Lecture 11: Three-dimensional topology optimization. Domain Decomposition methods and parallel computing.

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ETH zürich

Outline

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1. Challenges in three-dimensional topology optimization
2. A glimpse on domain decomposition methods and PETSc
3. Implementing a topology optimization test case

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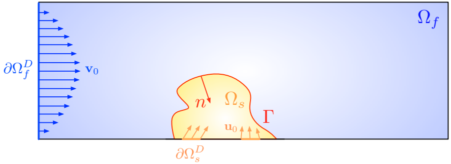

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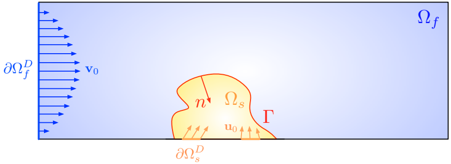

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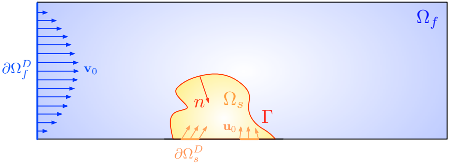

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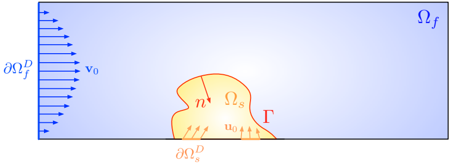

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- Thermo-elasticity with fluid-structure interaction for $\boldsymbol{u}$ iin Ω_s :

$$\begin{aligned} -\mathrm{div}(\sigma_s(\boldsymbol{u}, T_s)) &= \boldsymbol{f}_s && \text{in } \Omega_s \\ \sigma_s(\boldsymbol{u}, T_s)\cdot\boldsymbol{n} &= \sigma_f(\boldsymbol{v}, p)\cdot\boldsymbol{n} && \text{on } \Gamma. \end{aligned}$$

The coupled physics model

Our goal: solve generic topology optimization problems of the type

$$\begin{aligned} \min_{\Gamma} \quad & J(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma), T(\Gamma), \boldsymbol{u}(\Gamma)) \\ \text{s.c.} \quad & g_i(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma), T(\Gamma), \boldsymbol{u}(\Gamma)) = 0, \; 1 \leq i \leq p \cdot \\ & h_i(\Gamma, \boldsymbol{v}(\Gamma), p(\Gamma), T(\Gamma), \boldsymbol{u}(\Gamma)) \leq 0, \; 1 \leq i \leq q \end{aligned}$$

where $\boldsymbol{u}(\Gamma)$, $\boldsymbol{v}(\Gamma)$, $p(\Gamma)$, $T(\Gamma)$ are the solutions to PDE models.

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- In practice, the implementation must be **completely** revised.

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- Remeshing becomes also quite expensive in 3D due to the number of combinatorial operations (while it is inexpensive in 2D).

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In the present class, we focus on making FEM operations in parallel.

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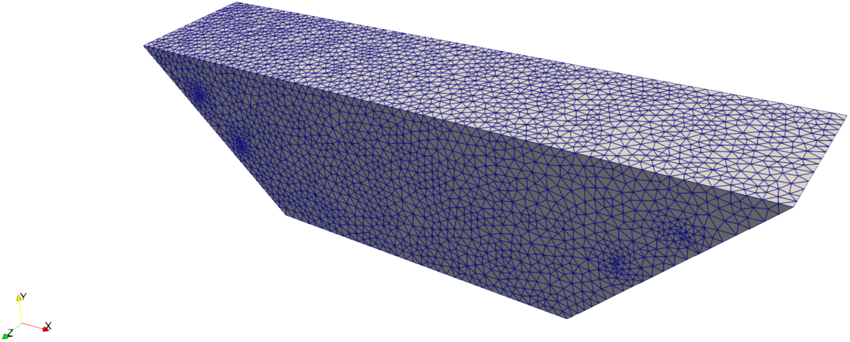

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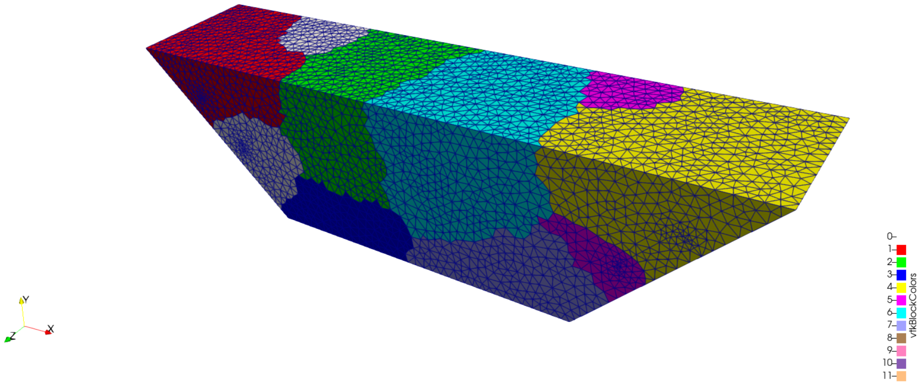

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Everything must be thought in parallel $\to$ completely revised implementation.

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- Script is run simultaneously by several processus. Communication of data between processes is achieved by several commands: `mpiComm`, `mpiGroup`, `mpiRequest`, `broadcast`, `mpiAllGather`...

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- The inverse of each of the matrices A_i can itself be computed with iterative methods, with **physics dependent** preconditioners.

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One can also use approximate algebraic methods on the submatrices A_i: incomplete LU, MUMPS solver, a fixed number of iterations of CG or GMRES, etc...

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2. the interface FreeFEM/PETSc written by Pierre Jolivet which allows to perform all the domain decomposition and preconditioning with minimum knowledge.