Analysis II - Lösung Serie 1

1. Gegebene Fläche:

$$z\,(x,y) = (x-2y)\,(x+y) = x^2 - x\,y - 2y^2$$

   (a) Plots:

   (b) Höhenlinien

$$\begin{aligned} z = 0 \text{ für } \quad & \text{i) } x - 2y = 0 \Leftrightarrow y = \tfrac{1}{2}x \\ & \text{ii) } x + y = 0 \Leftrightarrow y = -x \\ z = const \neq 0 \qquad & \underbrace{(x-2y)}_{\widetilde{x}}\underbrace{(x+y)}_{\widetilde{y}} = const \\ \Rightarrow \widetilde{y} = \frac{const}{\widetilde{x}} \quad & \text{gedrehte Hyberbeln} \end{aligned}$$

   (c) Parametrisierung der Fläche:

$$\underline{r}\,(x,y) = (x, y, z\,(x,y))^T$$

$$\text{Normale } \underline{n}\,(x,y) = \underline{r}_x \times \underline{r}_y = \begin{pmatrix} 1 \\ 0 \\ z_x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ z_y \end{pmatrix} = \begin{pmatrix} -z_x \\ -z_y \\ 1 \end{pmatrix}$$

$$\Rightarrow \underline{n}\,(x,y) = (-2x + y, 4y + x, 1)^T$$

$$\text{Punkt } P = (1, 2, z\,(1,2))^T = (1, 2, -9)^T$$
$$\underline{n}\,(1,2) = (0, 9, 1)^T$$

Gerade durch P in Richtung $\underline{n}$ ist gegeben durch $(1,2,-9)^T + t\,(0,9,1)^T$, (x,y)-Ebene ist gegeben durch $(x,y,0)$, damit muss gelten:

$$\begin{pmatrix} 1 \\ 2 \\ -9 \end{pmatrix} + t \begin{pmatrix} 0 \\ 9 \\ 1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$
$$\Rightarrow t = 9 \quad \Rightarrow \quad x = 1,\ y = 83$$

(d) Differentialgleichung der Falllinien:

$$y'(x) = \frac{z_y(x,y)}{z_x(x,y)} = \frac{-x-4y(x)}{2x-y(x)} = \frac{x\left(-1-4\frac{y(x)}{x}\right)}{x\left(2-\frac{y(x)}{x}\right)}$$

$$\text{Transformation } u(x) := \frac{y(x)}{x} \;\Rightarrow\; y' = x\,u' + u$$

$$\begin{aligned} x\,u' + u &= \frac{-1-4u}{2-u} \\ x\frac{du}{dx} &= \frac{-1-4u}{2-u} - u = \frac{u^2-6u-1}{2-u} \\ \frac{2-u}{u^2-6u-1}du &= \frac{1}{x}dx \qquad (\star) \end{aligned}$$

Partialbruchzerlegung

$$\begin{aligned} u^2-6u-1=0 \;&\Rightarrow\; u_{1,2} = 3\pm\sqrt{10} \\ \frac{2-u}{u^2-6u-1} &= \frac{A}{u-u_1} + \frac{B}{u-u_2} \end{aligned}$$

Multiplikation mit u^2-6u-1

$$\Rightarrow 2-u = A\left(u-3+\sqrt{10}\right) + B\left(u-3-\sqrt{10}\right)$$

$$\text{Setze } u = 3+\sqrt{10} \;\Rightarrow\; -1-\sqrt{10} = 2\sqrt{10}A \;\Rightarrow\; A = -\frac{1+\sqrt{10}}{2\sqrt{10}}$$

$$\text{Setze } u = 3-\sqrt{10} \;\Rightarrow\; -1+\sqrt{10} = -2\sqrt{10}B \;\Rightarrow\; B = -\frac{-1+\sqrt{10}}{2\sqrt{10}}$$

Integration von $(\star)$

$$\begin{aligned} \int \frac{2-u}{u^2-6u-1}du &= \int \frac{1}{x}dx \\ -\frac{1}{2\sqrt{10}}\int\left(\frac{1+\sqrt{10}}{u-3-\sqrt{10}} + \frac{-1+\sqrt{10}}{u-3+\sqrt{10}}\right)du &= \int\frac{1}{x}dx \\ \frac{1+\sqrt{10}}{2\sqrt{10}}\ln\left(u-3-\sqrt{10}\right) + \frac{-1+\sqrt{10}}{2\sqrt{10}}\ln\left(u-3+\sqrt{10}\right) &= -\ln(C\,x) \\ \ln\left(\left(u-3-\sqrt{10}\right)^{\frac{1+\sqrt{10}}{2\sqrt{10}}}\left(u-3+\sqrt{10}\right)^{\frac{-1+\sqrt{10}}{2\sqrt{10}}}\right) &= \ln\left(\frac{C}{x}\right) \end{aligned}$$

Exponentieren und Rücktransformieren $u = \frac{y}{x}$

$$\Rightarrow \left(\frac{y}{x}-3-\sqrt{10}\right)^{\frac{1+\sqrt{10}}{2\sqrt{10}}}\left(\frac{y}{x}-3+\sqrt{10}\right)^{\frac{-1+\sqrt{10}}{2\sqrt{10}}} x = \widetilde{C} \quad \text{Falllinien}$$

2. Gegebene Fläche:

$$z(x,y) = \sqrt{4+2x^2+y^2}$$

(a) Tangentialebene gegeben durch $\underline{n}\cdot(x,y,z)^T = d$ mit Normalenvektor $\underline{n} = (-z_x, -z_y, 1)^T$.

$$z_x(x,y) = \frac{2x}{\sqrt{4+2x^2+y^2}} = \frac{2x}{z} \quad z_y(x,y) = \frac{y}{\sqrt{4+2x^2+y^2}} = \frac{y}{z}$$

$$
\begin{aligned}
\underline{n}\,(x,y) &= \left(-\frac{2x}{z}, -\frac{y}{z}, 1\right)^T \\
P &= (2,2,z\,(2,2))^T = (2,2,4)^T \\
\underline{n}\,(2,2) &= \left(-1, -\frac{1}{2}, 1\right)^T
\end{aligned}
$$

Da $P \in E$ gilt

$$
\begin{aligned}
d = \underline{n} \cdot P &= -2 - \frac{1}{2}2 + 4 = 1 \\
\Rightarrow E:\; -x - \frac{1}{2}y + z &= 1
\end{aligned}
$$

(x,y)-Ebene hat $z = 0$, also

$$
-x - \frac{1}{2}y = 1 \;\Rightarrow\; y\,(x) = -2x - 2
$$

(b) Höhenlinien

$$
z\,(x,y) = const \;\Rightarrow
$$

$$
\begin{aligned}
4 + 2x^2 + y^2 &= C = const \\
2x^2 + y^2 &= C - 4 \quad \text{Ellipsen}
\end{aligned}
$$

(c) Entlang der Geraden gilt

$$
z\,(x, y\,(x)) = z\,(x, x+1) = \sqrt{4 + 2x^2 + (x+1)^2} \rightarrow \text{minimal}
$$

$$
\Rightarrow 4 + 2x^2 + (x+1)^2 \rightarrow \text{minimal}
$$

$$
\begin{aligned}
\frac{d}{dx}\left(4 + 2x^2 + (x+1)^2\right) &= 0 \\
6x + 2 &= 0
\end{aligned}
$$

$$
\Rightarrow\; x_0 = -\frac{1}{3} \quad y_0 = \frac{2}{3}
$$

(d) Falllinien:

$$
y'\,(x) = \frac{z_y\,(x,y)}{z_x\,(x,y)} = \frac{y}{2x} \;\Rightarrow\; \frac{2}{y}dy = \frac{1}{x}dx
$$

Integration liefert

$$
\begin{aligned}
2\ln y &= \ln\,(C\,x) \\
y^2 &= \widetilde{C}\,x
\end{aligned}
$$

$$
P = (2,2) \quad \Rightarrow\; 4 = \widetilde{C}\,2 \quad \Rightarrow\; \widetilde{C} = 2
$$

Für die x-Achse gilt $y = 0$

$$
\Rightarrow\; x = 0
$$