omega \mathrm{d}x = V_{target}.$$

Figure: Setting of the cantilever optimization problem.

Linear elasticity

- Ideally, the industry seeks to consider rather the *mass minimization* problem

$$\begin{aligned} \min_{\Omega \subset D} \quad & \mathrm{Vol}(\Omega) \\ s.t. \ & e(\boldsymbol{u}) : e(\boldsymbol{u})(x) \leq \mathtt{VM}_0 \text{ for any } x \in \Omega, \end{aligned}$$

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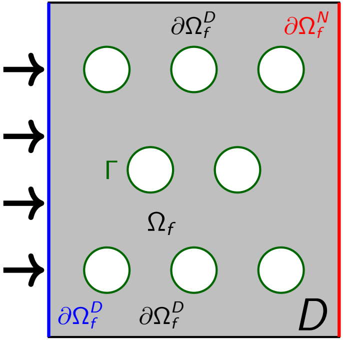
  where the constraint imposes a Von-Mises upper bound on the strain energy to prevent premature fatigue. This point-wise stress constraint is delicate to implement.
- Non-linear models or more complex constitutive laws can be considered to account for plasticity or more realistic phenomena.

Outline

1. The membrane's equation, reminders on FEM and weak formulations
2. Linear elasticity
3. Fluid flows
4. Heat diffusion
5. Coupled physics:
   5.1 convective heat transfer
   5.2 fluid-structure interactions
   5.3 thermoelasticity

Fluid flows

The motion of a flow can be modelled by the (transient) Navier-Stokes equations:



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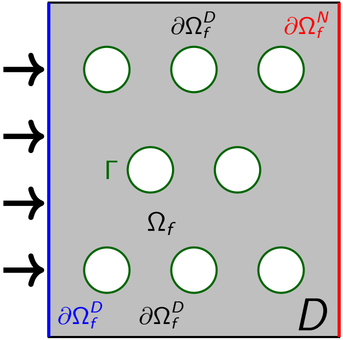

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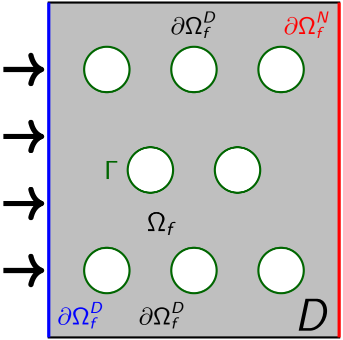
- $\boldsymbol{v}$ is the fluid velocity field, p is the (static) pressure, ρ is the fluid density.
- $\sigma_f(\boldsymbol{v}, p)$ is the fluid stress tensor. For Newtonian fluids, we have

$$\sigma_f(\boldsymbol{v}, p) = 2\nu e(\boldsymbol{v}) - pI, \quad e(\boldsymbol{v}) = \frac{\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T}{2}$$

where ν is the (dynamic) fluid viscosity, so that $-\mathrm{div}(\sigma_f(\boldsymbol{v}, p)) = -\nu \Delta \boldsymbol{v} + \nabla p$.

Fluid flows



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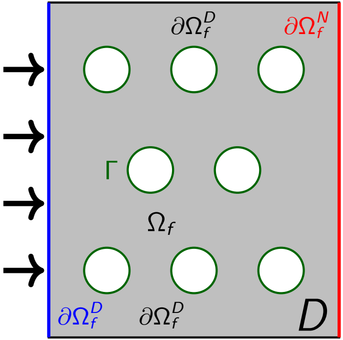

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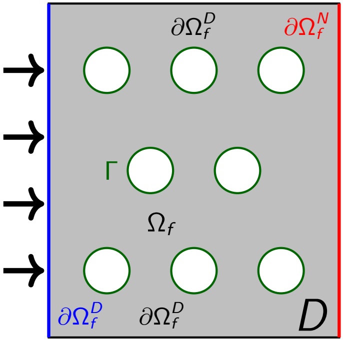

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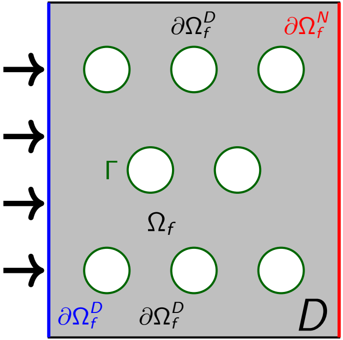

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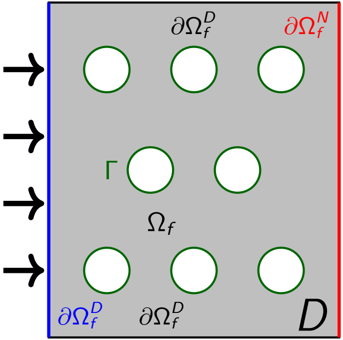

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- when formulating the design problem, one generally is not interested in the transient behavior of the system but rather on a steady-state ($\partial_t \boldsymbol{v} = 0$), or on a time-period in usage condition.

Fluid flows

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$$\begin{cases} \partial_t \boldsymbol{v} - \operatorname{div}(\sigma_f(\boldsymbol{v}, p)) + \rho \nabla \boldsymbol{v}\, \boldsymbol{v} = \boldsymbol{f}_f & \text{in } \Omega_f \\ \operatorname{div}(\boldsymbol{v}) = 0 & \text{in } \Omega_f \\ \boldsymbol{v} = \boldsymbol{v}_0 & \text{on } \partial\Omega_f^D \\ \sigma_f(\boldsymbol{v}, p)\boldsymbol{n} = 0 & \text{on } \partial\Omega_f^N \\ \boldsymbol{v} = 0 & \text{on } \Gamma, \end{cases}$$

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We may assume in many practical cases that $\partial_t \boldsymbol{v} = 0$, $\boldsymbol{v}$ does not depend on time.

Fluid flows

- The Reynolds number is the ratio between convection $\rho \nabla \boldsymbol{v}\boldsymbol{v}$ and diffusion $\nu \Delta \boldsymbol{v}$. It can be estimated as
$$\mathrm{Re} := \frac{\rho \mathrm{L} ||\mathrm{v}_0||}{\nu}$$
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- There is no universal definition because L can be chosen differently !
- However, when Re is large (typically $\mathrm{Re} \geqslant 10^3$), convection effects dominate and the numerical simulation maybe very difficult to solve due to **turbulence** and **boundary layer effects**

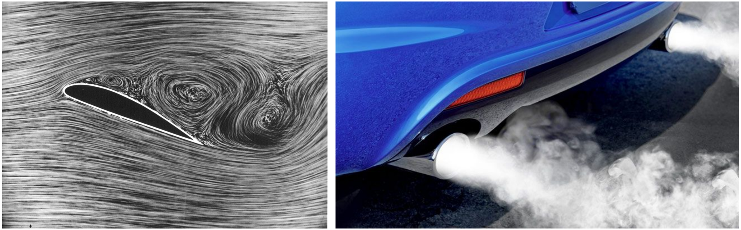

Figure: Example of turbulent flows featuring vortices. Images from Wikipedia and https://studiousguy.com/turbulent-flow-examples/

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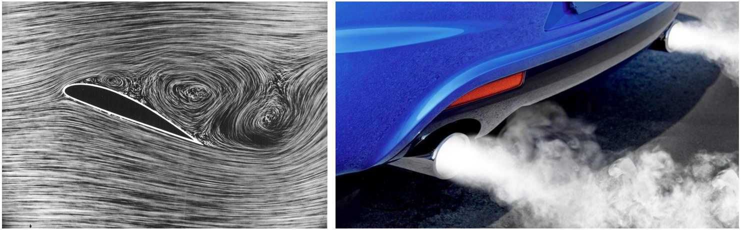

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- In the aeronautic industry, a flight encounters Reynolds number of the order of 10^6.
- A variety of **turbulence models** (Large Eddy Simulation (LES), Reynold Average Navier Stokes (RANS)...) are commonly used in the aeronautic industry to obtain approximate simulation of turbulent flows.

Fluid flows

For our applications, we will assume moderate Reynolds number ($\mathrm{Re} \leq 500$). The steady-state Navier-Stokes equations are nonlinear and requires a different numerical treatment:

- the nonlinearity can be solved using the Newton method. $(\boldsymbol{v}, p)$ are solutions to the following variational problem: find $(\boldsymbol{v}, p) \in \boldsymbol{v}_0 + V_{\boldsymbol{v},p}(\Gamma)$ such that

$$\forall (\boldsymbol{w}, q) \in V_{\boldsymbol{v},p}(\Gamma) \quad \int_{\Omega_f} \left[\sigma_f(\boldsymbol{v}, p) : \nabla \boldsymbol{w} + \rho \boldsymbol{w} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{v} - q \mathrm{div}(\boldsymbol{v})\right] \mathrm{d}x = \int_{\Omega_f} \boldsymbol{f}_f \cdot \boldsymbol{w} \mathrm{d}x;$$

where $V_{\boldsymbol{v},p}(\Gamma) = \{(\boldsymbol{w}, q) \in H^1(\Omega_f, \mathbb{R}^d) \times L^2(\Omega_f)/\mathbb{R} \,|\, \boldsymbol{w} = 0 \text{ on } \partial\Omega_f\}$.

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- The next iterate $(\boldsymbol{v}_{k+1}, p_{k+1})$ is then obtained by setting $\boldsymbol{v}_{k+1} := \boldsymbol{v}_k + \delta \boldsymbol{v}_k, \quad p_{k+1} := p_k + \delta p_k.$

Remarks:

- One needs different discretization spaces for the velocity and pressure for numerical stability, typically $\mathbb{P}1b/\mathbb{P}1$.

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- Existence and uniqueness of a solution to the steady-state NS equation is guaranteed for low Reynolds number (by a perturbation argument of the Stokes system).
- For large Reynolds number, there may exist several solutions to the NS equations (due to nonlinearity). The relevance of the physical modelling can then be questioned because the trajectories of the solutions generally converge to a limit cycle (or worse, to a global attractor) and not to a steady state.

Fluid flows

Common design functionals:

- The **drag** or sum of the friction forces:

$$\mathrm{Drag}(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma)) := \int_{\Omega_f} \sigma_f(\boldsymbol{v}, p) : \nabla \boldsymbol{v} \mathrm{d}x = \int_{\Omega_f} 2\nu e(\boldsymbol{v}) : e(\boldsymbol{v}) \mathrm{d}x.$$

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- The **lift** or forces generated by the flow around an obstacle along one direction:

$$\mathrm{Lift}(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma)) := -\int_{\Gamma} \boldsymbol{e}_y \cdot \sigma_f(\boldsymbol{v}, p) \cdot \boldsymbol{n} \mathrm{d}s,$$

Typical aerodynamic design problem:

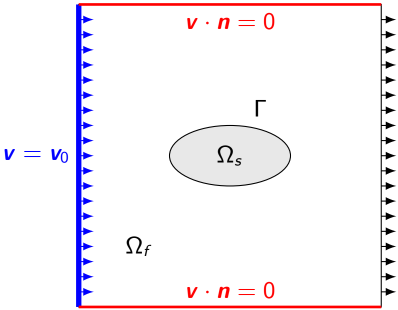

Figure: Setting of the aerodynamic design problem.

$$\min \quad -\mathrm{Lift}(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma))$$

$$s.t. \begin{cases} \mathrm{Drag}(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma)) \leq \mathtt{DRAG}_0 \\ \mathrm{Vol}(\Omega_f) = V_0 \\ \boldsymbol{X}(\Omega_s) := \dfrac{1}{|\Omega_s|} \displaystyle\int_{\Omega_s} \boldsymbol{x} \mathrm{d}x = \boldsymbol{x}_0. \end{cases}$$

Remarks:

- the functional $\mathrm{Drag}(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma))$ is also a measure analogous to the (static) **pressure drop**:

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- This problem also arises for the Lift functional, but can be bypassed by resorting to an equivalent volume formulation.

$$\mathrm{Lift}(\Gamma) = \int_{\Omega_f} (\mathcal{X}\boldsymbol{f}_f \cdot \boldsymbol{e}_y - \rho\mathcal{X}\boldsymbol{e}_y \cdot \nabla\boldsymbol{v} \cdot \boldsymbol{v} - \nabla\mathcal{X} \cdot \sigma_f(\boldsymbol{v}, p) \cdot \boldsymbol{e}_y)\mathrm{d}x$$

where $\mathcal{X}$ is a scalar field satisfying $\mathcal{X} = 1$ on $\partial\Omega_s$.

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4. Heat diffusion
5. Coupled physics:
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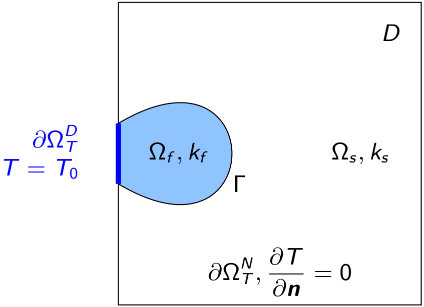

Figure: A bi-material distrubution of two conductive media with conductivity k_s and k_v.

$$\begin{cases} -\mathrm{div}(k_f \nabla T_f) = Q_f & \text{in } \Omega_f \\ -\mathrm{div}(k_s \nabla T_s) = Q_s & \text{in } \Omega_s \\ T = T_0 & \text{on } \partial\Omega_T^D \\ -k_f \dfrac{\partial T_f}{\partial \boldsymbol{n}} = h & \text{on } \partial\Omega_T^N \cap \partial\Omega_f \\ -k_s \dfrac{\partial T_s}{\partial \boldsymbol{n}} = h & \text{on } \partial\Omega_T^N \cap \partial\Omega_s \\ T_f = T_s & \text{on } \Gamma \\ -k_f \dfrac{\partial T_f}{\partial \boldsymbol{n}} = -k_s \dfrac{\partial T_s}{\partial \boldsymbol{n}} & \text{on } \Gamma, \end{cases}$$

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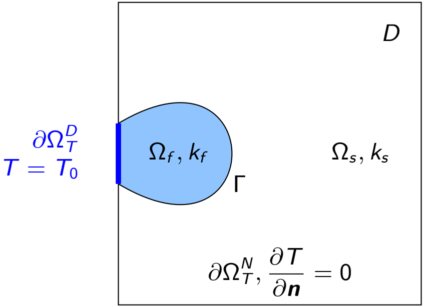
Heat diffusion



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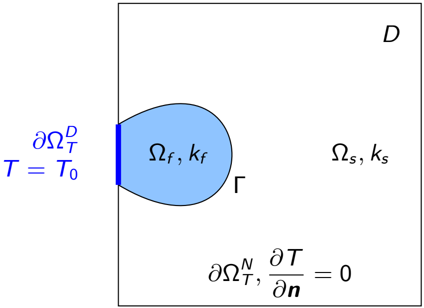

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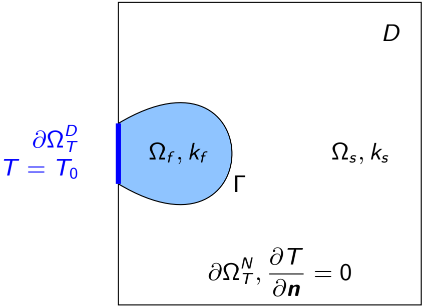

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- T_0: prescribed temperature on isothermal boundary $\partial\Omega_T^D$.
- h: heat loss on the boundary $\partial\Omega_T^N$. If $h = 0$, we say that the boundary is adiabatic.

Heat diffusion

The variational formulation reads find $T \in T_0 + V_T(\Gamma)$ such that, for any $S \in V_T(\Gamma)$,

$$\int_{\Omega_s} k_s \nabla T \cdot \nabla S \mathrm{dx} + \int_{\Omega_f} k_f \nabla T \cdot \nabla S \mathrm{dx} = \int_{\Omega_s} Q_s S \mathrm{dx} + \int_{\Omega_f} Q_f S \mathrm{dx} + \int_{\partial\Omega_T^N} hS \mathrm{d}s.$$

where

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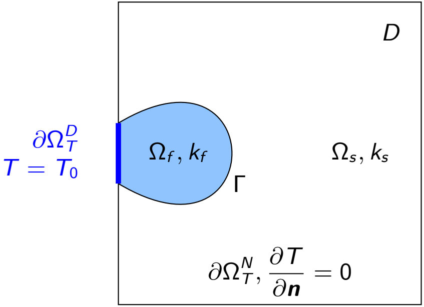
Not harder to solve than a Laplacian.

Heat diffusion

Minimizing the average temperature with a cooling material Ω_f:



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$$\min_{\Gamma} \quad J(\Gamma, T(\Gamma)) = \int_D T \mathrm{d}x$$
$$s.t. \quad \mathrm{Vol}(\Omega_f) \leq V_{target}.$$

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- The heat density carried out through convective transport by an incompressible flow $\boldsymbol{v}$ is given by

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- Once $\boldsymbol{v}$ is determined by the Navier-Stokes equations. . .

$$\begin{cases} -\mathrm{div}(\sigma_f(\boldsymbol{v}, p)) + \rho \nabla \boldsymbol{v}\, \boldsymbol{v} = \boldsymbol{f}_f & \text{in } \Omega_f \\ \mathrm{div}(\boldsymbol{v}) = 0 & \text{in } \Omega_f \\ \boldsymbol{v} = \boldsymbol{v}_0 & \text{on } \partial\Omega_f^D \\ \sigma_f(\boldsymbol{v}, p)\boldsymbol{n} = 0 & \text{on } \partial\Omega_f^N \\ \boldsymbol{v} = 0 & \text{on } \Gamma \end{cases}$$

- then one can solve the advection-diffusion equation

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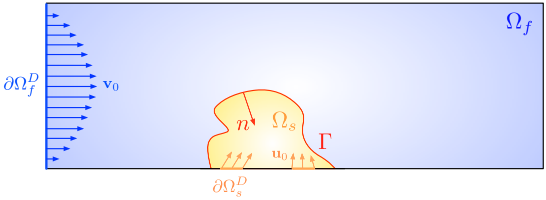
-k_s \dfrac{\partial T_s}{\partial \boldsymbol{n}} = h & \text{on } \partial\Omega_T^N \cap \partial\Omega_s \\
T_f = T_s & \text{on } \Gamma \\
-k_f \dfrac{\partial T_f}{\partial \boldsymbol{n}} = -k_s \dfrac{\partial T_s}{\partial \boldsymbol{n}} & \text{on } \Gamma,
\end{cases}
$$



An optimal heat transfer test case:

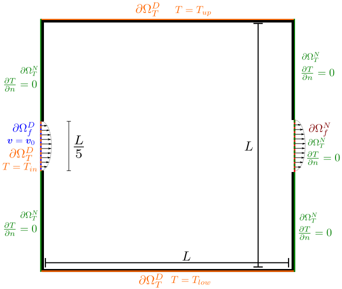


Figure: Setting of a convective heat transfer test case.

$$\min_{\Gamma} \quad J(\Gamma, \boldsymbol{v}(\Gamma), T(\Gamma)) := -\int_{\Omega_f} \rho c_p \boldsymbol{v} \cdot \nabla T \mathrm{d}x$$

$$s.t. \begin{cases} \mathtt{DP}(p(\Gamma)) := \int_{\partial\Omega_f^D} p \mathrm{d}s - \int_{\partial\Omega_f^N} p \mathrm{d}s \leq \mathtt{DP}_{static} \\ \mathrm{Vol}(\Omega_f) = V_{target}. \end{cases}$$

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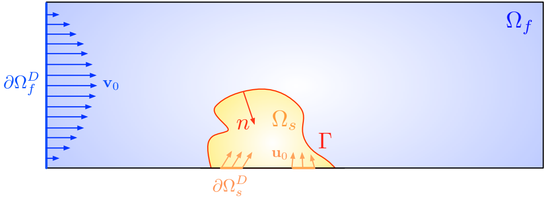
$J(\Gamma, \boldsymbol{v}(\Gamma), T(\Gamma)) := -\int_{\Omega_f} \rho c_p \boldsymbol{v} \cdot \nabla T \mathrm{d}x$ is the opposite of the heat transferred from the inlet to the outlet, indeed:

$$\int_{\Omega_f} \rho c_p \boldsymbol{v} \cdot \nabla T \mathrm{d}x = \int_{\partial\Omega_f} \rho c_p T(\boldsymbol{v} \cdot \boldsymbol{n})\mathrm{d}s = \int_{\partial\Omega_f^N} \rho c_p T(\boldsymbol{v} \cdot \boldsymbol{n})\mathrm{d}s + \int_{\partial\Omega_f^D} \rho c_p T_0(\boldsymbol{v}_0 \cdot \boldsymbol{n})\mathrm{d}s.$$

Fluid-structure interactions



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- $\boldsymbol{v} = 0$ on the deformed solid interface: $\boldsymbol{v}(x + \boldsymbol{u}(x)) = 0$ for $x \in \Gamma$;

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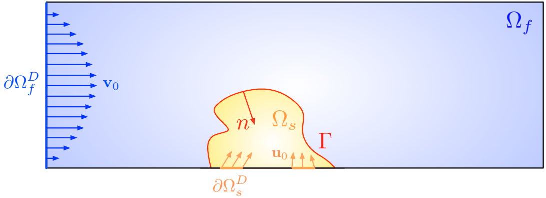

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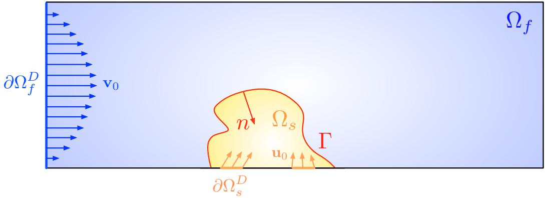

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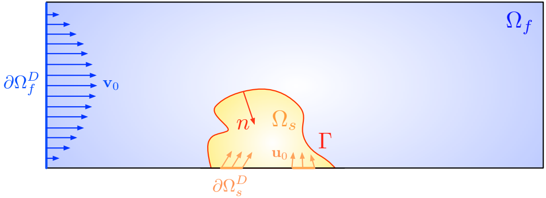

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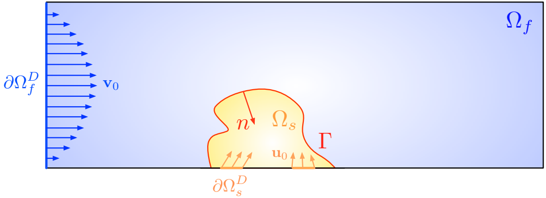

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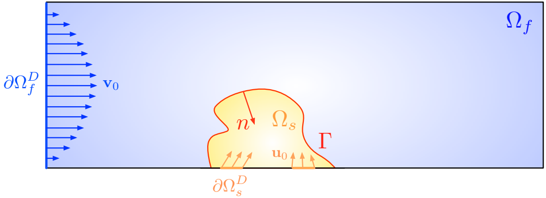

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The model is weakly coupled.

Fluid-structure interactions

One can still maximize the rigidity of the solid structure, which is forced by the incoming flow:

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The variable $\boldsymbol{u}(\Gamma)$ depends implicitly on $\boldsymbol{v}(\Gamma)$!

Thermoelasticity

- Thermal dilation due to an elevated temperature T can be modelled by an extra term in the Hooke's law:

$$\sigma_s(\boldsymbol{u}, T_s) := Ae(\boldsymbol{u}) - \alpha(T_s - T_{\mathrm{ref}})I \text{ with } Ae(\boldsymbol{u}) = 2\mu e(\boldsymbol{u}) + \lambda \mathrm{Tr}(e(\boldsymbol{u}))I.$$

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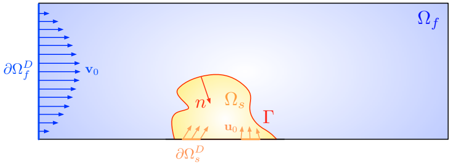
- Thermoelasticity can be coupled to the convection-diffusion model which determines the temperature field T_s

A three-physics weakly coupled model



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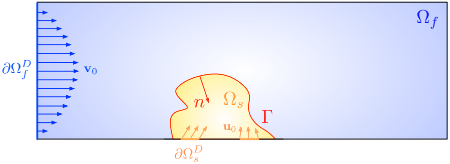

- Incompressible Navier-Stokes equations for velocity and pressure $(\boldsymbol{v}, p)$ inΩ_f

$$-\mathrm{div}(\sigma_f(\boldsymbol{v}, p)) + \rho\nabla\boldsymbol{v}\,\boldsymbol{v} = \boldsymbol{f}_f \text{ in } \Omega_f$$

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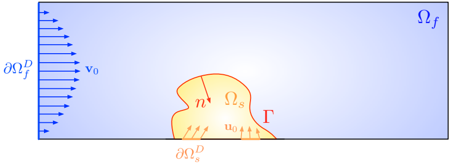

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$$\begin{aligned} -\mathrm{div}(k_f \nabla T_f) + \rho \boldsymbol{v} \cdot \nabla T_f &= Q_f \quad \text{in } \Omega_f \\ -\mathrm{div}(k_s \nabla T_s) &= Q_s \quad \text{in } \Omega_s \end{aligned}$$

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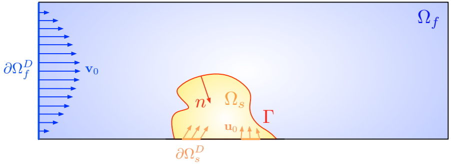

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- Linear elasticity with fluid-structure interaction for mechanical deformation $\boldsymbol{u}$ in Ω_s :

$$\begin{aligned} -\mathrm{div}(\sigma_s(\boldsymbol{u}, T_s)) &= \boldsymbol{f}_s \quad && \text{in } \Omega_s \\ \sigma_s(\boldsymbol{u}, T_s) \cdot \boldsymbol{n} &= \sigma_f(\boldsymbol{v}, p) \cdot \boldsymbol{n} \quad && \text{on } \Gamma. \end{aligned}$$

Exercise

Implement one of the physical models of your choice in FreeFEM: write a PDE solver that computes $\boldsymbol{u}$, T, or $(\boldsymbol{v}, p)$ for a simple domain geometry.

- `https://doc.freefem.org`
- `https://modules.freefem.org/`