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Computational Electromagnetics Course

Exercise Sheet 9: Maxwell Eigenvalue and Cavity Problems.

1. As given in definition 5.2a of the lecture notes, the Fortin projector $F_h : H_0(\mathrm{curl}, \Omega) \to W_{0,0}^{1,\perp}(\mathcal{T}_h)$ is a continuous linear operator defined by

$$\int_\Omega \mathrm{curl}\, F_h \boldsymbol{v} \cdot \mathrm{curl}\, \boldsymbol{v}_h' \mathrm{d}x = \int_\Omega \mathrm{curl}\, \boldsymbol{v} \cdot \mathrm{curl}\, \boldsymbol{v}_h' \,\mathrm{d}x \qquad \forall \boldsymbol{v}_h' \in W_{0,0}^{1,\perp}(\mathcal{T}_h) \quad (1)$$

A continuous self adjoint linear operator $T : (L^2(\Omega))^3 \to H_0^\perp(\mathrm{curl}, \Omega) \subset (L^2(\Omega))^3$ can be defined by

$$\int_\Omega \mathrm{curl}\, T \boldsymbol{v} \cdot \mathrm{curl}\, \boldsymbol{v}_h' \mathrm{d}x = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{v}_h' \,\mathrm{d}x \qquad \forall \boldsymbol{v}_h' \in H_0^\perp(\mathrm{curl}, \Omega) \qquad (2)$$

and the discrete equivalent of T denoted by $T_h : (L^2(\Omega))^3 \to W_{0,0}^{1,\perp}(\mathcal{T}_h)$ can also be defined

$$\int_\Omega \mathrm{curl}\, T_h \boldsymbol{v} \cdot \mathrm{curl}\, \boldsymbol{v}_h' \mathrm{d}x = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{v}_h' \,\mathrm{d}x \qquad \forall \boldsymbol{v}_h' \in W_{0,0}^{1,\perp}(\mathcal{T}_h) \qquad (3)$$

Using these definitions show that

$$T_h = F_h \circ T \qquad (4)$$

is true.

2. In the proof of Theorem 5.2C $\boldsymbol{u}^* \in H_0^\perp(\mathrm{curl}, \Omega)$ is fixed by setting $\mathrm{curl}\, \boldsymbol{u}^* \in W_{0,0}^2(\mathcal{T}_h)$ and

$$\int_\Omega \mathrm{curl}\, \boldsymbol{u}^* \cdot \mathrm{curl}\, \boldsymbol{u}_h' \mathrm{d}x = \int_\Omega \mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{u}_h' \mathrm{d}x \qquad \forall \boldsymbol{u}_h' \in W_{0,0}^{1,\perp}(\mathcal{T}_h) \quad (5)$$

Show that

$$\boldsymbol{u}^* = H \circ F_h \boldsymbol{u} \qquad (6)$$

is true where F_h is the Fortin projector and H is the Hodge mapping defined in Definition 4.3.4E of the lecture notes.

3. The duality technique for solving the Maxwell eigenvalue problem involves the solution the saddle point problem: for $\boldsymbol{g} \in (L^2(\Omega))^3$ find $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{g}) \in H_0(\text{curl}, \Omega)$, $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{g}) \in \text{curl}\, H_0(\text{curl}, \Omega)$ such that

$$\int_\Omega \boldsymbol{w} \cdot \boldsymbol{w}' \,\mathrm{d}x + \int_\Omega \boldsymbol{w}' \cdot \boldsymbol{p} \,\mathrm{d}x = 0 \tag{7}$$

$$\int_\Omega \text{curl}\, \boldsymbol{w} \cdot \boldsymbol{p}' \,\mathrm{d}x = \int_\Omega \boldsymbol{g} \cdot \boldsymbol{p}' \mathrm{d}x \tag{8}$$

for all $\boldsymbol{w}' \in H_0(\text{curl}, \Omega)$ and for all $\forall \boldsymbol{p}' \in \text{curl}\, H_0(\text{curl}, \Omega)$. Show that this saddle point problem has a unique solution.

4. The presence of kern (curl) in equation (5.2a) of the lecture notes will cripple inverse iteration eigenvalue solvers which aim to compute a few of the smallest resonant frequencies. A remedy to this is to add a symmetric positive definite form d_h resulting in the discrete regularised matrix eigenvalue problem

$$\begin{pmatrix} A & B^T \\ B & -D \end{pmatrix} \begin{pmatrix} \vec{e}_h \\ \vec{p}_h \end{pmatrix} = \begin{pmatrix} \omega^2 M \vec{e}_h \\ 0 \end{pmatrix} \tag{9}$$

Elimination of $\vec{p}_h$ yields

$$(A + B^T D^{-1} B)\vec{e}_h = \omega^2 M \vec{e}_h \tag{10}$$

Show that $(A + B^T D^{-1} B)$ is symmetric positive definite matrix. Also, give the stencils for the matrices on tenor product type meshes.

5. Lemma 6.2C states that the embedding $H_0(\text{curl}, \Omega) \to (L^2(\Omega))^3$ is not compact. Prove this lemma.