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Empirically observed log returns of risky assets are not normally distributed, but exhibit significant skewness and kurtosis. Observed log-returns occasionally appear to change discontinuously. Certain price processes with no continuous component have been found to allow for a considerably better fit of observed log returns than the classical BS model. Pricing derivative contracts on such underlyings becomes more involved (mathematically and numerically) since partial integro-differential equations must be solved.
Outline

Lévy processes
Lévy models
Pricing equation
Variational formulation
Localization
Discretization
Definition 1
An adapted, càdlàg stochastic process $X = \{X_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$ such that $X_0 = 0$ is called a Lévy process if it has the following properties.
Definition 1
An adapted, càdlàg stochastic process $X = \{X_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$ such that $X_0 = 0$ is called a Lévy process if it has the following properties.

1. Independent increments: $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$.
2. Stationary increments: $X_t - X_s$ has the same distribution as $X_{t-s}$, $0 \leq s < t < \infty$.
3. Stochastically continuous: $\lim_{t \to s} X_t = X_s$, where the limit is taken in probability.
Associate to \( X = \{ X_t : t \in [0, T]\} \) a random measure \( J_X \) on \([0, T] \times \mathbb{R}\),

\[
J_X(\omega, \cdot) = \sum_{t \in [0, T], \Delta X_t \neq 0} 1(t, \Delta X_t),
\]

which is called jump measure.
Associate to $X = \{X_t : t \in [0, T]\}$ a random measure $J_X$ on $[0, T] \times \mathbb{R}$,

$$J_X(\omega, \cdot) = \sum_{t \in [0, T]} 1(t, \Delta X_t),$$

which is called jump measure.

For any measurable subset $B \subset \mathbb{R}$, $J_X([0, t] \times B)$ counts then the number of jumps of $X$ occurring between 0 and $t$ whose amplitude belongs to $B$. The intensity of $J_X$ is given by the Lévy measure.
Definition 2
Let $X$ be a Lévy process. The measure $\nu$ on $\mathbb{R}$ defined by

$$
\nu(B) = \mathbb{E}[\#\{t \in [0, 1] : \Delta X_t \neq 0, \Delta X_t \in B\}], \quad B \in \mathcal{B}(\mathbb{R}),
$$

is called the Lévy measure of $X$. $\nu(B)$ is the expected number, per unit time, of jumps whose size belongs to $B$. 

Definition 2
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is called the Lévy measure of $X$. $\nu(B)$ is the expected number, per unit time, of jumps whose size belongs to $B$.

The Lévy measure satisfies

$$
\int_{\mathbb{R}} 1 \wedge z^2 \nu(dz) < \infty.
$$
Theorem 3 (Lévy-Itô decomposition)

Let $X$ be a Lévy process and $\nu$ its Lévy measure. Then, there exist a $\gamma$, $\sigma$ and a standard Brownian motion $W$ such that

$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{|x| \geq 1} \mathbb{1}_{|\Delta X_s| \geq 1} J_X(d\tau, dx) + \lim_{\epsilon \to 0} \int_0^t \int_{\epsilon \leq |x| \leq 1} x (J_X(d\tau, dx) - \nu(dx) d\tau),$$

where $J_X$ the jump measure of $X$. 
Theorem 3 (Lévy-Itô decomposition)

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$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{|x| \geq 1} x J_X(ds, dx)$$

$$+ \lim_{\epsilon \downarrow 0} \int_0^t \int_{\epsilon \leq |x| \leq 1} x (J_X(ds, dx) - \nu(dx)ds)$$

$$= \gamma t + \sigma W_t + \sum_{0 \leq s \leq t} \Delta X_s 1_{\{|\Delta X_s| \geq 1\}}$$

$$+ \lim_{\epsilon \downarrow 0} \int_0^t \int_{\epsilon \leq |x| \leq 1} x \tilde{J}_X(ds, dx),$$

where $J_X$ the jump measure of $X$. 

(1)
The triplet \((\sigma^2, \nu, \gamma)\) is the **characteristic triplet** of the process \(X\). It can also be derived from the Lévy-Khinchin representation.

**Theorem 4 (Lévy-Khinchin representation)**

*Let \(X\) be a Lévy process with characteristic triplet \((\sigma^2, \nu, \gamma)\).*
The triplet \((\sigma^2, \nu, \gamma)\) is the characteristic triplet of the process \(X\). It can also be derived from the Lévy-Khinchin representation.

**Theorem 4 (Lévy-Khinchin representation)**

Let \(X\) be a Lévy process with characteristic triplet \((\sigma^2, \nu, \gamma)\). Then for \(t \geq 0\),

\[
\mathbb{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R},
\]

with \(\psi(\xi) = -i\gamma \xi + \frac{1}{2} \sigma^2 \xi^2 + \int_{\mathbb{R}} \left(1 - e^{i\xi z} + i\xi z 1_{\{|z| \leq 1\}}\right) \nu(dz).
\]

(2)
In (2), the integral exists since the integrand is bounded outside of any neighborhood of 0 and

\[ 1 - e^{i\xi z} + i\xi z 1_{\{|z| \leq 1\}} = \mathcal{O}(z^2) \quad \text{as} \quad |z| \to 0. \]
In (2), the integral exists since the integrand is bounded outside of any neighborhood of 0 and

$$1 - e^{i\xi z} + i\xi z 1_{\{|z| \leq 1\}} = O(z^2) \quad \text{as} \quad |z| \to 0.$$ 

Can replace the truncation function $1_{\{|z| \leq 1\}}$ by any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(z) = 1 + O(|z|)$ as $|z| \to 0$ and $f(z) = O(1/|z|)$ as $|z| \to \infty$. 

Different choices of $f$ do not affect $\sigma^2$ and $\nu$. But $\gamma$ depends on the choice of the truncation function.
In (2), the integral exists since the integrand is bounded outside of any neighborhood of \(0\) and

\[ 1 - e^{i \xi z} + i \xi z \mathbb{1}_{\{|z| \leq 1\}} = \mathcal{O}(z^2) \quad \text{as} \quad |z| \to 0. \]

Can replace the truncation function \(1_{\{|z| \leq 1\}}\) by any bounded measurable function \(f : \mathbb{R} \to \mathbb{R}\) satisfying \(f(z) = 1 + \mathcal{O}(|z|)\) as \(|z| \to 0\) and \(f(z) = \mathcal{O}(1/|z|)\) as \(|z| \to \infty\).

Different choices of \(f\) do not affect \(\sigma^2\) and \(\nu\). But \(\gamma\) depends on the choice of the truncation function.
If \( \int_{|z| \leq 1} |z| \nu(dz) < \infty \), we can use the zero function as \( f \) and get

\[
\psi(\xi) = -i \gamma_0 \xi + \frac{1}{2} \sigma^2 \xi^2 + \int_{\mathbb{R}} (1 - e^{i\xi z}) \nu(dz),
\]  

(3)
If $\int |z| \nu(dz) < \infty$, we can use the zero function as $f$ and get

$$\psi(\xi) = -i\gamma_0 \xi + \frac{1}{2} \sigma^2 \xi^2 + \int_{\mathbb{R}} (1 - e^{i\xi z}) \nu(dz), \quad (3)$$

If $\int |z| \nu(dz) < \infty$, then, letting $f$ be the constant function 1, we obtain

$$\psi(\xi) = -i\gamma_c \xi + \frac{1}{2} \sigma^2 \xi^2 + \int_{\mathbb{R}} (1 - e^{i\xi z} + i\xi z) \nu(dz), \quad (4)$$

with triplet $(\sigma^2, \nu, \gamma_c)_c$ where $\gamma_c$ is called the center of $X$ since $\mathbb{E}[X_t] = \gamma_c t$. 

Computational Methods for Quant. Finance
If $\int_{\mathbb{R}} \nu(dz) < \infty$, i.e., $X$ is a compound Poisson process, we can re-write (3) as

$$
\psi(\xi) = -i\gamma_0 \xi + \frac{1}{2} \sigma^2 \xi^2 + \lambda \int_{\mathbb{R}} (1 - e^{i\xi z}) \nu_0(dz),
$$

with $\lambda = \int_{\mathbb{R}} \nu(dz)$ and $\nu_0 = \nu / \lambda$. We say $X$ is of finite activity with jump intensity $\lambda$ and jump size distribution $\nu_0$. 
If \( \int_{\mathbb{R}} \nu(dz) < \infty \), i.e., \( X \) is a compound Poisson process, we can re-write (3) as

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\]

with \( \lambda = \int_{\mathbb{R}} \nu(dz) \) and \( \nu_0 = \nu/\lambda \). We say \( X \) is of finite activity with jump intensity \( \lambda \) and jump size distribution \( \nu_0 \).

We use the representation (4) instead of (2) throughout and omit the subscript \( c \) for simplicity.
No arbitrage considerations require Lévy processes employed in mathematical finance to be martingales.

**Lemma 5**

Let \( X \) be a Lévy process with characteristic triplet \((\sigma^2, \nu, \gamma)\). Assume, \( \int_{|z|>1} |z| \nu(dz) < \infty \) and \( \int_{|z|>1} e^z \nu(dz) < \infty \). Then, \( e^X \) is a martingale with respect to the filtration \( \mathcal{F} \) of \( X \) if and only if

\[
\frac{\sigma^2}{2} + \gamma + \int_{\mathbb{R}} (e^z - 1 - z) \nu(dz) = 0.
\]
A Lévy process of jump diffusion has the following form

\[ X_t = \gamma_0 t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \]

where \( N \) is a Poisson process with intensity \( \lambda \) counting the jumps of \( X \) and \( Y_i \) are the jump sizes which are modelled by i.i.d. random variables with distribution \( \nu_0 \). The Lévy measure is given by \( \nu = \lambda \nu_0 \).
Merton model: jumps in the log-price $X$ are assumed to have a Gaussian distribution, i.e., $Y_i \sim \mathcal{N}(\mu, \delta^2)$ and therefore,

$$\nu_0(dz) = \frac{1}{\sqrt{2\pi \delta^2}} e^{-\frac{(z-\mu)^2}{2\delta^2}} dz. \quad (6)$$
Merton model: jumps in the log-price $X$ are assumed to have a Gaussian distribution, i.e., $Y_i \sim \mathcal{N}(\mu, \delta^2)$ and therefore,
\[ \nu_0(dz) = \frac{1}{\sqrt{2\pi}\delta^2} e^{-\frac{(z-\mu)^2}{2\delta^2}} dz. \] (6)

Kou model: the distribution of jump sizes is asymmetric exponential with a density of the form
\[ \nu_0(dz) = (p\beta_+ e^{-\beta_+ z} 1_{\{z>0\}} + (1-p)\beta_- e^{-\beta_- |z|} 1_{\{z<0\}}) dz, \] (7)
with $\beta_+, \beta_- > 0$ governing the decay of the tails for the distribution of the positive and negative jump sizes and $p \in [0, 1]$ representing the probability of an upward jump. The probability distribution of returns of this model has semi-heavy tails.
Pure jump modes

- A popular class: subordination of a Brownian motion with drift. Using an increasing process or subordinator $G = \{G_t : t \geq 0\}$, the resulting process is given by

$$X_t = \sigma W_{G_t} + \theta G_t, \quad \sigma > 0, \quad \theta \in \mathbb{R}, \quad t \in [0, T],$$

where $W$ is a standard Brownian motion.
Pure jump modes

- A popular class: subordination of a Brownian motion with drift. Using an increasing process or subordinator $G = \{G_t : t \geq 0\}$, the resulting process is given by

$$X_t = \sigma W_{G_t} + \theta G_t, \quad \sigma > 0, \quad \theta \in \mathbb{R}, \quad t \in [0, T],$$

where $W$ is a standard Brownian motion.

- Example: variance gamma process. Consider a gamma process $G$ with Lévy density $k_G(s) = e^{-\frac{s}{\vartheta}} (\vartheta s)^{-1}1_{\{s > 0\}}$. Then, the Lévy measure of $X$ is given for $B \in \mathcal{B}(\mathbb{R})$ by

$$\nu(B) = \int_B \int_0^\infty \frac{1}{\sqrt{2\pi s\sigma^2}} e^{-\frac{(z-\theta s)^2}{2s\sigma^2}} \frac{1}{\vartheta s} e^{-\frac{s}{\vartheta}} dsdz = ..$$
\[
.. = \frac{1}{\vartheta \sqrt{2\pi \sigma^2}} \int_B e^{\theta z/\sigma^2} \int_0^\infty s^{-\frac{1}{2}} - 1 e^{-\frac{z^2}{2\sigma^2}} \frac{1}{s} - \left( \frac{\theta^2}{2\sigma^2} + \frac{1}{\vartheta} \right) s \, ds \, dz.
\]

▶ One obtains

\[
\nu(dz) = \left( c \frac{e^{-\beta_+ |z|}}{|z|} 1_{\{z>0\}} + c \frac{e^{-\beta_- |z|}}{|z|} 1_{\{z<0\}} \right) dz. \quad (8)
\]
\[ \mathbb{E} = \frac{1}{\sqrt{2\pi \sigma^2}} \int_B e^{\theta z/\sigma^2} \int_0^\infty \left( e^{-(z^2/2\sigma^2 + \theta^2/2\sigma^2 + \vartheta)} - 1 - s^{-1/2} e^{-z^2/2\sigma^2} \left( s^2 - \frac{\theta^2}{2\sigma^2} + \frac{1}{\vartheta} \right) s \right) dsdz. \]

▶ One obtains

\[ \nu(dz) = \left( c e^{-\beta+|z|/|z|} 1\{z>0\} + c e^{-\beta-|z|/|z|} 1\{z<0\} \right) dz. \tag{8} \]

▶ The variance gamma process is a special case of the \textit{tempered stable process} (for \( c = c_+ = c_- \) also called \textit{CGMY process} or \textit{KoBoL}) which has a Lévy density of the form

\[ \nu(dz) = \left( c_+ \frac{e^{-\beta+|z|}}{|z|^{1+\alpha}} 1\{z>0\} + c_- \frac{e^{-\beta-|z|}}{|z|^{1+\alpha}} 1\{z<0\} \right) dz, \tag{9} \]

for \( c_+, c_-, \beta_+, \beta_- > 0 \) and \( 0 \leq \alpha < 2 \).
Admissible market models

Assumption 6

Let $X$ be a Lévy process with characteristic triplet $(\sigma^2, \nu, \gamma)$ and Lévy density $k(z)$ where $\nu(dz) = k(z)dz$.

1. There are constants $\beta_- > 0$, $\beta_+ > 1$ and $C > 0$ such that
$$k(z) \leq Ce^{-\beta_-|z|}, \quad z < -1, \quad k(z) \leq Ce^{-\beta_+z}, \quad z > 1. \quad (10)$$
Admissible market models

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$$ k(z) \leq Ce^{-\beta_- |z|}, \quad z < -1, \quad k(z) \leq Ce^{-\beta_+ z}, \quad z > 1. \quad (10) $$

2. There exist constants $0 < \alpha < 2$ and $C_+ > 0$ such that
$$ k(z) \leq C_+ \frac{1}{|z|^{1+\alpha}}, \quad 0 < |z| < 1. \quad (11) $$
Admissible market models

**Assumption 6**

Let $X$ be a Lévy process with characteristic triplet $(\sigma^2, \nu, \gamma)$ and Lévy density $k(z)$ where $\nu(dz) = k(z)dz$.

1. **There are constants** $\beta_- > 0$, $\beta_+ > 1$ and $C > 0$ such that
   
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   (10)

2. **There exist constants** $0 < \alpha < 2$ and $C_+ > 0$ such that
   
   \[ k(z) \leq C_+ \frac{1}{|z|^{1+\alpha}}, \quad 0 < |z| < 1. \]  
   (11)

3. **If $\sigma = 0$, additionally assume that there is a** $C_- > 0$ such that
   
   \[ \frac{1}{2}(k(z) + k(-z)) \geq C_- \frac{1}{|z|^{1+\alpha}}, \quad 0 < |z| < 1. \]  
   (12)
Outline

Lévy processes

Lévy models

Pricing equation

Variational formulation

Localization

Discretization
As in the Black-Scholes case we assume the risk-neutral dynamics of the underlying asset price is given by
\[ S_t = S_0 e^{rt + X_t}, \]
where \( X \) is a Lévy process with characteristic triplet \((\sigma^2, \nu, \gamma)\) under a non-unique EMM.
As in the Black-Scholes case we assume the risk-neutral dynamics of the underlying asset price is given by
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where \( X \) is a Lévy process with characteristic triplet \( (\sigma^2, \nu, \gamma) \) under a non-unique EMM.

Lemma 5: the martingale condition implies
\[ \gamma = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1 - z) \nu(dz). \quad (13) \]
As in the Black-Scholes case we assume the risk-neutral dynamics of the underlying asset price is given by \( S_t = S_0 e^{r t + X_t} \), where \( X \) is a Lévy process with characteristic triplet \((\sigma^2, \nu, \gamma)\) under a non-unique EMM.

Lemma 5: the martingale condition implies

\[
\gamma = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1 - z) \nu(dz).
\]  

We want to compute the value of the option in log-price with payoff \( g \) which is the expectation

\[
v(t, x) = \mathbb{E} \left[ e^{-r(T-t)} g(e^{rT+X_T}) \mid X_t = x \right].
\]
To show that $v(t, x)$ is a solution of a deterministic partial integro-differential equation (PIDE), we need

**Proposition 7**

Let $X$ be a Lévy process with characteristic triplet $(\sigma^2, \nu, \gamma)$ where the Lévy measure satisfies (10). Denote by $A$ the integro-differential operator
To show that \( v(t, x) \) is a solution of a deterministic \textit{partial integro-differential equation} (PIDE), we need

**Proposition 7**

\textit{Let} \( X \) \textit{be a Lévy process with characteristic triplet} \((\sigma^2, \nu, \gamma)\) \textit{where the Lévy measure satisfies} (10). \textit{Denote by} \( \mathcal{A} \) \textit{the integro-differential operator}

\[
(\mathcal{A}f)(x) := \frac{1}{2} \sigma^2 \partial_{xx} f(x) + \gamma \partial_x f(x) \\
+ \int_{\mathbb{R}} (f(x + z) - f(x) - z \partial_x f(x)) \nu(dz), \quad (15)
\]
To show that \( v(t, x) \) is a solution of a deterministic partial integro-differential equation (PIDE), we need

**Proposition 7**

Let \( X \) be a Lévy process with characteristic triplet \((\sigma^2, \nu, \gamma)\) where the Lévy measure satisfies (10). Denote by \( A \) the integro-differential operator

\[
(Af)(x) := \frac{1}{2} \sigma^2 \partial_{xx} f(x) + \gamma \partial_x f(x) + \int_{\mathbb{R}} \left( f(x + z) - f(x) - z \partial_x f(x) \right) \nu(dz),
\]

for functions \( f \in C^2(\mathbb{R}) \) with bounded derivatives. Then, the process \( M_t := f(X_t) - \int_0^t (Af)(X_s) ds \) is a martingale with respect to the filtration of \( X \).
Theorem 8

Let \( v \in C^{1,2}(J \times \mathbb{R}) \cap C^0(\overline{J} \times \mathbb{R}) \) with bounded derivatives in \( x \) be a solution of
Theorem 8

Let $v \in C^{1,2}(J \times \mathbb{R}) \cap C^0(\overline{J} \times \mathbb{R})$ with bounded derivatives in $x$ be a solution of

$$
\partial_t v + \mathcal{A}v - rv = 0, \quad \text{in } J \times \mathbb{R}, \quad v(T, x) = g(e^x), \quad \text{in } \mathbb{R}, \quad (16)
$$
Theorem 8

Let $v \in C^{1,2}(J \times \mathbb{R}) \cap C^0(\overline{J} \times \mathbb{R})$ with bounded derivatives in $x$ be a solution of

$$
\partial_t v + Av - rv = 0, \quad \text{in} \ J \times \mathbb{R}, \quad v(T, x) = g(e^x), \quad \text{in} \ \mathbb{R}, \quad (16)
$$

where $A$ as in (15) with drift $r + \gamma$. Then, $v(t, x)$ can also be represented as

$$
v(t, x) = \mathbb{E} \left[ e^{-r(T-t)} g(e^{rT+X_T}) \mid X_t = x \right].
$$
Using the Lévy-Khinchin representation (Theorem 4), we derive from (15), with $\xi \in \mathbb{R}$,

\begin{align*}
(-A)e^{i\xi x} &= \frac{1}{2} \sigma^2 \xi^2 e^{i\xi x} - \gamma i \xi e^{i\xi x} \\
&\quad - \int_\mathbb{R} (e^{i\xi(x+z)} - e^{i\xi x} - iz \xi e^{i\xi x}) \nu(dz) \\
&= \psi(\xi) e^{i\xi x}.
\end{align*}

(17)
\( \mathcal{A} \) as a PDO

- Using the Lévy-Khinchin representation (Theorem 4), we derive from (15), with \( \xi \in \mathbb{R} \),

\[
(-\mathcal{A})e^{i\xi x} = \frac{1}{2} \sigma^2 \xi^2 e^{i\xi x} - \gamma i\xi e^{i\xi x} \\
- \int_{\mathbb{R}} (e^{i\xi(x+z)} - e^{i\xi x} - iz\xi e^{i\xi x}) \nu(dz)
\]

\[
= \psi(\xi)e^{i\xi x}.
\]

- For \( f \in \mathcal{S}(\mathbb{R}) \) we can write

\[
f(x) = \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi) d\xi,
\]

where \( \hat{f}(\xi) := (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx \) is the Fourier transform of \( f \).
By applying (17) to the representation (18), using Fubini’s theorem and the convergence theorem of Lebesgue, we obtain
By applying (17) to the representation (18), using Fubini's theorem and the convergence theorem of Lebesgue, we obtain

\[ (-Af)(x) = \int_{\mathbb{R}} (-Ae^{i\xi x}) \hat{f}(\xi) d\xi = \int_{\mathbb{R}} \psi(\xi) e^{i\xi x} \hat{f}(\xi) d\xi. \]  

(19)

Thus, \(-A\) is a pseudodifferential operator (PDO) with symbol \(\psi\).
Back to the pricing PIDE. As always, we change to \textit{time-to-maturity} $t \to T - t$. Additionally, we remove the drift $\gamma$ and the interest rate $r$. Thus, we set

$$u(t, x) =: e^{rt}v(T - t, x - (\gamma + r)t),$$

which satisfies

$$\partial_t u - A^J u = 0, \quad \text{in } (0, T) \times \mathbb{R}, \quad (20)$$

with initial condition $u(0, x) = g(e^x)$ in $\mathbb{R}$ and

$$(A^J f)(x) := \frac{1}{2}\sigma^2 \partial_{xx} f(x) + \int_{\mathbb{R}} (f(x + z) - f(x) - z\partial_x f(x))\nu(dz). \quad (21)$$
Remark 9

If the Lévy measure satisfies $\int_{|z| \leq 1} |z| \nu(dz) < \infty$, we remove the drift $\gamma_0$ which is given due to the martingale condition by

$$\gamma_0 = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^z - 1) \nu(dz).$$

and obtain the generator

$$(\mathcal{A}^J f)(x) = \frac{1}{2} \sigma^2 \partial_{xx} f(x) + \int_{\mathbb{R}} (f(x + z) - f(x)) \nu(dz).$$
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Sobolev spaces of fractional order

- Need Sobolev spaces of fractional order $H^s(\mathbb{R})$, $s \in \mathbb{R}$.
- Recall: for $m \in \mathbb{N}_0$ we defined the norm
  $$\|u\|_{H^m(\mathbb{R})}^2 := \sum_{j=0}^{m} \|D^j u\|_{L^2(\mathbb{R})}^2$$
- For any $j \in \mathbb{N}_0$, we can write, due to Plancherel’s theorem
  $$\|D^j u\|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |D^j u(x)|^2 dx = 2\pi \int_{\mathbb{R}} |\widehat{D^j u}(\xi)|^2 d\xi$$
  $$= 2\pi \int_{\mathbb{R}} |\xi|^{2j} |\widehat{u}(\xi)|^2 d\xi.$$  
- Define an equivalent $H^s$-norm for $s \geq 0$ via
  $$\|u\|_{H^s(\mathbb{R})}^2 := \int_{\mathbb{R}} (1 + |\xi|)^{2s} |\widehat{u}(\xi)|^2 d\xi.$$
Set

\[ H^s(\mathbb{R}) := \{ u \in L^2(\mathbb{R}) : \| u \|_{H^s(\mathbb{R})} < \infty \}. \]

and, for \( G := (a, b) \subset \mathbb{R} \),

\[ \tilde{H}^s(G) := \{ u \mid_G : u \in H^s(\mathbb{R}), u \mid_{\mathbb{R} \setminus \overline{G}} = 0 \}. \]

For \( 0 < \alpha < 2 \) as in (11)–(12), let

\[ \rho = \begin{cases} 
1 & \text{if } \sigma > 0, \\
\alpha/2 & \text{if } \sigma = 0.
\end{cases} \]
The variational formulation of the PIDE (20) reads

Find \( u \in L^2(J; H^\rho(\mathbb{R})) \cap H^1(J; L^2(\mathbb{R})) \) such that

\[
(\partial_t u, v) + a^J(u, v) = 0, \quad \forall v \in H^\rho(\mathbb{R}), \quad \text{a.e. in } J,
\]

\( u(0) = u_0, \)

where \( u_0(x) := g(e^x) \) and the bilinear form

\( a^J(\cdot, \cdot) : H^\rho(\mathbb{R}) \times H^\rho(\mathbb{R}) \to \mathbb{R} \) is given by

\[
a^J(\varphi, \phi) := \frac{1}{2} \sigma^2 (\varphi', \phi')
\]

\[
- \int_{\mathbb{R}} \int_{\mathbb{R}} (\varphi(x + z) - \varphi(x) - z\varphi'(x))\phi(x)\nu(dz)dx.
\]
Remark 10

Let \( \varphi, \phi \in C_0^{\infty}(\mathbb{R}) \). By the definition of the bilinear form \( a^J(\cdot, \cdot) \) and (19) we find

\[
a^J(\varphi, \phi) = \int_{\mathbb{R}} (-A\varphi)(x)\phi(x)dx
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(\xi)e^{ix\xi} \hat{\varphi}(\xi) d\xi \phi(x) dx
\]

\[
= \int_{\mathbb{R}} \psi(\xi) \hat{\varphi}(\xi) \int_{\mathbb{R}} e^{ix\xi} \phi(x) dx d\xi
\]

\[
= \int_{\mathbb{R}} \psi(\xi) \hat{\varphi}(\xi) \\overline{\phi(\xi)} d\xi.
\]
Have to show that $a^J(\cdot, \cdot)$ is continuous and satisfies a Gårding inequality on $\mathcal{V} = H^\rho(\mathbb{R})$. By Remark 10, it is sufficient to study the characteristic exponent $\psi$.

**Lemma 11**

*Let $X$ be a Lévy process with characteristic triplet $(\sigma^2, \nu, 0)$ where the Lévy measure satisfies (11), (12). Then, there exist constants $C_i > 0$, $i = 1, 2, 3$,

\[
\Re \psi(\xi) \geq C_1 |\xi|^{2\rho}, \quad |\psi(\xi)| \leq C_2 |\xi|^{2\rho} + C_3.
\]
Using Lemma 11, we have

**Theorem 12**

Let $X$ be a Lévy process with characteristic triplet $(\sigma^2, \nu, 0)$ where the Lévy measure satisfies (11), (12). Then, there exist constants $C_i > 0$, $i = 1, 2, 3$, such that there holds for all $\varphi, \phi \in H^\rho(\mathbb{R})$

$$|a^J(\varphi, \phi)| \leq C_1 \|\varphi\|_{H^\rho} \|\phi\|_{H^\rho},$$

$$a^J(\varphi, \varphi) \geq C_2 \|\varphi\|_{H^\rho}^2 - C_3 \|\varphi\|_{L^2}^2.$$
Outline

Lévy processes
Lévy models
Pricing equation
Variational formulation
Localization
Discretization
As in the Black-Scholes case the unbounded log-price domain $\mathbb{R}$ is truncated to a bounded domain $G = (-R, R), R > 0$.

Let $\tau_G = \inf\{t \geq 0 \mid X_t \in G^c\}$ be the first hitting time of the complement set $G^c = \mathbb{R} \setminus G$ by $X$.

Assume there exist $C > 0, q \geq 1$ such that

$$g(s) \leq C(s + 1)^q, \quad \text{for all } s \in \mathbb{R}_+.$$ \hspace{1cm} (24)

Denote the pricing function of the knock-out barrier option in log-price by $v_R$. 
Theorem 13

Suppose the payoff function $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfies (24). Let $X$ be a Lévy process with characteristic triplet $(\sigma^2, \nu, 0)$ where the Lévy measure satisfies (10) with $\beta_+, \beta_- > q$ and $q$ as in (24). Then, there exist $C(T, \sigma, \nu), \gamma_1, \gamma_2 > 0$, such that

$$|v(t, x) - v_R(t, x)| \leq C(T, \sigma, \nu) e^{-\gamma_1 R + \gamma_2 |x|}.$$ 

- Bilinear form on the bounded domain $G$ is given by $a^J_R(u, v) = a^J(\tilde{u}, \tilde{v})$, where

- $\tilde{u}$ is the extension by zero to all of $\mathbb{R}$ of any function $u$ with support in $G$. 
We restate the problem (22) on the bounded domain: 

Find \( u_R \in L^2(J; \tilde{H}^\rho(G)) \cap H^1(J; L^2(G)) \) such that 

\[
(\partial_t u_R, v) + a^J_J(u_R, v) = 0, \quad \forall v \in \tilde{H}^\rho(G), \text{ a.e. in } J, \\
u_R(0) = u_0|_G.
\]
Outline

Lévy processes
Lévy models
Pricing equation
Variational formulation
Localization
Discretization
Focus on pure jump models, i.e., set $\sigma = 0$.

Main problem: the singularity of the Lévy measure at $z = 0$.

Integrate by parts to obtain, for $f \in \tilde{H}^2(G)$,

$$\int_{0}^{\infty} (f(x + z) - f(x) - zf'(x))k(z)dz$$

$$= (f(x + z) - f(x) - zf'(x))k^{(-1)}(z) \bigg|_{z=0}^{z=\infty}$$

$$= 0$$

$$- \int_{0}^{\infty} (f'(x + z) - f'(x))k^{(-1)}(z)dz$$

$$= - (f'(x + z) - f'(x))k^{(-2)}(z) \bigg|_{z=0}^{z=\infty}$$

$$= 0$$

$$+ \int_{0}^{\infty} f''(x + z)k^{(-2)}(z)dz.$$
Herewith, we denote by $k^{(-i)}(z)$ the $i$-th antiderivative of $k$ vanishing at $\pm\infty$, i.e.,

$$k^{(-i)}(z) = \begin{cases} 
\int_{-\infty}^{z} k^{(-i+1)}(x)dx & \text{if } z < 0, \\
-\int_{z}^{\infty} k^{(-i+1)}(x)dx & \text{if } z > 0.
\end{cases}$$

We obtain,

$$(\mathcal{A}^J f)(x) = \int_{\mathbb{R}} f''(x + z)k^{(-2)}(z)dz,$$  \hspace{1cm} (26)

Similarly, for $\varphi, \phi \in H^1_0(G)$,

$$a^J(\varphi, \phi) := \int_{G} \int_{G} \varphi'(y)\phi'(x)k^{(-2)}(y - x)dydx.$$ \hspace{1cm} (27)
Finite difference discretization

- Replace the domain $J \times G$ by discrete grid points $(t_m, x_i)$ and approximate the partial derivatives in (26) by difference quotients at the grid points.

- Let the space grid points be given by

$$x_i := -R + ih, \ i = 0, 1, \ldots, N + 1, \ h := \frac{2R}{N + 1} = \Delta x,$$

which are equidistant with mesh width $h$, and the time levels by

$$t_m := m\Delta t, \ m = 0, 1, \ldots, M, \ \Delta t := \frac{T}{M}.$$
Define the weights for \( j = 0, 1, 2, \ldots, \)

\[
\nu_j^+ := \int_{jh}^{(j+1)h} k(-2)(z)dz = k(-3)((j + 1)h) - k(-3)(jh),
\]

\[
\nu_j^- := \int_{-(j+1)h}^{-jh} k(-2)(z)dz = k(-3)(-jh) - k(-3)(-(j + 1)h),
\]

which satisfy

\[
\sum_{j=0}^{\infty} (\nu_j^+ + \nu_j^-) = \int_{\mathbb{R}} k(-2)(z)dz < \infty
\]
We discretize the generator (26) for \( f \in C^4_0(G) \) at the mesh point \( x_i, i = 1, \ldots, N \), by

\[
\int_{0}^{R-x_i} \partial_{xx} f(x_i + z) k^{(-2)}(z) dz = \sum_{j=0}^{N-i} \partial_{xx} f(x_i + jh) \nu_j^+ + O(h)
\]

\[
= \sum_{j=0}^{N-i} (\delta_{xx} f)_{i+j} \nu_j^+ + O(h) + O(h^2),
\]

\[
\int_{-R+x_i}^{0} \partial_{xx} f(x_i + z) k^{(-2)}(z) dz = \sum_{j=0}^{i-1} (\delta_{xx} f)_{i-j} \nu_j^- + O(h).
\]
Define $G^J \in \mathbb{R}^{N \times N}$, for $i = 1, \ldots, N$, by

$G^J_{i,i} = \frac{1}{h^2} \left( 2(\nu_0^+ + \nu_0^-) - \nu_1^+ - \nu_1^- \right)$,

$G^J_{i,i+1} = \frac{1}{h^2} \left( 2\nu_1^+ - \nu_2^+ - \nu_0^+ - \nu_0^- \right)$,

$G^J_{i,i-1} = \frac{1}{h^2} \left( 2\nu_1^- - \nu_2^- - \nu_0^- - \nu_0^+ \right)$,

$G^J_{i,i+j} = \frac{1}{h^2} \left( 2\nu_j^+ - \nu_{j+1}^+ - \nu_{j-1}^+ \right)$, \quad $j = 2, \ldots, N - i$,

$G^J_{i,i-j} = \frac{1}{h^2} \left( 2\nu_j^- - \nu_{j+1}^- - \nu_{j-1}^- \right)$, \quad $j = 2, \ldots, i - 1$. 

Computational Methods for Quant. Finance
Using the $\theta$-scheme for the time discretization, we obtain the fully discrete scheme

\[
\begin{align*}
\text{Find } u^{m+1} \in \mathbb{R}^N \text{ such that for } m = 0, \ldots, M - 1, \\
(I + \theta \Delta t G^J) u^{m+1} &= (I - (1 - \theta) \Delta t G^J) u^m, \\
u^0 &= u_0.
\end{align*}
\]
Consider again the mesh (28) with uniform mesh width $h$ and the finite element space $V_N = S^1_T \cap H^1_0(G)$.

Need to compute the stiffness matrix $A^J_{i,j} = a^J(b_j, b_i)$, where the bilinear form $a^J(\cdot, \cdot)$ is given by (27).

Define, for $j = 0, 1, 2, \ldots$,

$$k^+_{j} := \int_0^h \int_{jh}^{(j+1)h} k^{(-2)}(y - x) \, dy \, dx,$$

$$k^-_{j} := \int_{jh}^{(j+1)h} \int_0^h k^{(-2)}(y - x) \, dy \, dx.$$
Now, for $j \geq i$,

\[ a^J(b_j, b_i) = \int_{x_i-1}^{x_i+1} \int_{x_{j-1}}^{x_j+1} b'_j(y) b'_i(x) k^{(-2)}(y - x) \, dy \, dx \]

\[ = \frac{1}{h^2} \left( \int_{x_i-1}^{x_i} \int_{x_{j-1}}^{x_j} k^{(-2)}(y - x) \, dy \, dx - \int_{x_i}^{x_{j-1}} \int_{x_i}^{x_j} k^{(-2)}(y - x) \, dy \, dx \right. \]

\[ - \int_{x_i}^{x_i+1} \int_{x_{j-1}}^{x_j} k^{(-2)}(y - x) \, dy \, dx + \left. \int_{x_i}^{x_{j-1}} \int_{x_i}^{x_j} k^{(-2)}(y - x) \, dy \, dx \right) \]

\[ = \frac{1}{h^2} \left( \int_0^h \int_{(j-i)h}^{(j-i+1)h} k^{(-2)}(y - x) \, dy \, dx - \int_0^h \int_{(j-i+1)h}^{(j-i+2)h} k^{(-2)}(y - x) \, dy \, dx \right. \]

\[ - \int_0^h \int_{(j-i)h}^{(j-i-1)h} k^{(-2)}(y - x) \, dy \, dx + \left. \int_0^h \int_{(j-i)h}^{(j-i+1)h} k^{(-2)}(y - x) \, dy \, dx \right). \]
Define \( A^J \in \mathbb{R}^{N \times N} \), for \( i = 1, \ldots, N \), by

\[
A^J_{i,i} = \frac{1}{h^2} \left( 2k_0^+ - k_1^+ - k_1^- \right),
\]

\[
A^J_{i,i+j} = \frac{1}{h^2} \left( 2k_j^+ - k_{j+1}^+ - k_{j-1}^+ \right), \quad j = 1, \ldots, N - i,
\]

\[
A^J_{i,i-j} = \frac{1}{h^2} \left( 2k_j^- - k_{j+1}^- - k_{j-1}^- \right), \quad j = 1, \ldots, i - 1.
\]

Using the \( \theta \)-scheme for the time discretization, we obtain the matrix problem,

Find \( u^{m+1} \in \mathbb{R}^N \) such that for \( m = 0, \ldots, M - 1 \)

\[
(M + \theta \Delta t A^J)u^{m+1} = (M - (1 - \theta) \Delta t A^J)u^m,
\]

\[
u_0^N = u_0.
\]
The variables \( k_j^+, k_j^- \) can again be computed exactly

\[
k_0^+ = \int_0^h \int_0^h k^{(-2)}(y - x) \, dy \, dx
\]
\[
= \int_0^h \left( \int_0^x k^{(-2)}(y - x) \, dy + \int_x^h k^{(-2)}(y - x) \, dy \right) \, dx
\]
\[
= \int_0^h \left( k^{(-3)}(0^-) - k^{(-3)}(-x) + k^{(-3)}(h - x) - k^{(-3)}(0^+) \right) \, dx
\]
\[
= h \left( k^{(-3)}(0^-) - k^{(-3)}(0^+) \right)
\]
\[
+ k^{(-4)}(-h) - k^{(-4)}(0^-) - k^{(-4)}(0^+) + k^{(-4)}(h),
\]

and, for \( j = 1, \ldots, N - 1, \)

\[
k_j^+ = -2k^{(-4)}(jh) + k^{(-4)}((j - 1)h) + k^{(-4)}((j + 1)h).
\]
Example

Variance gamma model (8) with $c = 4$, $\beta_- = 20/3$, $\beta_+ = 40/3$. For $T = 1$ and $K = 100$ we compute the $L^\infty$-error at maturity $t = T$ on the subset $G_0 = (K/2, 3/2K)$. 

![Graph showing error comparison between FDM and FEM methods.](image-url)
American options

- Value $v$ of an American option in log-price

$$v(t, s) := \sup_{\tau \in T_{t,T}} \mathbb{E}\left[ e^{-r(\tau-t)} g(e^{r\tau} + x + X_{\tau-t}) \mid X_t = x \right],$$

- Provided some smoothness, $v$ is solution of a parabolic integro-differential inequality

$$\partial_t u - A^J u \geq 0 \quad \text{in } J \times \mathbb{R}$$
$$u(t, x) \geq \tilde{g}(t, x) \quad \text{in } J \times \mathbb{R}$$
$$(\partial_t u - A^J u)(\tilde{g} - u) = 0 \quad \text{in } J \times \mathbb{R}$$
$$u(0, x) = g(e^x) \quad \text{in } \mathbb{R},$$

where $u(t, s) = e^{rt} v(T - t, x - (\gamma + r)t)$ and

$$\tilde{g}(t, x) := e^{rt} g(e^x - (\gamma + r)t).$$
Example

American put \((T = 1, K = 100)\) in the Variance gamma model \((8)\) with \(c = 4, \beta_- = 20/3, \beta_+ = 40/3\). Observe: The smooth pasting condition does generally not hold for Lévy models. Also: the exercise boundary values in a Lévy model may not reach the option’s strike price, i.e., \(s^*(t) < K\) for all \(t \leq T\).