Computational Methods for Quant. Finance II

Finite difference and finite element methods

Lecture 1

Scope of the course

Analysis and implementation of numerical methods for pricing options.

Models: Black-Scholes, stochastic volatility, exponential Lévy.

Options: European, American, Asian, barrier, compound ...

In this course: Focus on deterministic (PDE based) methods

- Finite difference methods (FDM)
- Finite element methods (FEM)

This course will be complemented by the course Monte Carlo methods in autumn 2009.

Organization of the course

14 lectures (2 hours) + 13 exercise classes (1 hour).

- No lectures on April, 14.
- Testat: 70% of solved homework assignments (theoretical exercises + MATLAB programming).

Examination: On Tuesday, May 26, 15–17.
Written, closed-book examination includes theoretical and MATLAB programming problems.
Examination takes place on ETH-workstations running MATLAB under LINUX. Own computer will NOT be required.

Outline

Definitions and notation

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ be a multiindex. Set $|\alpha| = \sum_{i=1}^d \alpha_i$.
For $u : G \to \mathbb{R}$, $x = (x_1, \dots, x_d) \in G \subset \mathbb{R}^d$ define

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} u.$$

Let $k \in \mathbb{N}_0$. Then

$$D^k u(x) := \{ D^\alpha u(x) \ : \ |\alpha| = k \}$$

is the set of all partial derivatives of order k. If $k = 1$, we regard the elements of $D^1 u(x) =: Du(x)$ as being arranged in a vector

$$Du = (\partial_{x_1} u, \dots, \partial_{x_d} u).$$

If $k = 2$, we regard the elements of $D^2u(x)$ as being arranged in a matrix

$$D^2 u = \begin{pmatrix} \partial_{x_1}\partial_{x_1} u & \cdots & \partial_{x_1}\partial_{x_d} u \\ & \ddots & \\ \partial_{x_d}\partial_{x_1} u & \cdots & \partial_{x_d}\partial_{x_d} u \end{pmatrix}.$$

In the following: write $\partial_{x_i x_j}$ instead of $\partial_{x_i}\partial_{x_j}$. Hence, the Laplacian Δu of u can be written as

$$\Delta u := \sum_{i=1}^{d} \partial_{x_i x_i} u = \mathrm{tr}(D^2 u).$$

A partial differential equation (PDE) is an equation involving an unknown function of two or more variables and certain of its derivatives.

Let $G \in \mathbb{R}^d$ be open, $x = (x_1, \ldots, x_d)$ and $\mathbb{N} \ni k \geq 1$.

Definition

An expression of the form

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0, \quad x \in G$$

is called a k-th order PDE, where

$$F : \mathbb{R}^{d^k} \times \mathbb{R}^{d^{k-1}} \times \cdots \times \mathbb{R}^d \times \mathbb{R} \times G \to \mathbb{R}$$

is given and

$$u : G \to \mathbb{R}$$

is the unknown.

Let $a_{ij}(x), b_i(x), c(x)$ and $f(x)$ be given functions. For a linear 2nd order PDE in $d+1$ variables, F has the form

$$F(D^2u, Du, u, x) = -\sum_{i,j=0}^{d} a_{ij}(x)\partial_{x_i x_j} u + \sum_{i=0}^{d} b_i(x)\partial_{x_i} u + c(x)u - f(x).$$

Assume that the matrix $A(x) := \{a_{ij}(x)\}_{i,j=0}^{d}$ is symmetric with real eigenvalues $\lambda_0(x) \leq \lambda_1(x) \leq \cdots \leq \lambda_d(x)$.

Definition

Let $S = \{0, \ldots, d\}$. At $x \in \mathbb{R}^{d+1}$, the PDE is called

(i) elliptic $\Leftrightarrow \lambda_i(x) \neq 0, \forall i \ \wedge \ \operatorname{sign}(\lambda_0(x)) = \ldots = \operatorname{sign}(\lambda_d(x))$

(ii) parabolic $\Leftrightarrow \exists! j \in S \ : \lambda_j(x) = 0 \wedge \ \operatorname{rank}(A(x), b(x)) = d+1$

(iii) hyperbolic $\Leftrightarrow \lambda_i(x) \neq 0, \forall i \ \wedge \exists! j \in S \ : \ \operatorname{sign}\lambda_j(x) \neq \operatorname{sign}\lambda_k(x)$, $k \in S \setminus \{j\}$

The PDE is called elliptic, parabolic, hyperbolic on G, if it is elliptic, parabolic, hyperbolic $\forall x \in G$.

Examples

- The heat equation $\partial_t u - \Delta u = f(t,x)$ is parabolic (set $x_0 = t$).
- The Poisson equation $\Delta u = f(x)$ is elliptic.
- The wave equation $\partial_{tt} u - \Delta u = f(t,x)$ is hyperbolic (set $x_0 = t$)
- The Black Scholes equation for the value of a European option $v(t,s)$

  $$\partial_t v - \frac{1}{2}\sigma^2 s^2 \partial_{ss} v - rs\partial_s v + rv = 0$$

  with $\sigma, r \geq 0$ is parabolic at $(t,s) \in (0,T) \times (0,R)$ and degenerates to a ordinary differential equation as $s \to 0$.

Note: PDEs can have infinitely many solutions. To obtain a unique solution, we have to pose boundary conditions.

Outline

The PDE

Let $G = (a, b) \subset \mathbb{R}$ be a open interval and let $J := (0, T)$, $T > 0$.
Find $u : J \times G \to \mathbb{R}$ such that

$$\begin{cases} \partial_t u - \partial_{xx} u = f(t,x) & \text{in} \quad J \times G \\ u = 0 & \text{on} \quad J \times \partial G \\ u(0,\cdot) = u_0 & \text{in} \quad G \end{cases}$$

Remark

(i) *The equation $u(0,\cdot) = u_0$ in G is the initial condition.*

(ii) *The equation $u = 0$ on $J \times \partial G$ is the boundary condition. Here it is of Dirichlet type and homogeneous.*

Goal: approximate $u(t, x)$.

Discretization of the domain

Computational domain $J \times G$ is replaced by discrete grid:

$$\{(t_m, x_i)\}, \quad i = 0, \ldots, N+1, \quad m = 0, \ldots, M,$$

where x_i are space grid points with space step size h and t_m are the time levels with time step size k:

$$x_i = a + ih, \quad h = \frac{b-a}{N+1}, \qquad t_m = mk, \quad k = \frac{T}{M}.$$

We represent the exact solution $u(t,x)$ by its values on the grid:

$$u(t,x) \;\longrightarrow\; \{u_i^m = u(t_m, x_i)\}, \quad i = 0, \ldots, N+1, \; m = 0, \ldots, M.$$

The goal is to approximate the values $\{u_i^m\}$. Values of the solution between grid points are then found by some interpolation.

Difference Quotients (= Finite Differences)

We want to approximate the derivatives of u using only its values on the grid. First, let us consider a function $f(x)$ of one variable.

Assume that $f \in C^2$. Then, using Taylor's formula,

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{h}{2} f''(\xi), \quad \xi \in [x, x+h] \, .$$

If $f_i = f(x_i)$ are the values of f on the grid $\{x_i\}$, we obtain

$$f'(x_i) = \frac{f_{i+1} - f_i}{h} + O(h) =: \left(\delta_x^+ f\right)_i + O(h).$$

Similarly, for $f \in C^4$

$$f''(x_i) = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} + O(h^2) =: \left(\delta_{xx} f\right)_i + O(h^2).$$

FD scheme

Let $\theta \in [0,1]$. We replace the PDE $\partial_t u - \partial_{xx} u = f$ by the set of algebraic equations

$$\left\{ \begin{array}{rcll} \mathcal{E}_i^m & = & \theta f_i^{m+1} + (1-\theta) f_i^m & i = 1, \dots, N,\ m = 0, \dots, M-1 \\ u_i^0 & = & u_0(x_i) & i = 1, \dots, N \\ u_k^m & = & 0 & k \in \{0, N+1\},\ m = 0, \dots, M \end{array} \right. ,$$

where $\mathcal{E}_i^m$ is the finite difference operator

$$\begin{aligned} \mathcal{E}_i^m &:= k^{-1}\left(u_i^{m+1} - u_i^m\right) - \left[\theta(\delta_{xx} u)_i^{m+1} + (1-\theta)(\delta_{xx} u)_i^m\right] \\ &= \frac{u_i^{m+1} - u_i^m}{k} \\ &\qquad - \left[\theta \frac{u_{i+1}^{m+1} - 2u_i^{m+1} + u_{i-1}^{m+1}}{h^2} + (1-\theta) \frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{h^2}\right]. \end{aligned}$$

FD scheme in matrix form

Introduce the column vectors

$$\underline{u}^m = (u_1^m, \ldots, u_N^m)^\top, \quad \underline{\mathcal{E}}^m = (\mathcal{E}_1^m, \ldots, \mathcal{E}_N^m)^\top, \quad \underline{f}^m = (f_1^m, \ldots, f_N^m)^\top$$

and the tridiagonal $N \times N$ matrix

$$\mathbf{G} = h^{-2}\text{tridiag}(-1, 2, -1) \; .$$

Then the FD-scheme $\underline{\mathcal{E}}^m = \theta \underline{f}^{m+1} + (1-\theta)\underline{f}^m$ becomes, in matrix form: Given $\underline{u}^0 = (u_0(x_1), \ldots, u_0(x_N))^\top \in \mathbb{R}^N$, for $m = 0, \ldots, M-1$ find $\underline{u}^{m+1} \in \mathbb{R}^N$ such that

$$\big(\underbrace{\mathbf{I} + \theta k \mathbf{G}}_{=:\mathbf{B}}\big)\underline{u}^{m+1} + \big(\underbrace{-\mathbf{I} + (1-\theta)k\mathbf{G}}_{=:-\mathbf{C}}\big)\underline{u}^m = k[\underbrace{\theta \underline{f}^{m+1} + (1-\theta)\underline{f}^m}_{=:\underline{F}^m}],$$

or

$$\mathbf{B}\underline{u}^{m+1} = \mathbf{C}\underline{u}^m + k\underline{F}^m, \quad m = 0, \ldots, M-1.$$

Variational formulation

We do not require the PDE to be hold pointwise. Take a smooth test function $v \in C_0^\infty(G)$ satisfying $v(a) = v(b) = 0$. Multiply the PDE with v and integrate by parts:

$$\begin{aligned} \int_G \partial_t uv \,\mathrm{d}x - \int_G \partial_{xx} uv \,\mathrm{d}x &= \int_G fv \,\mathrm{d}x \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_G uv \,\mathrm{d}x - \underbrace{[\partial_x u(t,x)v(x)]_{x=a}^{x=b}}_{=0} + \int_G \partial_x u \partial_x v \,\mathrm{d}x &= \int_G fv \,\mathrm{d}x \end{aligned}$$

The variational or weak formulation of the heat equation reads: Find u such that $u(0) = u_0$ and such that $\forall v \in C_0^\infty(G)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_G u(t,x)v(x)\mathrm{d}x + \int_G u'(t,x)v'(x)\mathrm{d}x = \int_G f(t,x)v(x)\mathrm{d}x.$$

Galerkin discretization

Let V_N be a finite (N) dimensional subspace of $H_0^1(G)$.

The idea is to approximate $u(t,x)$ by an element $u_N(t,x) \in V_N$, for each $t \in J$.
Find $u_N(t,x) \in V_N$ such that $u_N(0,x) = u_{0,N}(x)$ and such that $\forall v_N \in V_N$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_G u_N(t,x)v_N(x)\mathrm{d}x + \int_G u_N'(t,x)v_N'(x)\mathrm{d}x = \int_G f(t,x)v_N(x)\mathrm{d}x.$$

Let $\{b_j\}_{j=1}^N$ be a basis of V_N. Then $u_N(t,x) = \sum_{j=1}^N u_{N,j}(t)b_j(x)$, where

$$\underline{u}_N(t) = (u_{N,1}(t), u_{N,2}(t), \ldots, u_{N,N}(t))^\top$$

is a vector of unknown functions. Similarly, $\forall v_N \in V_N$ $v_N(x) = \sum_{i=1}^N v_{N,i}b_i(x)$.

Hence (we skip the argument x in $\int_G$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_G u_N(t)v_N + \int_G u_N'(t)v_N' = \int_G f(t)v_N, \quad \forall v_N \in V_N$$

$$\Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_G \Big(\sum_j u_{N,j}(t)b_j\Big)\Big(\sum_i v_{N,i}b_i\Big) + \int_G \Big(\sum_j u_{N,j}(t)b_j\Big)'\Big(\sum_i v_{N,i}b_i\Big)' = \int_G f(t)\sum_i v_{N,i}b_i$$

$$\Leftrightarrow \sum_i v_{N,i}\Big[\sum_j \dot{u}_{N,j}\int_G b_jb_i + u_{N,j}\int_G b_j'b_i' - \int_G f(t)b_i\Big] = 0$$

$$\Leftrightarrow \underline{v}_N^\top\Big[\mathbf{M}\underline{\dot{u}}_N(t) + \mathbf{A}\underline{u}_N(t) - \underline{f}_N(t)\Big] = 0, \quad \forall \underline{v}_N \in \mathbb{R}^N$$

$$\Leftrightarrow \mathbf{M}\underline{\dot{u}}_N(t) + \mathbf{A}\underline{u}_N(t) = \underline{f}_N(t)\ .$$

Semi discrete scheme

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_N(t), v_N) + a\big(u_N(t), v_N\big) = (f(t), v_N), \quad \forall v_N \in V_N$$

is equivalent to the ODE

$$\mathbf{M}\underline{\dot{u}}_N(t) + \mathbf{A}\underline{u}_N(t) = \underline{f}_N(t),$$

where $\mathbf{M}$ (mass matrix) and $\mathbf{A}$ (stiffness matrix) are $N \times N$ matrices with

$$\mathbf{M}_{ij} = \int_G b_j(x) b_i(x) \mathrm{d}x, \quad \mathbf{A}_{ij} = \int_G b_j'(x) b_i'(x) \mathrm{d}x.$$

Similarly, $\underline{f}_N(t) \in \mathbb{R}^N$ with entries

$$f_{N,i}(t) = \int_G f(t,x) b_i(x) \mathrm{d}x.$$

Fully discrete scheme

We discretize in time. Write

$$\underline{u}_N^m := u_N(t_m), \quad \underline{f}_N^m := f_N(t_m),$$

where the time levels t_m, $m = 0, \ldots, M$ are as before. Proceeding exactly as in the FDM, the fully discrete scheme reads:

Given $\underline{u}_N^0 = (u_0(x_i))_{i=1}^N \in \mathbb{R}^N$, for $m = 0, \ldots, M-1$ find $\underline{u}_N^{m+1} \in \mathbb{R}^N$ such that

$$\mathbf{M}k^{-1}\left(\underline{u}_N^{m+1} - \underline{u}_N^m\right) + \mathbf{A}\left(\theta \underline{u}_N^{m+1} + (1-\theta)\underline{u}_N^m\right) = \theta \underline{f}_N^{m+1} + (1-\theta)\underline{f}_N^m.$$

Choice of V_N

Setting $\mathbf{B} := \mathbf{M} + k\theta\mathbf{A}$, $\mathbf{C} := \mathbf{M} - k(1-\theta)\mathbf{A}$ and $\underline{F}_N^m := \theta\underline{f}_N^{m+1} + (1-\theta)\underline{f}_N^m$ this can be written as

$$\mathbf{B}\underline{u}_N^{m+1} = \mathbf{C}\underline{u}_N^m + k\underline{F}_N^m, \quad m = 0, \dots, M-1.$$

It remains to chose a space V_N. Probably the simplest choice: V_N is the space of piecewise linear, continuous functions.

Let

$$\mathcal{T} := \{a = x_0 < x_1 < \cdots < x_{N+1} = b\} = \{K_i\}_{i=1}^{N+1}$$

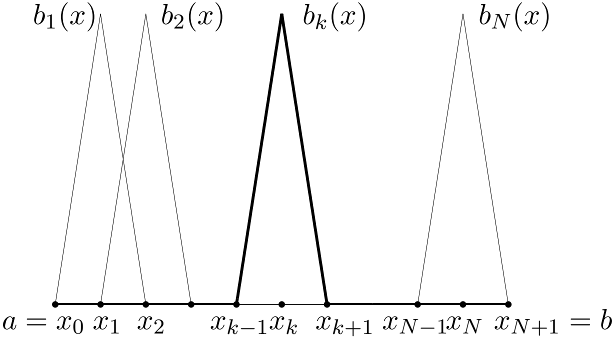
be an equidistant mesh on G with $K_i := (x_{i-1}, x_i)$. (x_i as before). Set

$$V_N = S_0^1 := \left\{u \in C_0^0(G) \ : u|_{K_i} \text{ is affin linear on } K_i \in \mathcal{T}\right\}.$$

Note: $\dim V_N = N$.

A basis $\{b_i\}_{i=1}^N$ of V_N is given by the so-called hat-functions

$$b_i : [a, b] \to \mathbb{R}_{\geq 0}, \quad b_i(x) = \max\{0, 1 - h^{-1}|x - x_i|\}, i = 1, \ldots, N.$$



With this basis, we find for the mass- and stiffness matrix

$$\mathbf{M} = h/6 \,\mathrm{tridiag}(1, 4, 1), \qquad \mathbf{A} = h^{-1}\mathrm{tridiag}(-1, 2, -1).$$

For both FDM and FEM, we have to solve M systems of N linear equations of the form

$$\mathbf{B}\underline{u}^{m+1} = \mathbf{C}\underline{u}^{m} + k\underline{F}^{m}, \quad m = 0, \ldots, M-1.$$

where $\underline{F}^m = \theta \underline{f}^{m+1} + (1-\theta)\underline{f}^m$ and

	FDM	FEM
$\underline{u}^m$	vector of $u_i^m \approx u(t_m, x_i)$	coeff. vector of $u_N(t_m, x)$
$\mathbf{B}$	$\mathbf{I} + k\theta\mathbf{G}$	$\mathbf{M} + k\theta\mathbf{A}$
$\mathbf{C}$	$\mathbf{I} - k(1-\theta)\mathbf{G}$	$\mathbf{M} - k(1-\theta)\mathbf{A}$
	$\mathbf{G} = h^{-2}\mathrm{tridiag}(-1, 2, -1)$	$\mathbf{A} = h^{-1}\mathrm{tridiag}(-1, 2, -1)$
$\underline{f}^m$	$f(t_m, x_i)$	$\underline{f}_i^m = \int_G f(t_m, x) b_i(x) \mathrm{d}x$

Matlab coding (FDM)

```
function error = heateq_fdm(a,b,T,N,theta)
h = (b-a)/(N+1); k = h; e = ones(N,1);
I = speye(N); G = h^(-2)*spdiags([-e 2*e -e],-1:1,N,N);
B = I+k*theta*G; C = I-k*(1-theta)*G
x = [a+h:h:b-h]'; u0 = x.*sin(pi*x);
f = -(1-pi^2)*x.*sin(pi*x)-2*pi*cos(pi*x);
u = zeros(N,T/k+1); u(:,1) = u0;
for j = 1:T/k
    F = k*f*(theta*exp(-j*k)+(1-theta)*exp(-(j-1)*k));
    u(:,j+1) = B\(C*u(:,j)+F);
    err(j) = norm(u(:,j+1)-exp(-k*j)*u0);
end
error = sqrt(h)*max(err)
```

Example

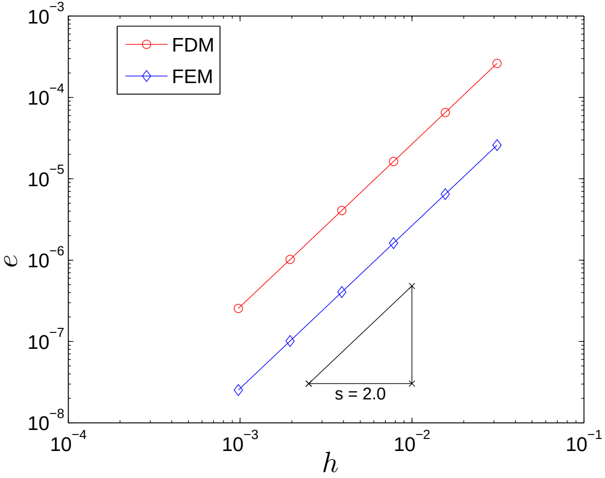
Let $G = (0,1)$, $T = 1$, and $u(x,t) = e^{-t} x \sin(\pi x)$.

We measure the discrete $L^\infty(0,T;L^2(G))$-error defined by

$$e := \sup_m h^{\frac{1}{2}} \|\underline{\epsilon}^m\|_{\ell_2}, \quad \|\underline{\epsilon}^m\|_{\ell_2}^2 := \sum_i |u(t_m, x_i) - u_i^m|^2 .$$

For $\theta = 0.5$ and $k = h$, we obtain, in terms of the mesh width h, convergence both of FDM and FEM of second order, i.e.,

$$e = O(h^s), \quad s = 2 .$$


FDM
FEM
s = 2.0
e
h
10^−3
10^−4
10^−5
10^−6
10^−7
10^−8
10^−4
10^−3
10^−2
10^−1