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Computational Electromagnetics Course

Exercise Sheet 6: Potential Based Weak Forms and Least Squares Formulations.

1. For potential based weak forms a gauge condition is required. This is because the gradient of a scaler field can be added to a variable of the weak variational form without effecting the actual variational statement. For example, in the weak variational statement

$$a_{1/\mu}(\mathrm{d}\boldsymbol{a}, \mathrm{d}\boldsymbol{a}') + \frac{\mathrm{d}^2}{\mathrm{d}t^2} a_\epsilon(\boldsymbol{a}, \boldsymbol{a}') + \frac{\mathrm{d}}{\mathrm{d}t} a_\epsilon(\mathrm{d}v, \boldsymbol{a}') + \int_{\partial\Omega} \boldsymbol{h} \wedge \boldsymbol{a}' = \int_\Omega \boldsymbol{j} \wedge \boldsymbol{a}' \qquad \forall \boldsymbol{a}' \in \mathcal{D}F^{1,0}(\Omega) \tag{1}$$

the gradient of a 0–form can be added to $\boldsymbol{a}$ without effecting the equation. Show that the above variational statement is gauge invariant.

2. In Section 3.2 of the lecture notes the general gauge condition for a potential based weak form is given as

$$\mathrm{d} *_\kappa \boldsymbol{a} = - *_\gamma \frac{\mathrm{d}}{\mathrm{d}t} v \tag{2}$$

where κ and γ are arbitrary metric tensors. Three specific options are proposed

- choose $\kappa = \epsilon$ and $\gamma_{\partial\Omega} v = 0$ which results in the Lorenz gauged variational formulation given in equation (3.2) of the lecture notes;
- choose $\gamma = 0$ and $\gamma_{\partial\Omega} v = 0$ which results in the Coulomb gauge;
- choose $v = 0$ which results in the so called temporal gauge.

For each of the above gauge conditions, show that they uniquely determine the potentials v and $\boldsymbol{a}$.

3. The idea of the least squares formulation is to impose the strong form of the topological laws as a constraint and minimise the residual energies in the material laws. This results in the following minimisation procedure: minimise

$$J(\boldsymbol{e}, \boldsymbol{h}) = E_{mag}\left(*_\mu \boldsymbol{h} + \int (\mathrm{d}\boldsymbol{e})\ \mathrm{d}t \right) + E_{el}\left(*_\epsilon \boldsymbol{e} + \int (\mathrm{d}\boldsymbol{h} - \boldsymbol{j})\ \mathrm{d}t \right) \tag{3}$$

The necessary conditions for a minimum are

$$\begin{aligned} &\frac{\partial J}{\partial \boldsymbol{e}}(\boldsymbol{e},\boldsymbol{h})\boldsymbol{e}' = 0 \\ &\frac{\partial J}{\partial \boldsymbol{e}}(\boldsymbol{e},\boldsymbol{h})\boldsymbol{h}' = 0 \end{aligned} \qquad \forall \boldsymbol{e}', \boldsymbol{h}' \tag{4}$$

Based on vector proxies state the full variational statement that arises from equations (3) and (4).

4. Definition 3.4E in the lecture notes defines the Hilbert space $H_\ell(\mathrm{d}0, \Omega)$ as

$$H_\ell(\mathrm{d}0, \Omega) := \{\omega \in H_\ell(\mathrm{d}, \Omega)\ \mathrm{d}\omega = 0\} \tag{5}$$

where $H_\ell(\mathrm{d}, \Omega)$ is in turn defined by

$$H_\ell(d; \Omega) := \overline{\mathcal{D}F^{\ell,1}(\Omega)}^{||\cdot||_{d;\Omega}} \tag{6}$$

Show that $H_\ell(\mathrm{d}0, \Omega) \subset H_\ell(\mathrm{d}, \Omega)$ is a closed subset.

5. Definition 3.4F in the lecture notes defines $H_{\ell,0}(\mathrm{d}, \Omega)$ as

$$H_{\ell,0}(\mathrm{d}, \Omega) := \overline{\{\omega \in \mathcal{D}F^{\ell,1}(\Omega), \mathrm{supp}(\omega) \subset \Omega\}}^{||\cdot||_{\mathrm{d},\Omega}} \tag{7}$$

Show that $H_{\ell,0}(\mathrm{d}, \Omega) \subset H_\ell(\mathrm{d}, \Omega)$ is a closed subset.