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Computational Electromagnetics Course

Exercise Sheet 4: Discrete Hodge Operators.

1. The de Rham map is defined according to

$$S^\ell : \mathcal{F}^\ell(\Omega) \to \mathcal{C}^\ell(\mathcal{T}_h),\ (S^\ell\omega)(\mathcal{F}) = \int_{\mathcal{F}} \omega\ \forall \mathcal{F} \in \mathcal{S}_\ell(\mathcal{T}_h) \tag{1}$$

   A corollary of this is

$$d_h \circ S^\ell = S^{\ell+1} \circ d \qquad \text{in } \mathcal{C}^{\ell+1}(\mathcal{T}_h) \tag{2}$$

   where d_h denotes the discrete external derivative of co–chains and d the regular external derivative of integral forms. Show that this corollary is true.

2. If $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{h}$, $\boldsymbol{j}$ and $\boldsymbol{d}$ are solutions of Maxwell's equations then the following relationships are true

$$D^1 S^1 \boldsymbol{e} = -\frac{d}{dt} S^2 \boldsymbol{b} \tag{3}$$

$$D^1 S^1 \boldsymbol{h} = \frac{d}{dt} S^2 \boldsymbol{d} + S^2 \boldsymbol{j} \tag{4}$$

   Show that above statements are correct by using Corollary 2.2E of the lecture notes.

3. For the discrete Maxwell's equations the number of equations may not be equal to the number or unknowns. Two strategies are possible, the primary elimination strategy which leads to the discrete wave equation for the electric field and the secondary elimination strategy which leads to the discrete magnetic wave equation

$$(\tilde{D}^1)^T M^2_{1/\epsilon} \tilde{D}^1 \vec{h} = -\frac{d}{dt} \tilde{M}^1_\mu \vec{h} + (\tilde{D}^1)^T M^2_{1/\epsilon} \vec{j} \tag{5}$$

   where D^1 is an incidence matrix and M^1_μ, $M^2_{1/\epsilon}$ are mass matrices. Show that the above discrete equation can be deduced from equations (2.2b), (2.3c), (2.3g) and (2.2a).

4. For the equation system

$$\left.\begin{array}{rcl} du & = & s \\ dj & = & \rho + f \\ j & = & *_\sigma s \\ \rho & = & *_\tau u \end{array}\right\} \quad \text{in } \Omega \subset A^3,\ \gamma_{\partial\Omega} u = 0 \tag{6}$$

where u is a 0–form, s a 1–form, j a 2–form ρ a 3–form (unknowns), f a 3–form (source term) and $*_\sigma$, $*_\tau$ are Hodge operators, do the following:

(a) Express the above equation system in terms of vector proxies.

(b) Give the abstract discrete material laws (the pairing and mass matrices) for the above equation system.

(c) Give an expression for the discrete integration by parts formula for the above equation system.

(d) Following section 2.4 of the lecture notes, give the abstract error analysis for the above equation system.

5. Give an example of when one can not construct an orthogonal dual mesh which satisfies the Delaunay property for the finite volume solution approach described in section 2.5 of the lecture notes.

6. For the barycentric dual mesh approach to finite volumes, discuss why a consistent diagonal mass matrix cannot be constructed in general.

7. Consider the local pairing matrices for general hexahedra in the mimetic finite difference method. Show that the local paring matrices are independent of the shape of the hexahedron.