Analysis II - Lösung Serie 12

1. Zuerst werden die Randbedingungen homogenisiert durch

$$v(x,y,t) = u(x,y,t) - \frac{x}{a}\left(1-e^{-\gamma t}\right) \qquad u(x,y,t) = v(x,y,t) + \frac{x}{a}\left(1-e^{-\gamma t}\right)$$

$$\Rightarrow \quad \partial_t u = \partial_t\left(v + \frac{x}{a}\left(1-e^{-\gamma t}\right)\right) = \partial_t v - \gamma\frac{x}{a}e^{-\gamma t} \quad \text{und} \quad \Delta u = \Delta v$$

Das Problem lautet damit

$$\partial_t v = D\Delta v + q + \gamma\frac{x}{a}e^{-\gamma t} \quad \text{in} \quad [0,a]\times[0,b]$$

$$v = 0 \quad \text{auf} \quad \Gamma_0, \Gamma_2 \qquad \frac{\partial v}{\partial n} = 0 \quad \text{auf} \quad \Gamma_1$$

$$v(x,y,0) = u(x,y,0) - \frac{x}{a}\left(1-e^{-\gamma 0}\right) = 0$$

Für die Eigenfunktionen wird

$$\Delta\varphi + \lambda\varphi = 0 \quad \text{in} \quad [0,a]\times[0,b]$$

$$\varphi = 0 \quad \text{auf} \quad \Gamma_0, \Gamma_2 \qquad \frac{\partial \varphi}{\partial n} = 0 \quad \text{auf} \quad \Gamma_1$$

mit dem Ansatz $\varphi(x,y) = X(x)\,Y(y)$ gelöst, wobei aus den Randbedingungen folgt

$$X(0) = X(a) = 0 \quad \text{und} \quad Y'(0) = Y'(b) = 0$$

Inklusive Normierung ergibt sich

$$\varphi_{k,l}(x,y) = \frac{2}{\sqrt{ab}}\sin\left(\frac{k\pi}{a}x\right)\cos\left(\frac{l\pi}{b}y\right) \qquad \lambda_{k,l} = \left(\frac{k\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2 \qquad k = 1,2,.. \quad l = 0,1,..$$

Entwicklung des "Quellterms" $q + \gamma\frac{x}{a}e^{-\gamma t}$:

$$q+\gamma\frac{x}{a}e^{-\gamma t} = \sum_{k,l=1}^{\infty}\beta_{k,l}(t)\;\varphi_{k,l}(x,y) \quad \text{mit} \quad \beta_{k,l}(t) = \int_0^b\int_0^a\left(q+\gamma\frac{x}{a}e^{-\gamma t}\right)\frac{2}{\sqrt{ab}}\sin\left(\frac{k\pi}{a}x\right)\cos\left(\frac{l\pi}{b}y\right)dx\,dy$$

Es folgt

$$\beta_{k,l}(t) = \frac{2q}{\sqrt{ab}}\int_0^a\sin\left(\frac{k\pi}{a}x\right)dx\int_0^b\cos\left(\frac{l\pi}{b}y\right)\,dy + e^{-\gamma t}\frac{2\gamma}{\sqrt{ab}}\int_0^a\frac{x}{a}\sin\left(\frac{k\pi}{a}x\right)dx\int_0^b\cos\left(\frac{l\pi}{b}y\right)\,dy$$

wobei für die Integrale gilt

$$\int_0^a\sin\left(\frac{k\pi}{a}x\right)dx = \frac{a}{k\pi}\left(1-(-1)^k\right) \qquad \int_0^a\frac{x}{a}\sin\left(\frac{k\pi}{a}x\right)dx = -\frac{a}{k\pi}(-1)^k$$

$$\int_0^b\cos\left(\frac{l\pi}{b}y\right)\,dy = \begin{cases} b & l = 0 \\ 0 & l > 0\end{cases}$$

$$\Rightarrow \quad \beta_{k,0}(t) = \frac{2\sqrt{ab}}{k\pi}\left(q\left(1-(-1)^k\right) - \gamma e^{-\gamma t}(-1)^k\right) \qquad \beta_{k,l}(t) = 0 \text{ für } l > 0$$

Die Gleichung für $v(x,y,t)$ wird nun durch den Ansatz

$$v(x,y,t) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \alpha_{k,l}(t)\ \varphi_{k,l}(x,y)$$

gelöst:

$$\partial_t \sum_{k,l=1}^{\infty} \alpha_{k,l}(t)\ \varphi_{k,l}(x,y) - D\,\Delta \sum_{k,l=1}^{\infty} \alpha_{k,l}(t)\ \varphi_{k,l}(x,y) - \sum_{k,l=1}^{\infty} \beta_{k,l}(t)\ \varphi_{k,l}(x,y) = 0$$

$$\sum_{k,l=1}^{\infty} \left(\partial_t \alpha_{k,l}(t)\ \varphi_{k,l}(x,y) - D\,\alpha_{k,l}(t)\ \Delta\varphi_{k,l}(x,y) - \beta_{k,l}(t)\ \varphi_{k,l}(x,y)\right) = 0$$

$$\sum_{k,l=1}^{\infty} \left(\alpha'_{k,l}(t) + D\,\alpha_{k,l}(t)\ \lambda_{k,l} - \beta_{k,l}(t)\right)\varphi_{k,l}(x,y) = 0$$

$$\Rightarrow \qquad \alpha'_{k,l}(t) = -D\,\lambda_{k,l}\,\alpha_{k,l}(t) + \beta_{k,l}(t)$$

Die Lösung dieser DGL ist gegeben durch

$$\alpha_{k,0}(t) = \alpha_{k,0}(0) + \int_0^t e^{-D\lambda_{k,0}(t-\tau)}\beta_{k,0}(\tau)\,d\tau$$

$$\alpha_{k,l}(t) = \alpha_{k,l}(0)\ e^{-D\lambda_{k,l}\,t}$$

Die Anfangswerte $\alpha_{k,l}(0)$ ergeben sich aus der Anfangsbedingung für v

$$v(x,y,0) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \alpha_{k,l}(0)\ \varphi_{k,l}(x,y) \overset{!}{=} 0 \quad \Rightarrow \quad \alpha_{k,l}(0) = 0$$

Also ist $\alpha_{k,l}(t) = 0$ für $l > 0$ und

$$\alpha_{k,0}(t) = \int_0^t e^{-D\lambda_{k,0}(t-\tau)}\beta_{k,0}(\tau)\,d\tau = \tfrac{2\sqrt{ab}}{k\pi}\int_0^t e^{-D\left(\frac{k\pi}{a}\right)^2(t-\tau)}\left(q\left(1-(-1)^k\right) - \gamma e^{-\gamma\tau}(-1)^k\right)d\tau$$

$$= \tfrac{2\sqrt{ab}}{k\pi} e^{-D\left(\frac{k\pi}{a}\right)^2 t}\left(q\left(1-(-1)^k\right)\int_0^t e^{D\left(\frac{k\pi}{a}\right)^2\tau}d\tau - \gamma(-1)^k\int_0^t e^{\left(D\left(\frac{k\pi}{a}\right)^2-\gamma\right)\tau}d\tau\right)$$

$$= \frac{2\sqrt{ab}}{k\pi}\left(\tfrac{q\left(1-(-1)^k\right)}{D\left(\frac{k\pi}{a}\right)^2}\left(1-e^{-D\left(\frac{k\pi}{a}\right)^2 t}\right) - \tfrac{\gamma(-1)^k}{D\left(\frac{k\pi}{a}\right)^2-\gamma}\left(e^{-\gamma t} - e^{-D\left(\frac{k\pi}{a}\right)^2 t}\right)\right)$$

Insgesamt lautet die Lösung

$$v(x,y,t) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\alpha_{k,l}(t)\ \varphi_{k,l}(x,y) = \sum_{k=1}^{\infty}\alpha_{k,0}(t)\ \varphi_{k,0}(x,y)$$

$$= \sum_{k=1}^{\infty}\frac{4}{k\pi}\left(\tfrac{q\left(1-(-1)^k\right)}{D\left(\frac{k\pi}{a}\right)^2}\left(1-e^{-D\left(\frac{k\pi}{a}\right)^2 t}\right) - \tfrac{\gamma(-1)^k}{D\left(\frac{k\pi}{a}\right)^2-\gamma}\left(e^{-\gamma t} - e^{-D\left(\frac{k\pi}{a}\right)^2 t}\right)\right)\sin\left(\frac{k\pi}{a}x\right)$$

$$u(x,y,t) = v(x,y,t) + \frac{x}{a}\left(1-e^{-\gamma t}\right)$$

$$= \frac{x}{a}\left(1-e^{-\gamma t}\right)$$

$$+ \sum_{k=1}^{\infty}\frac{4}{k\pi}\left(\tfrac{q\left(1-(-1)^k\right)}{D\left(\frac{k\pi}{a}\right)^2}\left(1-e^{-D\left(\frac{k\pi}{a}\right)^2 t}\right) - \tfrac{\gamma(-1)^k}{D\left(\frac{k\pi}{a}\right)^2-\gamma}\left(e^{-\gamma t} - e^{-D\left(\frac{k\pi}{a}\right)^2 t}\right)\right)\sin\left(\frac{k\pi}{a}x\right)$$

und der Grenzwert

$$u(x,y,t) = \sum_{k=1}^{\infty}\frac{4}{k\pi}\tfrac{q\left(1-(-1)^k\right)}{D\left(\frac{k\pi}{a}\right)^2}\sin\left(\frac{k\pi}{a}x\right) + \frac{x}{a}$$

2. Die nötigen normierten Eigenfunktionen und Eigenwerte

$$\varphi'' + \lambda\varphi = 0 \quad \text{in } [0, L] \qquad \varphi(0) = \varphi(L) = 0$$

lauten

$$\varphi_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\,\pi}{L} x\right) \qquad \lambda_k = \left(\frac{k\,\pi}{L}\right)^2 \qquad k = 1, 2, ..$$

Entwicklung der Quelle

$$\delta \cos(\omega\, t) = \sum_{k=1}^{\infty} \beta_k(t)\; \varphi_k(x) \qquad \text{mit} \quad \beta_k(t) = \int_0^L \delta \cos(\omega\, t) \sqrt{\frac{2}{L}} \sin\left(\frac{k\,\pi}{L} x\right) dx$$

$$\beta_k(t) = \begin{cases} 2\delta \cos(\omega\, t) \frac{\sqrt{2L}}{k\,\pi} & k \text{ ungerade} \\ 0 & \text{sonst} \end{cases}$$

Mit dem Ansatz

$$u(x, t) = \sum_{k=1}^{\infty} \alpha_k(t)\; \varphi_k(x)$$

folgt

$$\sum_{k=1}^{\infty} \left(\alpha_k'(t)\; + \nu\, \lambda_k\, \alpha_k(t) - \beta_k\;(t)\right) \varphi_k(x) = 0$$

$$\alpha_k'(t) \;= -\nu\, \lambda_k\, \alpha_k(t) + \beta_k\;(t)$$

$$\Rightarrow \quad \alpha_k(t) = \alpha_k(0) + \int_0^t e^{-\nu\lambda_k (t-\tau)} \beta_k(\tau)\, d\tau$$

Die Konstanten $\alpha_k(0)$ folgen wieder aus der Anfangsbedingung

$$u(x, 0) = \sum_{k=1}^{\infty} \alpha_k(0)\; \varphi_k(x) \stackrel{!}{=} 0 \qquad \Rightarrow \quad \alpha_k(0) = 0$$

Also

$$\alpha_k(t) = \int_0^t e^{-\nu\lambda_k (t-\tau)} 2\delta \cos(\omega\, \tau)\, \frac{\sqrt{2L}}{k\,\pi} d\tau = 2\delta \frac{\sqrt{2L}}{k\,\pi} e^{-\nu\lambda_k\, t} \int_0^t e^{\nu\lambda_k\, \tau} \cos(\omega\, \tau)\, d\tau$$

$$= 2\delta \frac{\sqrt{2L}}{k\,\pi} \frac{1}{(\nu\lambda_k)^2 + \omega^2} \left(\nu\lambda_k \cos(\omega\, t) + \omega \sin(\omega\, t) - \nu\lambda_k e^{-\nu\lambda_k\, t}\right)$$

$$\Rightarrow \quad u(x, y, t) = 4\delta \sum_{k=1}^{\infty} \frac{1}{k\,\pi} \frac{\nu\lambda_k \cos(\omega\, t) + \omega \sin(\omega\, t) - \nu\lambda_k e^{-\nu\lambda_k\, t}}{(\nu\lambda_k)^2 + \omega^2} \sin\left(\frac{k\,\pi}{L} x\right)$$