Lösung Serie 6

1. Parametrisierung des Paraboloides

$$\underline{r}(s,\phi) = \begin{pmatrix} x(s,\phi) \\ y(s,\phi) \\ z(s,\phi) \end{pmatrix} = \begin{pmatrix} a\, s\, \cos\phi \\ b\, s\, \sin\phi \\ 1-s^2 \end{pmatrix}, \quad 0 \le s \le 1, \phi \le \frac{\pi}{2}$$

mit

$$\underline{r}_s = \begin{pmatrix} a\, \cos\phi \\ b\, \sin\phi \\ -2s \end{pmatrix}, \quad \underline{r}_\phi = \begin{pmatrix} -a\, s\, \sin\phi \\ b\, s\, \cos\phi \\ 0 \end{pmatrix}, \quad \underline{r}_s \times \underline{r}_\phi = \begin{pmatrix} 2\, b\, s^2\, \cos\phi \\ 2\, a\, s^2\, \sin\phi \\ a\, b\, s \end{pmatrix}$$

Der Volumenstrom ist dann

$$\begin{aligned}
\dot{V} &= \int_0^{\frac{\pi}{2}} \int_0^1 \underline{v}(\underline{r}(s,\phi)) \cdot \underline{r}_s \times \underline{r}_\phi \, ds\, d\phi \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \begin{pmatrix} s^2\, \cos^2\phi \\ s\, \sin\phi \\ 1-s^2 \end{pmatrix} \cdot \begin{pmatrix} 2\, b\, s^2\, \cos\phi \\ 2\, a\, s^2\, \sin\phi \\ a\, b\, s \end{pmatrix} ds\, d\phi \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \left(2\, b\, s^4\, \cos^3\phi + 2\, a\, s^3\, \sin^2\phi + a\, b\, s - a\, b\, s^3\right) ds\, d\phi \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{2}{5}\, b\, \cos^3\phi + \frac{1}{2}\, a\, \sin^2\phi + \frac{1}{2}\, a\, b - \frac{1}{4}\, a\, b\right) d\phi \\
&= \int_0^{\frac{\pi}{2}} \left(\frac{2}{5}\, b\, \cos^3\phi + \frac{1}{2}\, a\, \sin^2\phi + \frac{1}{4}\, a\, b\right) d\phi \\
&= \left(\frac{4}{15}\, b + \frac{\pi}{8}\, a + \frac{\pi}{8}\, a\, b\right) = \frac{4}{15}\, b + \frac{\pi}{8}\, (a + a\, b)
\end{aligned}$$

2. Parametrisierung der Fläche

$$\underline{r}(\phi,\theta) = \begin{pmatrix} x(\phi,\theta) \\ y(\phi,\theta) \\ z(\phi,\theta) \end{pmatrix} = \begin{pmatrix} \sqrt{2}\cos(\phi)\sin(\theta) \\ \cos(\theta) \\ \sqrt{2}\sin(\phi)\sin(\theta) \end{pmatrix}, \quad 0 \le \phi \le 2\pi,\ 0 \le \theta \le \frac{\pi}{2}$$

Bitte wenden!

mit

$$\underline{r}_\phi = \begin{pmatrix} -\sqrt{2}\sin(\phi)\sin(\theta) \\ 0 \\ \sqrt{2}\cos(\phi)\sin(\theta) \end{pmatrix} = \sqrt{2}\sin(\theta)\begin{pmatrix} -\sin(\phi) \\ 0 \\ \cos(\phi) \end{pmatrix}, \quad \underline{r}_\theta = \begin{pmatrix} \sqrt{2}\cos(\phi)\cos(\theta) \\ -\sin(\theta) \\ \sqrt{2}\sin(\phi)\cos(\theta) \end{pmatrix}$$

dann

$$\underline{r}_\phi \times \underline{r}_\theta = \sqrt{2}\sin(\theta)\begin{pmatrix} \cos(\phi)\sin(\theta) \\ \sqrt{2}\cos(\theta) \\ \sin(\phi)\sin(\theta) \end{pmatrix}$$

Für die Oberfläche gilt

$$\begin{aligned} A &= \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} d\theta \left\|\underline{r}_\phi \times \underline{r}_\theta\right\| = \\ &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\sqrt{2}\sin(\theta)\sqrt{\cos^2(\phi)\sin^2(\theta) + 2\cos^2(\theta) + \sin^2(\phi)\sin^2(\theta)} = \\ &= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\sqrt{2}\sin(\theta)\sqrt{\sin^2(\theta) + 2\cos^2(\theta)} = \\ &= 2\sqrt{2}\pi \int_0^{\pi/2} d\theta \sin(\theta)\sqrt{\cos^2(\theta) + 1} \end{aligned}$$

mit Substitution $q = \cos(\theta)$ erhält man

$$A = 2\sqrt{2}\pi \int_0^1 dq\sqrt{q^2+1}$$

Mit

$$\int \sqrt{x^2+1}dx = \frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left(x + \sqrt{x^2+1}\right)\right) + C$$

erhäl man

$$\begin{aligned} A &= 2\sqrt{2}\pi \int_0^1 dq\sqrt{q^2+1} = 2\sqrt{2}\pi\frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right) = \\ &= 2\pi + \sqrt{2}\pi\ln\left(1+\sqrt{2}\right) \end{aligned}$$

3. Parametrisierung der Fläche in Kugelkoordinaten

$$\underline{q}(r,\phi) = \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \end{pmatrix}, \quad \theta = \text{atan}\frac{R}{H},\ 0 \le r \le \sqrt{H^2+R^2},\ 0 \le \phi \le 2\pi$$

mit

$$\underline{q}_r = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}, \quad \underline{q}_\phi = \begin{pmatrix} -r\sin(\theta)\sin(\phi) \\ r\sin(\theta)\cos(\phi) \\ 0 \end{pmatrix},$$

Siehe nächstes Blatt!

$$\underline{q}_r \times \underline{q}_\phi = \begin{pmatrix} -r\sin(\theta)\cos(\theta)\cos(\phi) \\ -r\sin(\theta)\cos(\theta)\sin(\phi) \\ r\sin(\theta)^2 \end{pmatrix}$$

Die Dichte wächst linear in z von 1 für $z = 0$ bis 2 für $z = H$,d.h.

$$\varrho = 1 + \frac{z}{H} = 1 + \frac{r\cos(\theta)}{H}$$

wobei die letze Formel enspricht die Dichte in Kugelkordinaten. Für den Schwerpunkt der Manteloberfläche F gilt:

$$\underline{r}_s = \frac{\int_F \varrho \underline{r} dF}{M}$$

wo $M = \int_F \varrho dF$ die Mantelmasse ist.
Das Fläche element ist:

$$\|\underline{q}_r \times \underline{q}_\phi\| = r\sqrt{\sin(\theta)^2\cos(\theta)^2\cos(\phi)^2 + \sin(\theta)^2\cos(\theta)^2\sin(\phi)^2 + \sin(\theta)^4} = r\sin(\theta)$$

Für den Schwerpunkt erhÏt man:

$$\begin{aligned}
\underline{r}_s &= \frac{\int_F \varrho \underline{r} dF}{M} = \\
&= \frac{1}{M}\int_0^{2\pi}\int_0^{\sqrt{H^2+R^2}} \left(1 + \frac{r\cos(\theta)}{H}\right) \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \end{pmatrix} r\sin(\theta)\, dr\, d\phi \\
&= \frac{2\pi}{M}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \int_0^{\sqrt{H^2+R^2}} \left(1 + \frac{r\cos(\theta)}{H}\right) r^2\cos(\theta)\sin(\theta)\, dr = \\
&= \frac{2\pi}{M}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \int_0^{\sqrt{H^2+R^2}} \left(r^2 + \frac{r^3\cos(\theta)}{H}\right) \cos(\theta)\sin(\theta)\, dr = \\
&= \frac{2\pi}{M}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(\frac{r^3}{3} + \frac{r^4\cos(\theta)}{4H}\right)\Bigg|_{r=0}^{\sqrt{H^2+R^2}} \cos(\theta)\sin(\theta) = \\
&= \frac{2\,\pi\,\left(H^2+R^2\right)^{3/2}}{M}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(\frac{1}{3} + \frac{\sqrt{H^2+R^2}\cos(\theta)}{4H}\right)\cos(\theta)\sin(\theta)
\end{aligned}$$

Bitte wenden!

mit $\cos(\theta) = \frac{H}{\sqrt{H^2+R^2}}$ und $\sin(\theta) = \frac{R}{\sqrt{H^2+R^2}}$ erhält man

$$\underline{r}_s = \frac{2\,\pi\,\left(H^2+R^2\right)^{3/2}}{R} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(\frac{1}{3}+\frac{1}{4}\right) \frac{H}{\sqrt{H^2+R^2}} \frac{R}{\sqrt{H^2+R^2}} =$$

$$= \frac{7\,R\,H\pi\sqrt{H^2+R^2}}{6\,M} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Für die Mantelmasse erhält man:

$$M = \int_F \varrho dF = \int_0^{2\pi} \int_0^{\sqrt{H^2+R^2}} \left(1+\frac{r\cos(\theta)}{H}\right) r\,\sin(\theta) dr\,d\phi =$$

$$= 2\pi \int_0^{\sqrt{H^2+R^2}} \left(r+\frac{r^2\cos(\theta)}{H}\right) \sin(\theta) dr =$$

$$= 2\pi \left(\frac{H^2+R^2}{2}+\frac{\left(H^2+R^2\right)^{3/2}\cos(\theta)}{3H}\right) \sin(\theta) =$$

$$= 2\pi \left(\frac{H^2+R^2}{2}+\frac{\left(H^2+R^2\right)^{3/2}}{H}\frac{3H}{\sqrt{H^2+R^2}}\right) \frac{R}{\sqrt{H^2+R^2}} =$$

$$= \frac{5}{3}\pi\,R\sqrt{H^2+R^2}$$

Also

$$\underline{r}_s = \frac{7\,R\,H\pi\sqrt{H^2+R^2}}{6\,M} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{7\,R\,H\pi\sqrt{H^2+R^2}}{6\,\frac{5}{3}\pi\,R\,\sqrt{H^2+R^2}} = \frac{7}{10} H \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Der Grundkreis hat Oberflaäche $M' = 2\pi R^2$ und Schwerpunkt in $\underline{r}_s' = \begin{pmatrix} 0 \\ 0 \\ H \end{pmatrix}$.

So füer den Schwerpunt für die gesamte fläche gilt:

$$\underline{r}_s'' = \frac{M'\underline{r}_s' + M\underline{r}_s}{M'+M} =$$

$$= \frac{2\pi\,R^2\,H + \frac{7}{6}\pi R\,\sqrt{H^2+R^2}H}{2\pi\,R^2 + \frac{5}{3}\pi\,R\,\sqrt{H^2+R^2}} \underline{\mathbf{e}}_z =$$

$$= \frac{12\,R + 7\sqrt{H^2+R^2}}{12\,R+10\,\sqrt{H^2+R^2}} H\underline{\mathbf{e}}_z =$$

$$= \left(1 - \frac{3\sqrt{1+\left(\frac{H}{R}\right)^2}}{12+10\sqrt{1+\left(\frac{H}{R}\right)^2}}\right) H\underline{\mathbf{e}}_z$$