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Computational Electromagnetics Course

Exercise Sheet 5: Whitney Forms.

1. Theorem 2.7D in the lecture notes, which is the De Rahm theorem for Whitney forms, is stated below
$$\dim\ \mathrm{kern}(d) \cap W_0^\ell / dW_0^{\ell-1} = \beta_\ell(\Omega) \tag{1}$$
where W_0^ℓ, $W_0^{\ell-1}$ are spaces of Whitney forms and $\beta_\ell(\Omega)$ is the ℓth Betti number. An outline of the proof is given in the lecture notes, please try to fill in the gaps.

2. The barycentric coordinates $\lambda_1, \cdots, \lambda_4$ are associated with the four vertices of a non–degenerate tetrahedron $T = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4]$, $\boldsymbol{a}_i \in A_3$. They satisfy
$$\sum_{i=1}^{4} \lambda_i = 1 \tag{2}$$
for any point inside the tetrahedron and $\lambda_i(\boldsymbol{a}_j) = \delta_{ij}$. Establish that $\lambda_i(\boldsymbol{x}) = \frac{|F_i|}{6|T|} < \boldsymbol{x} - \boldsymbol{a}_j, \boldsymbol{n}_i >$ where $\boldsymbol{n}_i$ is the normal to face i.

3. The general extension formula for ℓ-co-chains on an n-simplex $T = [a_0, \cdots, a_n] \subset A_n$ is
$$W^\ell \vec{w}_{|I} = \ell! \sum_I \sum_{j=0}^{\ell} (-1)^j \lambda_{i_j} \wedge d\lambda_{i_o} \wedge \cdots \wedge d\lambda_{i_{j-1}} \wedge d\lambda_{i_{j+1}} \wedge \cdots \wedge d\lambda_{i_\ell} \vec{w}([a_{i_o}, \cdots, a_{i_\ell}]) \tag{3}$$
where $I = (i_0, \cdots, i_\ell)$, $0 \leq i_o \leq \cdots \leq i_\ell \leq n$. Express the local basis forms for tetrahedral 1–forms and Whitney 2–forms in terms of Euclidean vector proxies.

4. Corollary 2.7E in the lecture notes states that W^ℓ commutes with local traces in to sub-simplices. An implication of this is that the trace of a tetrahedral Whitney form on a face gives the Whitney form on a triangle. Show that this is true for different degree Whitney forms.

5. Section 2.7.3 of the lecture notes discusses the construction of non–simplicial Whitney forms. In this section it is shown how hexahedral (or brick) elements can be constructed by a tensor product approach. Using a tensor product approach show how Whitney 1–forms can be constructed on a prism.

6. Construct the local mass matrix for Whitney 1–forms on a brick under the assumption of constant matearial coefficents. Compare the resulting mass matrix with that obtained for the mimetic finite difference method.