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Computational Electromagnetics Course

Exercise Sheet 1: Maxwell's Equations and Differential Forms

1. For the 1–form given in Table 1.2.C of the lecture notes, determine the new vector proxy after a change of inner product to $\boldsymbol{u} * \boldsymbol{v} = A\boldsymbol{u} \cdot A\boldsymbol{v}$. Here, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3,3}$ regular and $\cdot$ denotes the Euclidean inner product.

2. Show that the pull back operation for Euclidean vector proxies of 1–forms on $M \subset \mathbb{R}^3$ is given by

$$Y_1(\phi^*\omega)(\hat{\boldsymbol{x}}) = D\phi(\hat{\boldsymbol{x}})^T Y_1(\omega)(\phi(\boldsymbol{x})) \tag{1}$$

   where $\phi : \hat{M} \subset \mathbb{R}^3 \to M$ is a diffeomorphism, $\boldsymbol{x} \in M$ and $\hat{\boldsymbol{x}} \in \hat{M}$.

   [Hint:- use the relationship

$$(\phi^*\omega)(\hat{\boldsymbol{x}})(\hat{\boldsymbol{v}}_1, \cdots, \hat{\boldsymbol{v}}_\ell) = \omega(\phi(\hat{\boldsymbol{x}}))(D\phi(\hat{\boldsymbol{x}})\hat{\boldsymbol{v}}_1, \cdots, D\phi(\hat{\boldsymbol{x}})\hat{\boldsymbol{v}}_\ell) \tag{2}$$

   for all $\hat{\boldsymbol{v}}_1, \cdots, \hat{\boldsymbol{v}}_\ell \in \mathbb{R}^3$, $\hat{\boldsymbol{x}} \in \hat{M}$]

3. For the differential ℓ-form $0 \leq \ell \leq n-1$ on $M \subset \mathbb{R}^n$, the exterior derivative is given by

$$d\omega(\boldsymbol{x})(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{\ell+1}) = \sum_{i=1}^{\ell+1} (-1)^{i-1} (D\omega(\boldsymbol{x})\boldsymbol{v}_i)(\boldsymbol{v}_1, \cdots, \check{\boldsymbol{v}}_i, \cdots, \boldsymbol{v}_{\ell+1}) \tag{3}$$

   where $\boldsymbol{x} \in M$, $\boldsymbol{v} \in \mathbb{R}^n$ and $\check{\boldsymbol{v}}_i$ denotes omit $\boldsymbol{v}_i$. Show that for the case of 0–forms the exterior derivative is equivalent to the gradient operator for Euclidean vector proxies.

4. Let M be an n-dimensional star-shaped domain in $\mathbb{R}^n$. The Poincaré mapping $k : \mathcal{DF}^\ell(M) \to \mathcal{DF}^{\ell-1}(M)$, $1 \leq \ell \leq n$ is defined by

$$k(\omega)(\boldsymbol{x})(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{\ell-1}) = \int_0^1 t^{\ell-1} \omega(t\boldsymbol{x})(\boldsymbol{x}, \boldsymbol{v}_1, \cdots, \boldsymbol{v}_{\ell-1}) \, \mathrm{d}t \tag{4}$$

   where $\omega \in \mathcal{DF}^\ell(\mathbb{R}^n)$, $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{x} \in M$.

   Prove that the relationship

$$d(k(\omega)) + k(d\omega) = \omega \tag{5}$$

   is true.

5. Construct cycle representatives of the homology spaces for the following manifolds

    - Exterior of a torus.
    - Interior of a Toroidal pipe.