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Computational Electromagnetics Course

Exercise Sheet 2: Elementary Calculus of Differential Forms

1. Definition 1.2 in the lecture notes states that for a submanifold $\mathcal{N} \subset \mathcal{M} \subset \mathbb{R}^n$ the trace operator $\gamma_N : \mathcal{F}^\ell(\mathcal{M}) \to \mathcal{F}^\ell(\mathcal{N})$ satisfies

$$\int_\Sigma \gamma_\mathcal{N}\omega = \int_\Sigma \omega \qquad \forall \Sigma \in S_\ell(\mathcal{N}) \subset S_\ell(\mathcal{M}) \tag{1}$$

Show that the pull back operator $\phi^* : \mathcal{F}^\ell(\mathcal{N}) \to \mathcal{F}^\ell(\hat{\mathcal{N}})$ diffeomorphisim, $\phi^* : \mathcal{F}^\ell(\mathcal{M}) \to \mathcal{F}^\ell(\hat{\mathcal{M}})$ diffeomorphisim, respectively on $\gamma_\mathcal{N}$ commutes, i.e.

$$\phi^* \circ \gamma_\mathcal{N} = \gamma_{\hat{\mathcal{N}}} \circ \phi^* \tag{2}$$

2. Corollary 1.4B states that for a pull back operator $\phi^* : \mathcal{F}^\ell(\mathcal{M}) \to \mathcal{F}^\ell(\hat{\mathcal{M}})$ diffeomorphisim, $\mathcal{M}, \hat{\mathcal{M}} \subset \mathbb{R}^n$, $0 \le \ell \le n$ and the trace operator $\gamma_\mathcal{N} : \mathcal{F}^\ell(\mathcal{M}) \to \mathcal{F}^\ell(\mathcal{N})$, $\mathcal{N} \subset \mathcal{M}$ satisfy

$$\phi^* \circ d = d \circ \phi^* \tag{3}$$

$$\gamma_\mathcal{N} \circ d = d \circ \gamma_\mathcal{N} \tag{4}$$

Show that both of these corollaries are true.

3. Corollary 1.4C states that the exterior derivative of the exterior derivative is zero. Show that

$$d \circ d = 0 \tag{5}$$

is true.

4. Theorem 1,4E in the lecture notes states that for a differentiable manifold $\mathcal{M} \subset \mathbb{R}^n$ with differential forms $\omega \in \mathcal{DF}^{\ell,1}(\mathcal{M})$ and $\eta \in \mathcal{DF}^{m,1}(\mathcal{M})$ $0 \le \ell, m \le n$ the exterior derivative of an exterior product is given by

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^\ell \omega \wedge d\eta \tag{6}$$

Give a statement of this expression in terms of the Euclidean vector proxies for different order differential forms in $\mathbb{R}^3$.

5. For a differentiable manifold $\mathcal{M} \subset \mathbb{R}^3$ and differential forms $\omega \in \mathcal{DF}^{\ell,1}(\mathcal{M})$ and $\eta \in \mathcal{DF}^{m,1}(\mathcal{M})$ the integration by parts formula over the boundary $\partial\Sigma$ is

$$\int_{\partial\Sigma} \omega \wedge \eta = \int_{\Sigma} d\omega \wedge \eta + (-1)^{\ell} \omega \wedge d\eta \tag{7}$$

where $\Sigma \in \mathcal{S}_{\ell+m+1}(\mathcal{M})$ and $\wedge$ denotes the exterior product. Prove the integration by parts formula in terms of vector proxies.

6. Show that

$$\int_{\Omega} \operatorname{curl} \boldsymbol{u} \, \mathrm{d}\Omega = \int_{\partial\Omega} \boldsymbol{u} \times \boldsymbol{n} \, \mathrm{d}S \tag{8}$$

where $\Omega \subset \mathbb{R}^3$, $\boldsymbol{n}$ is the outward normal vector and $\boldsymbol{u} \in (C^{\infty}(\bar{\Omega}))^3$. [Hint: does not fit the calculus of differential forms]