Lecture 12: shape optimization with geometric constraints.

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Geometric constraints

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$$\Omega_\theta := (I + \theta)\Omega = \{x + \theta(x) \,|\, x \in \Omega\}$$

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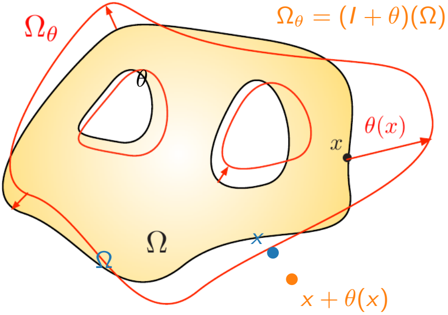


Figure: Deformation of a domain Ω with the method of Hadamard. A small vector field θ is used to deform Ω into $\Omega_\theta = (I + \theta)\Omega$.

Shape optimization problems

Shape/Topology optimization is the mathematical art of generating shapes that best fulfill a proposed objective.
Generically, a design optimization problem arises under the form

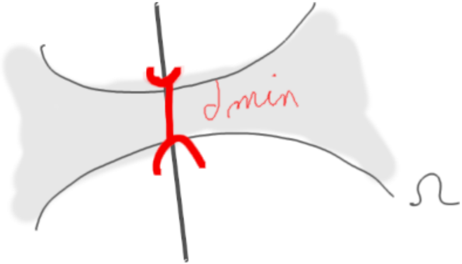
$$\min_{\Omega \subset D} \quad J(\Omega)$$
$$s.t. \begin{cases} G_i(\Omega) = 0, & 1 \leq i \leq p \\ H_j(\Omega) \leq 0, & 1 \leq j \leq q \end{cases}$$

where

- Ω is an **open domain** sought to be optimized
- J is an **objective function** to minimize (corresponding to a measure of the performance)
- G_i and H_j are respectively p and q **equality and inequality constraints** (corresponding e.g. to industrial specifications to meet)

Shape optimization problems

Today: how to take into account **geometric** constraints, e.g.:



- minimum thickness

Shape optimization problems

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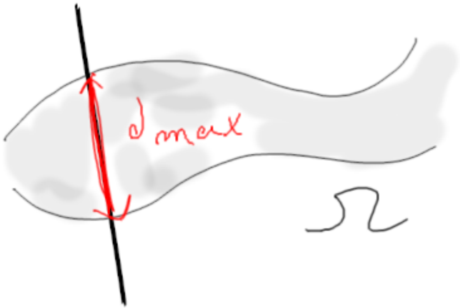

- minimum thickness
- maximum thickness

Shape optimization problems

Today: how to take into account **geometric** constraints, e.g.:

- minimum thickness
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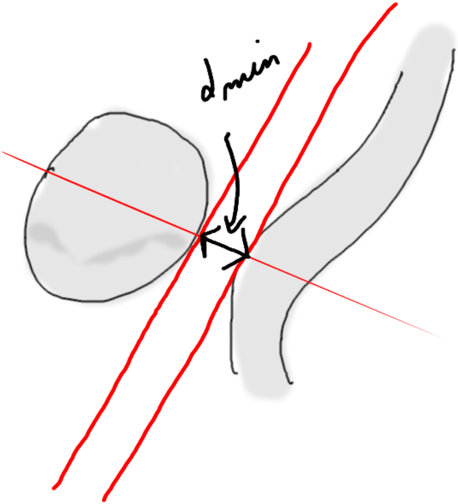
- minimum distance between to connected components



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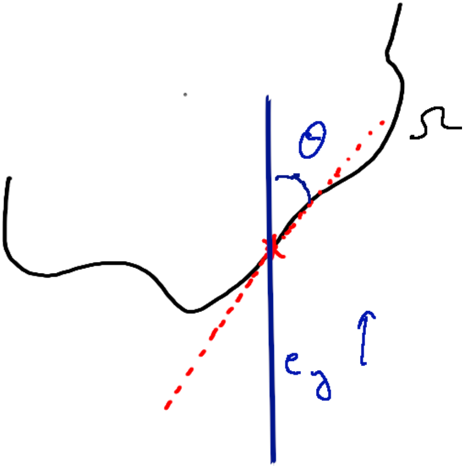

- minimum thickness
- maximum thickness
- minimum distance between to connected components
- minimum angle with respect to a direction (overhang)

Outline

1. The signed distance function
2. Formulation of geometric constraints
3. Shape derivatives of geometric constraints
4. Numerical examples

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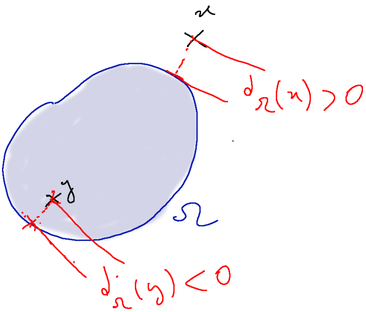
The signed distance function

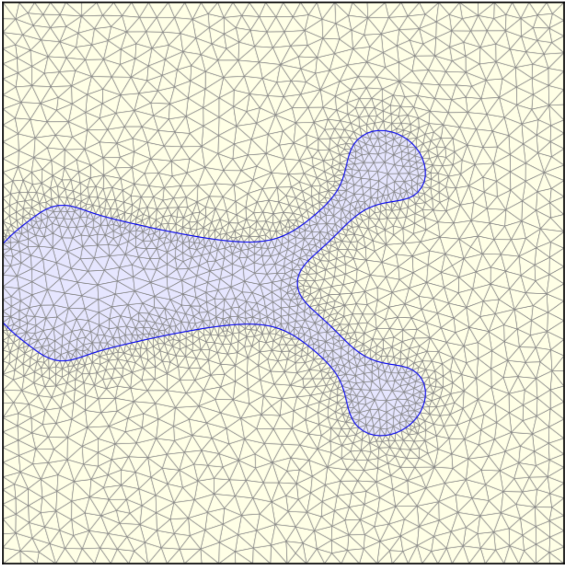
Definition 1

The signed distance function d_Ω to the domain $\Omega \subset D$ is defined by:

$$\forall x \in D, \;\; d_\Omega(x) = \begin{cases} -\min\limits_{y \in \partial\Omega} ||y - x|| & \text{if } x \in \Omega, \\ \min\limits_{y \in \partial\Omega} ||y - x|| & \text{if } x \in D \backslash \Omega. \end{cases}$$



An example: a meshed subdomain $\Omega \subset D$



The signed distance function

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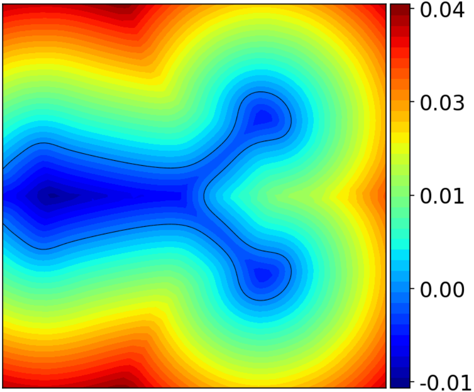
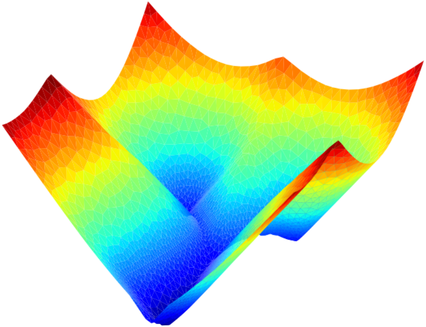




The signed distance function

Definition 2 (Skeleton set and projection)

1. The set of points $x \in \mathbb{R}^d$ for which the minimization problem

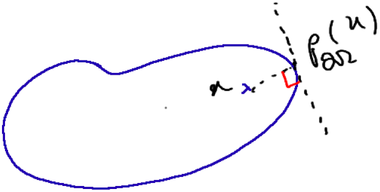
$$\min_{y \in \partial\Omega} ||x - y|| \tag{1}$$

admits several minimizers is called the *skeleton* of Ω and is denoted by Σ.

Definition 2 (Skeleton set and projection)

2. For any $x \in \mathbb{R}^d \backslash \Sigma$, the unique minimizer of eq. (1) is denoted $p_{\partial\Omega}(x)$ and is called the (orthogonal) *projection* of x onto $\partial\Omega$, in that case it holds

$$\forall x \in \mathbb{R}^d \backslash \Sigma,\ d_\Omega(x) = \begin{cases} -\|x - p_{\partial\Omega}(x)\| \text{ if } x \in \Omega, \\ \ \ \|x - p_{\partial\Omega}(x)\| \text{ if } x \notin \Omega. \end{cases}$$



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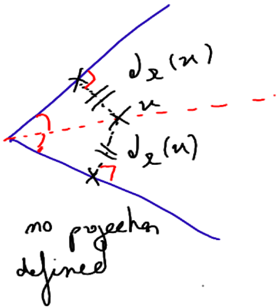

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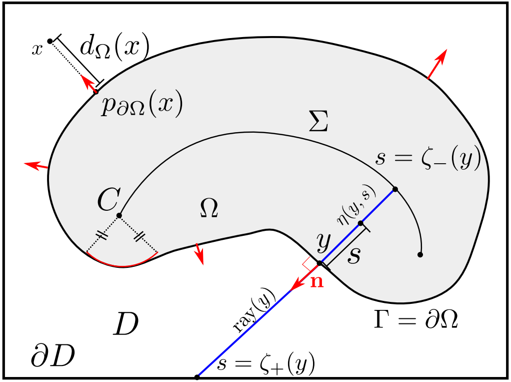

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Proposition 1 (Differentiability of d_Ω)

Assume Ω is a $\mathcal{C}^1$ domain with outward normal $\boldsymbol{n}$.

- *The signed distance function d_Ω is differentiable at any point $x \in \mathbb{R}^d \backslash \Sigma$, and it is not differentiable on Σ.*

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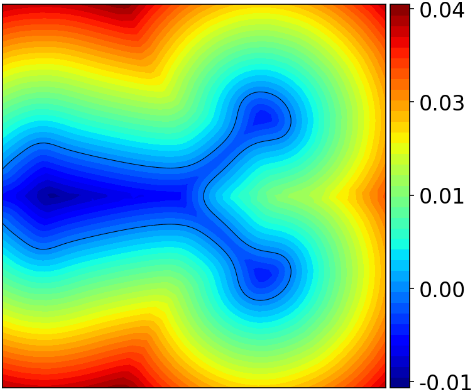

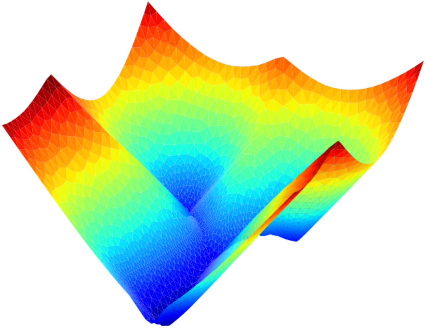

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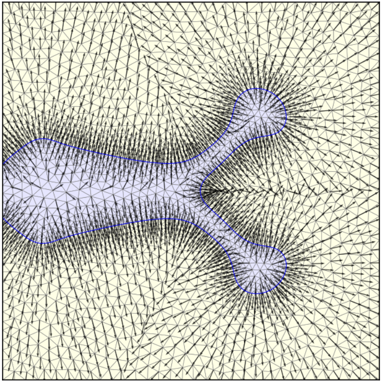
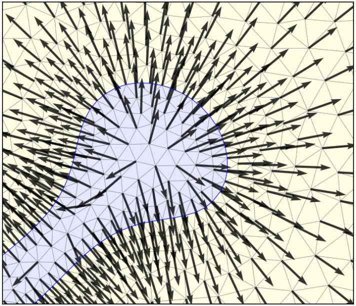
- *The gradient ∇d_Ω is an extension of the unit normal vector $\boldsymbol{n}$ to $\partial\Omega$ pointing outward Ω:*

$$\forall x \in \mathbb{R}^d \setminus \Sigma, \ \nabla d_\Omega(x) = \boldsymbol{n}(p_{\partial\Omega}(x)).$$

The signed distance function

An example: the gradient of the signed distance function ∇d_Ω:





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- *The gradient ∇d_Ω is an extension of the unit normal vector $\boldsymbol{n}$ to $\partial\Omega$ pointing outward Ω:*
$$\forall x \in \mathbb{R}^d \backslash \Sigma,\ \nabla d_\Omega(x) = \boldsymbol{n}(p_{\partial\Omega}(x)).$$
- *In particular, d_Ω solves the so-called "Eikonal" equation:*
$$\begin{cases} ||\nabla d_\Omega|| = 1 \ in \ \mathbb{R}^d \backslash \Sigma, \\ d_\Omega = 0 \ on \ \partial\Omega. \end{cases}$$

The signed distance function

Definition 3

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$$\forall y \in \partial\Omega,\ \ \zeta_+(y) = \sup\{s \geqslant 0 \,|\, \{y + t\nabla d_\Omega(y) \,|\, t \in [0, s)\} \cap (\overline{\Sigma} \cup \partial D) = \emptyset\},$$

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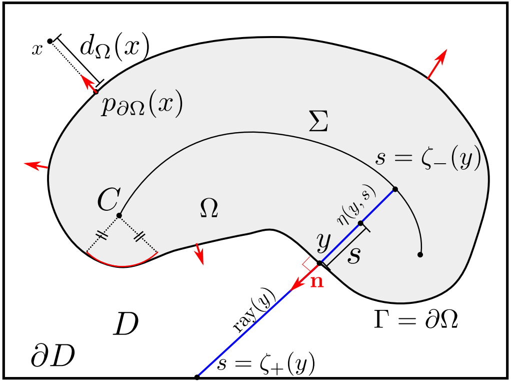
Then we also have

$$ray(y) = \{y + s\boldsymbol{n}(y) \,|\, \zeta_-(y) < s < \zeta_+(y)\}.$$

The signed distance function



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Formulation of geometric constraints

Some general principles:

- Geometric constraints are often **point-wise** constraints formulated from the **signed distance function**, e.g.

$$p(d_\Omega(x), \nabla d_\Omega(x)) \leq 0, \qquad \text{for all } x \in \Omega$$

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- Sometimes, enforcing strictly the constraint might not be desirable because the feasible region becomes tight, or because it may prevent topological changes (such as minimum thickness).
  A trade-off can be achieved by setting the value desired for the objective function as a constraint , e.g.:

$$\begin{aligned} \min_{\Omega \subset D} \quad & P(\Omega) \\ s.t. \quad & J(\Omega) \leq 0.9 \mathrm{J}_{opt} \end{aligned}$$

  where J_{opt} would be the optimal value without the constraint.

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Formulation of geometric constraints

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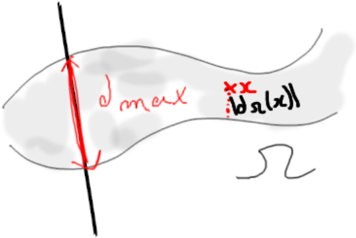

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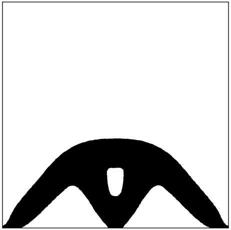
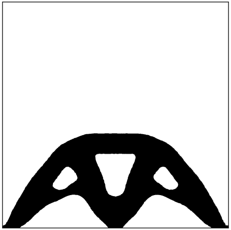
  for some p large enough.
- This means taking into account the averaged constraint

$$P_{\max}(\Omega) \leq d_{\max} \text{ with } P_{\max}(\Omega) := 2\left(\frac{1}{|\Omega|}\int_\Omega |d_\Omega|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$



(a) No maximum thickness constraint



(b) $d_{\max} = 0.07$.

Figure: Maximum thickness constraint for 2D arch.

- Minimum thickness can be modelled by the condition that the shape has a distance greater than $d_{\min}/2$ to its skeleton:

$$\forall y \in \partial\Omega, \qquad \zeta_-(y) < -d_{\min}/2.$$

Formulation of geometric constraints

Minimum thickness

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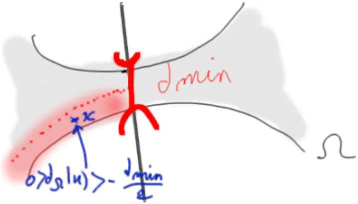

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  3. Enforcing the minimum thickness at all iterations would prevent topological changes to occur.
- It is better to rely on a more flexible formulation

Formulation of geometric constraints

Minimum thickness

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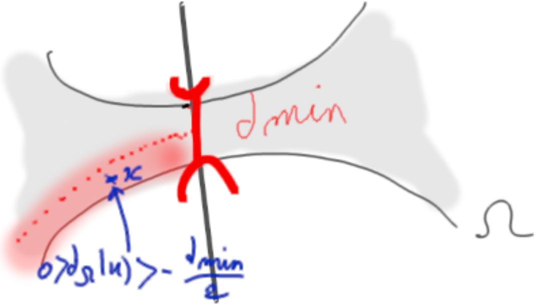

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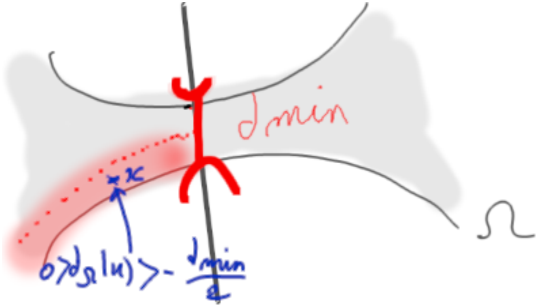

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- In order to find a good compromise between thickness and the original optimization problem, we use the reformulation

$$\begin{aligned} \min_{\Omega \subset D} \quad & P_{\min}(\Omega) \\ s.t. \quad & J(\Omega) \leq \alpha \mathrm{J}_{opt} \end{aligned}$$

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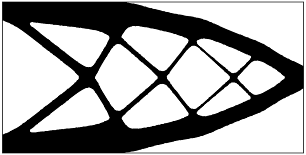
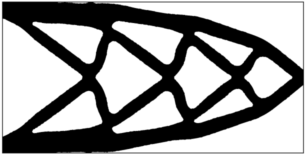
- the minimization will find shapes with good performances and which satisfy approximately the minimum thickness constraint.

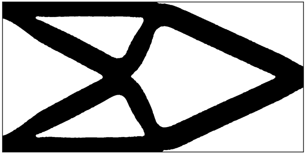
Formulation of geometric constraints
Minimum thickness



(a) No minimum thickness constraint.



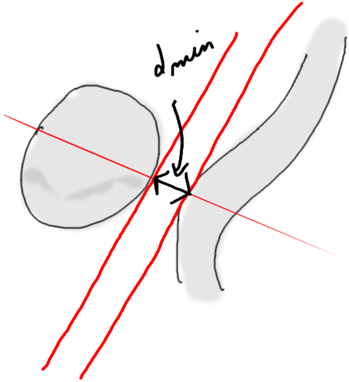
(b) $d_{\min} = 0.1$.



(c) $d_{\min} = 0.2$

Figure: Minimum thickness constraint for 2D cantilever.

Distance constraint:



Formulation of geometric constraints

Minimum distance constraint

An application: heat-exchangers:

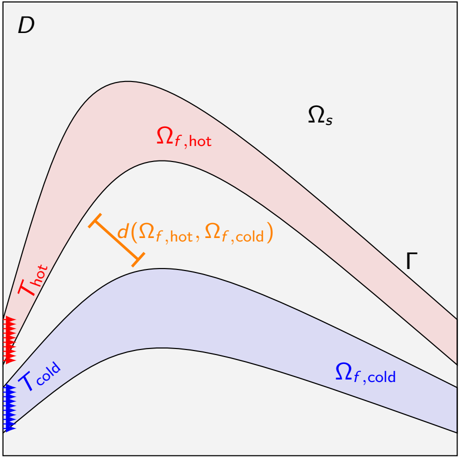


Figure: Minimum distance constraint for two-fluid heat-exchangers.

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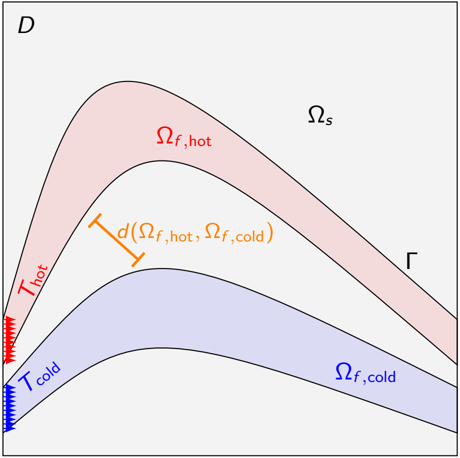

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We enforce it by imposing

$$\forall x \in \Omega_{f,\text{cold}}, \;\; d_{\Omega_{f,\text{hot}}}(x) \geqslant d_{\min},$$

where $d_{\Omega_{f,\text{hot}}}$ is the signed distance function to the domain $\Omega_{f,\text{hot}}$.

Formulation of geometric constraints

Minimum distance constraint

$$\forall x \in \Omega_{f,\text{cold}}, \;\; d_{\Omega_{f,\text{hot}}}(x) \geqslant d_{\min},$$

This constraint can be equivalently formulated as

$$\left|\left|\frac{1}{d_{\Omega_{f,\text{hot}}}}\right|\right|_{L^\infty(\Omega_{f,\text{cold}})} \leq \frac{1}{d_{\min}} \Leftrightarrow \left|\left|\frac{1}{d_{\Omega_{f,\text{hot}}}}\right|\right|^{-1}_{L^\infty(\Omega_{f,\text{cold}})} \geqslant d_{\min}.$$

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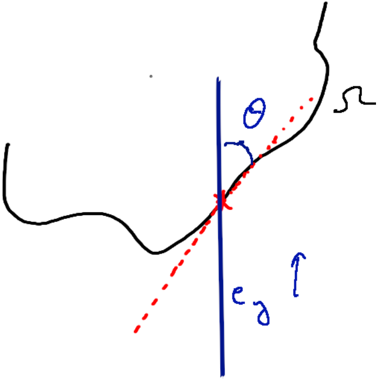
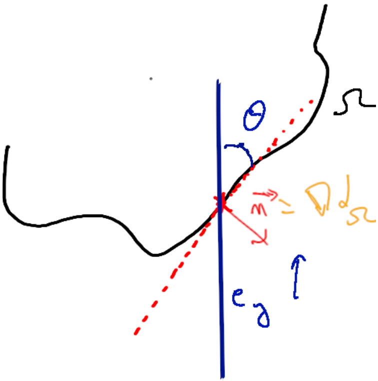
We can approximate it by

$$Q_{\mathrm{hot}\to\mathrm{cold}}(\Omega_f) \geqslant d_{\min}$$

were

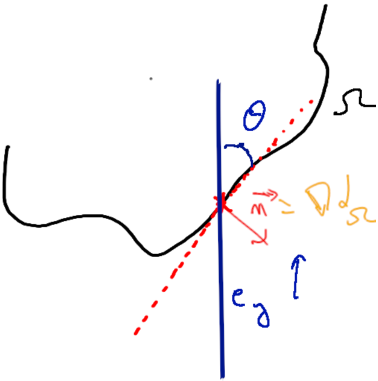
$$Q_{\mathrm{hot}\to\mathrm{cold}}(\Omega_f) := \left(\int_{\Omega_{f,\mathrm{cold}}} \frac{1}{|d_{\Omega_{f,\mathrm{hot}}}|^p}\mathrm{d}x\right)^{-\frac{1}{p}}.$$


θ
Ω
e_y


θ
Ω
$\vec{n} = \nabla d_\Omega$
e_y

Formulation of geometric constraints

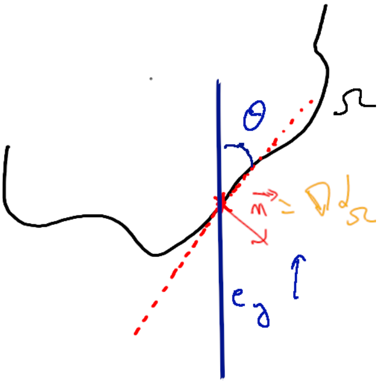
Overhang constraint



Overhang constraint: the angle θ between the tangent plane and the vertical direction must not be too large, e.g.

$$\theta \leq \beta$$

where β is the maximum angle allowed (e.g. $\beta = \pi/4$).



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$$\forall y \in \partial\Omega, \boldsymbol{n}(y) \cdot \boldsymbol{e}_y = \cos(\pi/2 + \theta) = -\sin(\theta) \geqslant -\sin(\beta).$$

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Remembering that $\boldsymbol{n}(y) = \nabla d_{\Omega(y)}$, we can formulate this in terms of d_Ω:

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This is equivalent to

$$\nabla d_\Omega \cdot \boldsymbol{e}_y + \sin\beta \geqslant 0,$$

for instance

$$P_\beta(\Omega) = 0 \text{ with } P_\beta(\Omega) := \int_\Omega \min\left(\nabla d_\Omega \cdot \boldsymbol{e}_y + \sin(\beta), 0\right)^2 \mathrm{d}x.$$

- Maximum thickness:

$$P_{\mathrm{max}}(\Omega) \leq d_{\mathrm{max}} \text{ with } P_{\mathrm{max}}(\Omega) := 2\left(\frac{1}{|\Omega|}\int_{\Omega} |d_{\Omega}|^{p} \mathrm{d}x\right)^{\frac{1}{p}}.$$

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Outline

1. The signed distance function
2. Formulation of geometric constraints
3. Shape derivatives of geometric constraints
4. Numerical examples

Shape derivatives of geometric constraints

All the previous formulations bring into play functionals formulated in terms of the signed distance function d_Ω , e.g.

$$P(\Omega) = \int_D j(d_\Omega) \mathrm{d}x.$$

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In order to compute the shape derivative of $P(\Omega)$, we need to compute the shape derivative $d'_\Omega(\boldsymbol{\theta})$.

Proposition 3

For any $x \notin \Sigma$, the map $\boldsymbol{\theta} \mapsto d_{(I+\boldsymbol{\theta})\Omega}(x)$ is Gâteaux-differentiable at $\boldsymbol{\theta}$ as an application from $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ into R^d and its derivative reads

$$d'_\Omega(\boldsymbol{\theta})(x) = -\boldsymbol{\theta}(p_{\partial\Omega}(x)) \cdot \boldsymbol{n}(p_{\partial\Omega}(x)).$$

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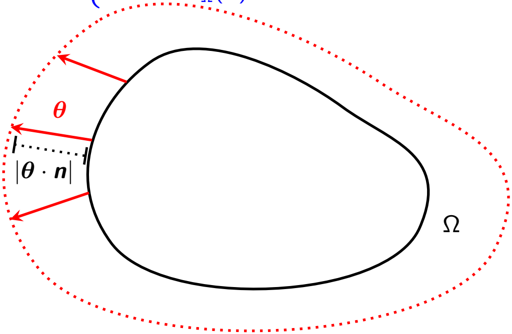
Equivalently, $d'_\Omega(\boldsymbol{\theta})$ is characterized by the boundary value problem

$$\begin{cases} \nabla d'_\Omega(\boldsymbol{\theta}) \cdot \nabla d_\Omega = 0 \text{ in } D \backslash \overline{\Sigma} \\ d'_\Omega(\boldsymbol{\theta}) = -\boldsymbol{\theta} \cdot \boldsymbol{n} \text{ on } \partial\Omega. \end{cases}$$



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The composition with the projection $p_{\partial\Omega}$ is not very easy to implement.

Proposition 4

For any $\boldsymbol{\theta} \in W^{1,\infty}(D, \mathbb{R}^d)$, *we have*

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$$= -\int_{\partial\Omega} \left(\int_{z \in ray(y)} j'(d_\Omega(z)) \prod_{i=1}^{d-1} (1 + \kappa_i(y) d_\Omega(z))\mathrm{d}z \right) \boldsymbol{\theta}(y) \cdot \boldsymbol{n}(y)\mathrm{d}\sigma(y)$$

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&= \int_{\partial\Omega} u(y)\boldsymbol{\theta}(y)\cdot\boldsymbol{n}(y)\mathrm{d}\sigma(y)
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$$

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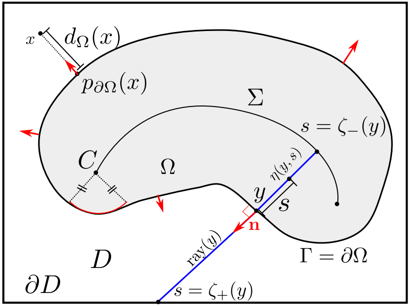

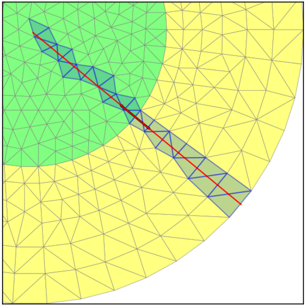

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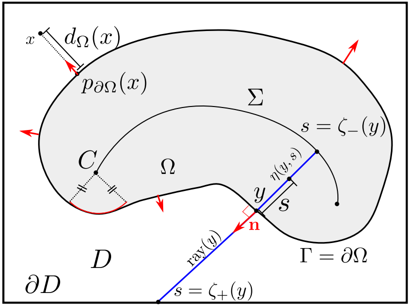

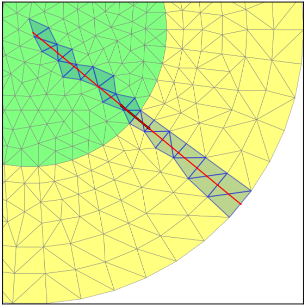

2. Estimating the principal curvatures $\kappa_i(y)$.

These two operations are quite delicate to implement !

It turns out that it is possible to compute u without integrating along the rays:

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Proposition 5

Let $\hat{u} \in V_\omega$ be the solution to the variational problem

$$\forall v \in V_\omega, \int_{\partial\Omega} \hat{u} v \mathrm{d}s + \int_D \omega(\nabla d_\Omega \cdot \nabla \hat{u})(\nabla d_\Omega \cdot \nabla v)\mathrm{d}x = -\int_D j'(d_\Omega) v \mathrm{d}x,$$

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$\omega > 0$ is a weight that can be chosen rather arbitrarily.

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Sketch of the proof.

Take $v = -d'_\Omega(\boldsymbol{\theta})$. Since $\nabla v \cdot \nabla d_\Omega = 0$ and $v = \boldsymbol{\theta} \cdot \boldsymbol{n}$ on $\partial\Omega$, one finds

$$\int_{\partial\Omega} \hat{u} \boldsymbol{\theta} \cdot \boldsymbol{n} \mathrm{d}s = \int_D j'(d_\Omega) d'_\Omega(\boldsymbol{\theta}) \mathrm{d}x.$$

whence $\hat{u} = u$ on $\partial\Omega$. □

Recap:

- The shape derivative of a geometric constraint of the form

$$P(\Omega) = \int_D j(d_\Omega(x))\mathrm{d}x$$

reads

$$P'(\Omega)(\boldsymbol{\theta}) = \int_D j'(d_\Omega)d'_\Omega(\theta)\mathrm{d}x = \int_{\partial\Omega} u\boldsymbol{\theta}\cdot\boldsymbol{n}\mathrm{d}s.$$

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Outline

1. The signed distance function
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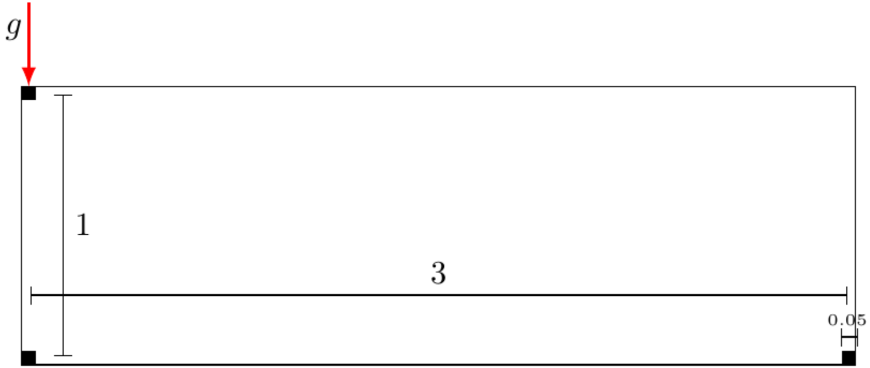
Numerical examples

Maximum thickness on MBB beam



Numerical examples

Maximum thickness on MBB beam

Optimized shape without maximum thickness constraint ($\max d_\Omega = 0.36$).

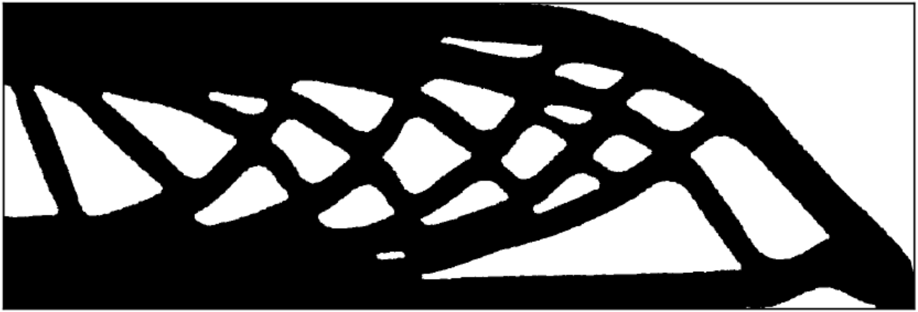
Figure: MBB beam without thickness constraint

(c) Optimized shape with maximum thickness constraint.

Figure: MBB beam with maximum thickness constraint

(a) Optimized shape without minimum thickness constraint.

Figure: MBB beam without minimum thickness constraint



(b) Optimized shape with minimum thickness constraint ($d_{\min} = 0.1$).

Figure: MBB beam with thickness constraint

(c) Optimized shape with minimum thickness constraint ($d_{\min} = 0.2$).

Figure: MBB beam with minimum thickness constraint

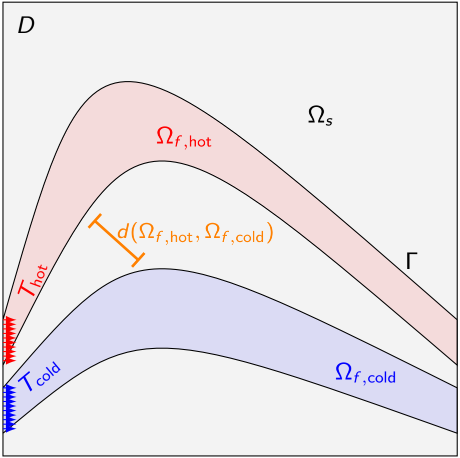


Figure: Settings of the heat exchanger topology optimization problem .

Non-penetration constraint:

$$d(\Omega_{f,\text{hot}}, \Omega_{f,\text{cold}}) \geqslant d_{\text{min}}.$$

We enforce it by imposing

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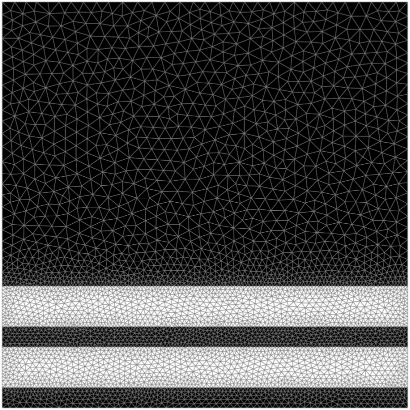
Iteration 0



Numerical examples

3D fluid-to-fluid heat exchanger

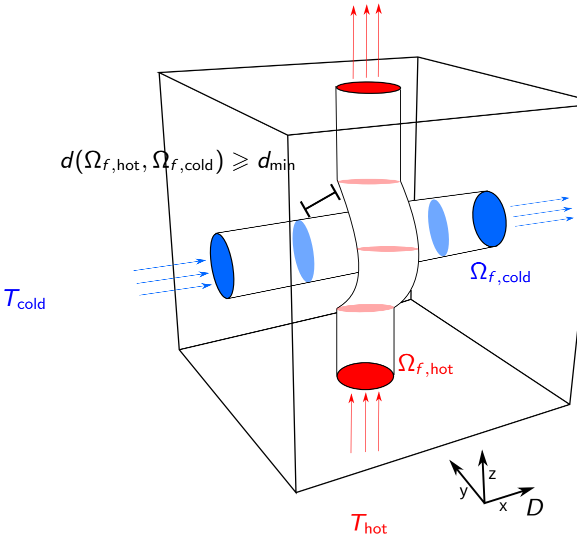


Figure: Schematic of the 3D setting.

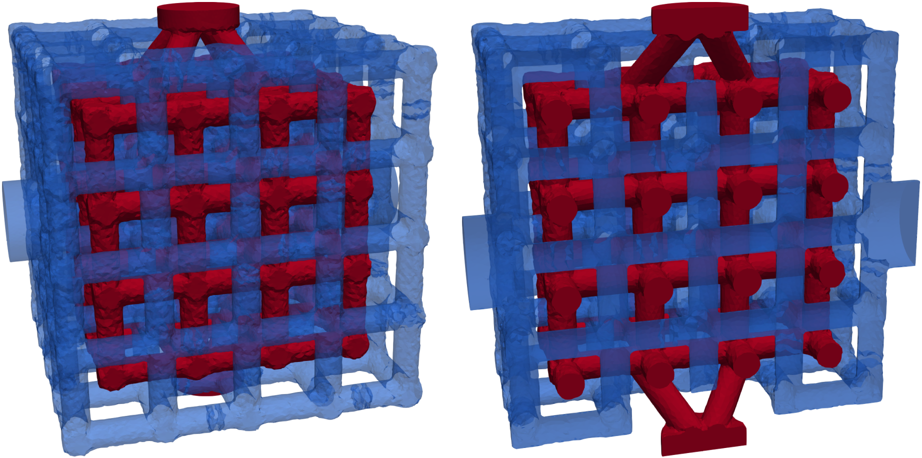


Figure: Initial distribution of fluid considered for the 3D heat exchanger test case.

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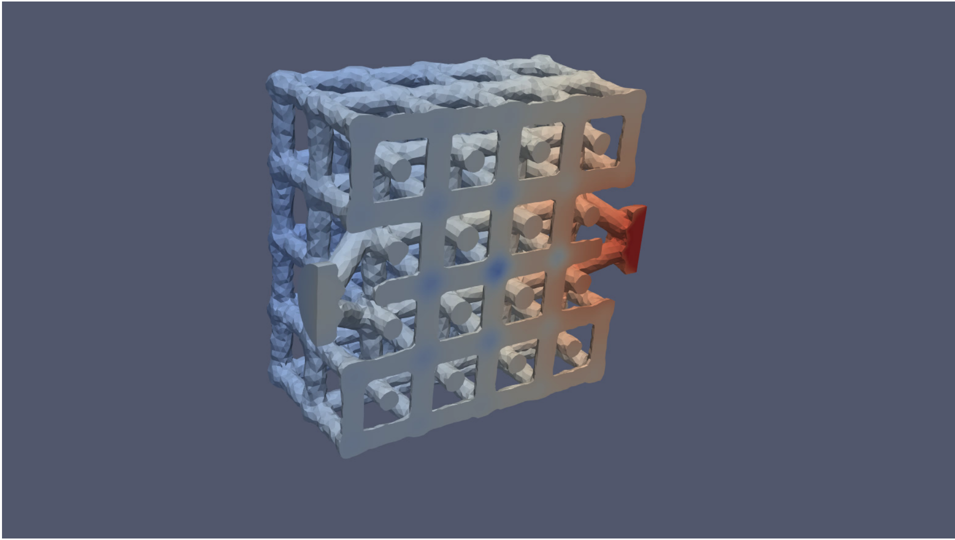

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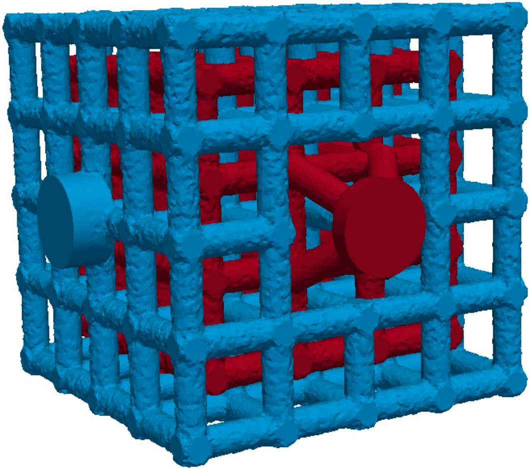

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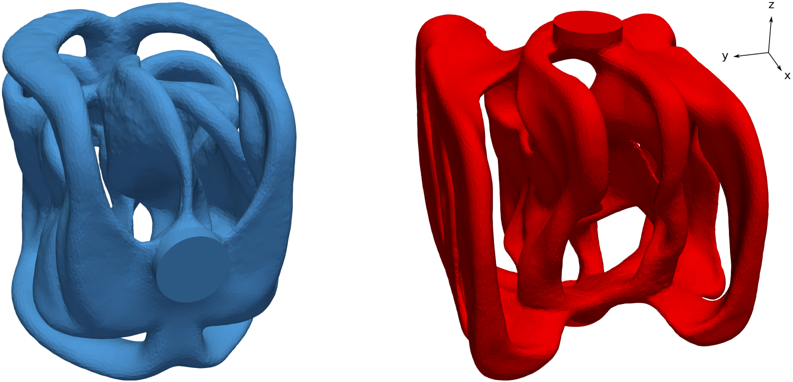

(a) Cold phase

(b) Hot phase

Figure: Separate plots of the topologically optimized cold and hot fluid phases in the configuration $d_{\text{min}} = 0.04$.

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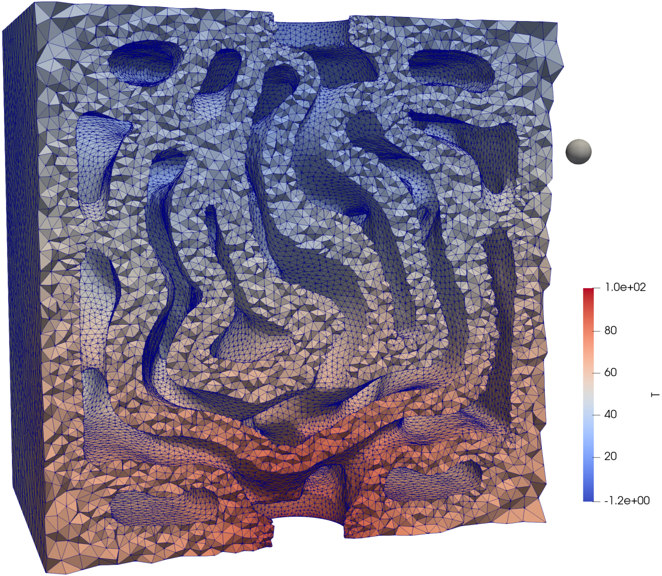

Figure: Cut of the resulting solid domain