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Computational Electromagnetics Course

Exercise Sheet 10: Higher Order Discrete Differential Forms

1. Euclidean vector proxies for higher order discrete differential forms on hexahedra are given in equation (7.1b) of the lecture notes. For convenience, they are repeated below

$$
\begin{aligned}
W_p^0(\hat{C}) &= Q_{p+1,p+1,p+1} && (1)\\
W_p^1(\hat{C}) &= Q_{p,p+1,p+1} \times Q_{p+1,p,p+1} \times Q_{p+1,p+1,p} && (2)\\
W_p^2(\hat{C}) &= Q_{p+1,p,p} \times Q_{p,p+1,p} \times Q_{p,p,p+1} && (3)\\
W_p^3(\hat{C}) &= Q_{p,p,p} && (4)
\end{aligned}
$$

   Show that these higher order discrete differential forms satisfy

$$dW_p^\ell(\hat{C}) \subset W_p^\ell(\hat{C}) \qquad (5)$$

   for $\ell = 0, 1, 2, 3$.

2. Lemma 7.2.1.B of the lecture notes states that: if $S = [A_{s_0}, \cdots, A_{s_m}]$, $\{s_0, \cdots, s_m\} \subset \{0, \cdots, n\}$, $\omega \in Z_p^\ell(S)$, $\ell \leq m \leq n$, $p > m - \ell$, then

$$\omega = \sum_{I \subset \{s_0, \cdots, s_m\}, |I| = \ell} \alpha_I(\lambda_{s_0}, \cdots, \lambda_{s_m}) \lambda_{I'} d\lambda_I \qquad (6)$$

   with $I' \cup I = \{s_0, \cdots, s_m\}$, $I' \cap I = \emptyset$, $\alpha_I \in \tilde{\mathcal{P}ol}_q(\mathbb{R}^{m+1})$, $q = p - (m + 1 - \ell)$. Give an explicit basis of $\mathcal{Z}_p^\ell(S)$ for $p = 1$, $\ell = 1, 2$, $m = 2$ on tetrahedra (that is $n = 3$). In addition, state the dimension of $\mathcal{Z}_p^\ell$ in each case.

3. To ensure that Whitney forms are included in the set of higher order discrete differential forms on tetrahedra one can employ the idea of direct splitting. Here the spaces $\mathcal{Z}_p^\ell(S)$ and $\mathcal{X}_p^\ell(S)$ are split as

$$\mathcal{Z}_p^\ell(S) := \mathcal{N}_p^\ell(S) \oplus \mathcal{Y}_p^\ell(S) \qquad \mathcal{X}_p^\ell(S) := \mathcal{Y}_{p+1}^\ell(S) \oplus d\mathcal{Y}_{p+1}^{\ell-1}(S) \qquad (7)$$

   show that $\mathcal{N}_p^\ell(S) = d\mathcal{Y}_{p+1}^{\ell-1}(S)$.

4. Section 7.2.3 of the lecture notes discusses the concrete realization of the higher order basis functions for 1–forms on tetrahedra. Discuss the details of how higher order discrete 1–forms can be constructed on triangles in 2-dimensions.