ETH Lecture 401-0663-00L Numerical Methods for CSE

Numerical Methods for Computational Science and Engineering

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Autumn Term 2016

URL: http://www.sam.math.ethz.ch/~hiptmair/tmp/NumCSE/NumCSE16.pdf

IV. Filtering Algorithms

4.1. Discrete Convolutions

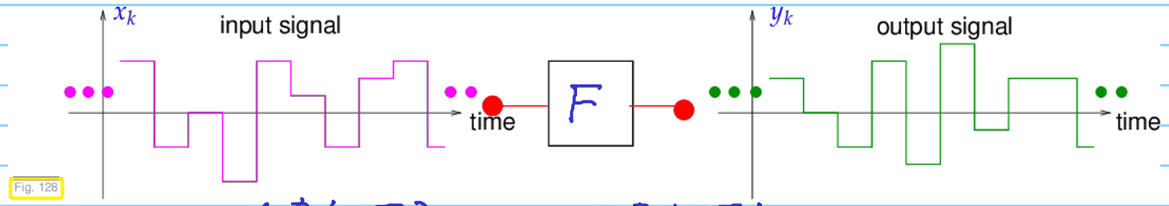


Fig. 128

$F : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$

Impulse response $(h_j) = F((\delta_{0j})_{j\in\mathbb{Z}})$

Definition 4.1.4. Finite channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is called finite, if every input signal of finite duration produces an output signal of finite duration,

$$\{\, \exists M \in \mathbb{N}: \ |j| > M \ \Rightarrow\ x_j = 0 \,\} \ \Rightarrow\ \exists N \in \mathbb{N}: \ |k| > N \ \Rightarrow\ (F((x_j)_{j\in\mathbb{Z}}))_k = 0\,, \tag{4.1.5}$$

Definition 4.1.11. Causal channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is called causal (or physical, or nonanticipative), if the output does not start before the input

$$\forall M \in \mathbb{N}: \ (x_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}),\ x_j = 0 \ \ \forall j \le M \ \Rightarrow\ F((x_j)_{j\in\mathbb{Z}})_k = 0 \ \ \forall k \le M\,. \tag{4.1.12}$$

Impulse response does not depend on t.

$\ell^\infty(\mathbb{Z}) \triangleq$ vector space of bounded signals

"Signals in pictures"

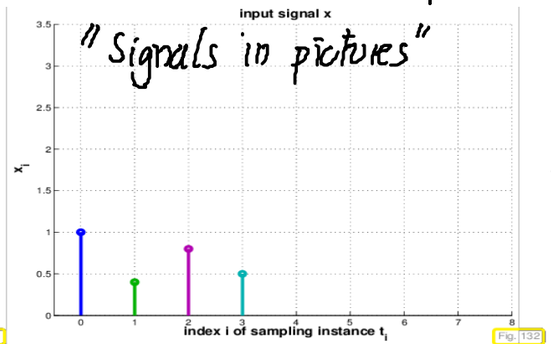


Fig. 131

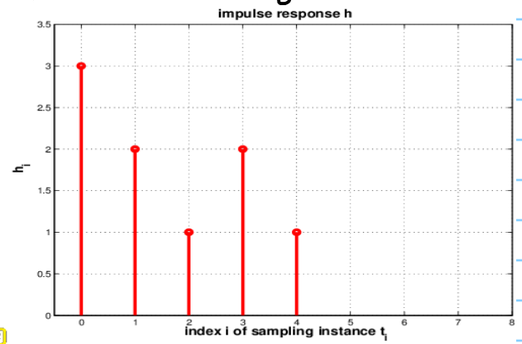


Fig. 132

(2)

Definition 4.1.7. Time-invariant channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is called time-invariant, if shifting the input in time leads to the same output shifted in time by the same amount; it *commutes with the time shift operator* from (4.1.6):

$$\forall (x_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}),\ \forall m \in \mathbb{Z}: \quad F(S_m((x_j)_{j\in\mathbb{Z}})) = S_m(F((x_j)_{j\in\mathbb{Z}}))\,. \tag{4.1.8}$$

time shift $S_m((x_j)_{j\in\mathbb{Z}}) = (x_{j-m})_{j\in\mathbb{Z}}$

$\triangleright$ $F((\delta_{jm})_{j\in\mathbb{Z}}) = (h_{j-m})_{j\in\mathbb{Z}}$

Definition 4.1.9. Linear channel/filter

A filter $F : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is called linear, if F is a linear mapping:

$$F(\alpha(x_j)_{j\in\mathbb{Z}} + \beta(y_j)_{j\in\mathbb{Z}}) = \alpha F((x_j)_{j\in\mathbb{Z}}) + \beta F((y_j)_{j\in\mathbb{Z}}) \quad \forall (x_j)_{j\in\mathbb{Z}}, (y_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}),\ \alpha,\beta \in \mathbb{R}\,. \tag{4.1.10}$$

LT - FIR channel

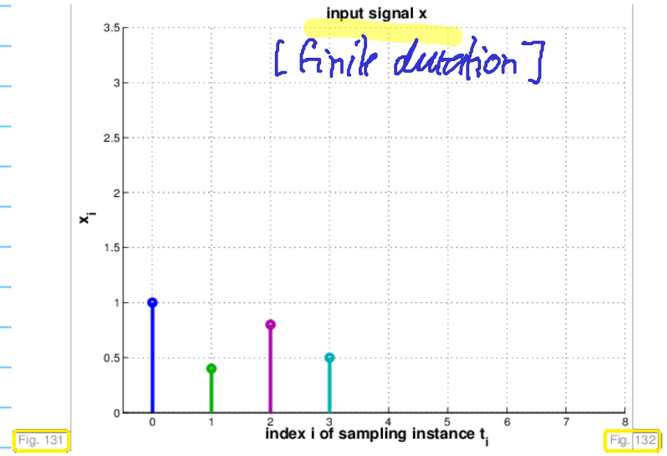
LT-FIR formula, i.e. $(0,\dots,h_0,\dots,h_{n-1},0,\dots)$, $n \in \mathbb{N}$

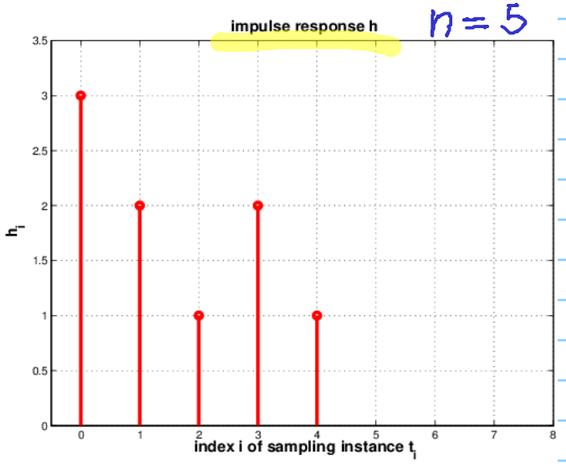
$$F((x_j)) = F\Big(\sum_{k\in\mathbb{Z}} x_k (\delta_{kj})_{j\in\mathbb{Z}}\Big)$$

$$= \sum_{k\in\mathbb{Z}} x_k F((\delta_{kj})_{j\in\mathbb{Z}})$$

$$= \sum_{k\in\mathbb{Z}} x_k (h_{j-k})_{j\in\mathbb{Z}}$$

$$(F((x_j)_{j\in\mathbb{Z}}))_j = \sum_{k\in\mathbb{Z}} x_k h_{j-k} \quad \text{[finite sum]}$$





$(0,\dots 0, x_0 \dots, x_{m-1}, 0 \dots)$ [finite]

$\Rightarrow$ Duration of output: $(m+n-2)\Delta t$!

③

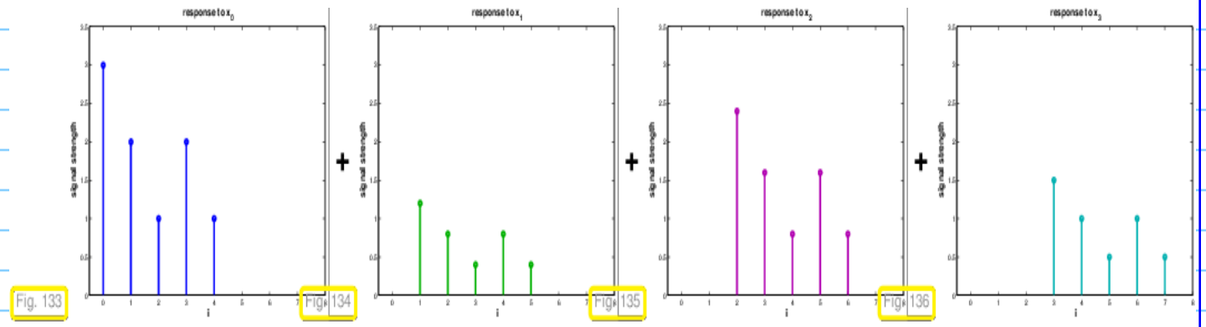


Fig. 133 Fig. 134 Fig. 135 Fig. 136

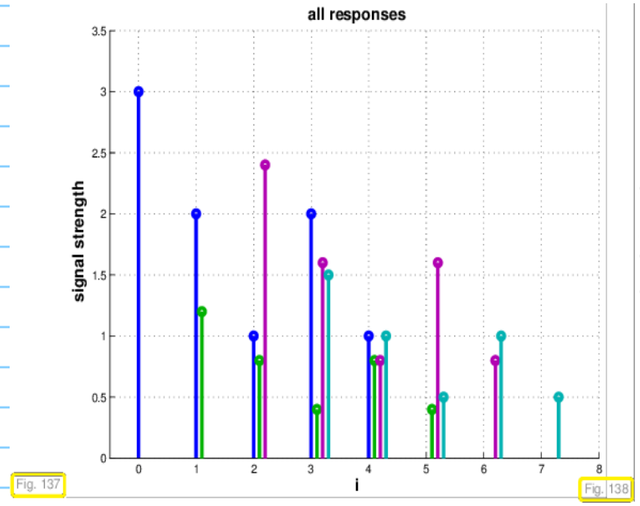


Fig. 137

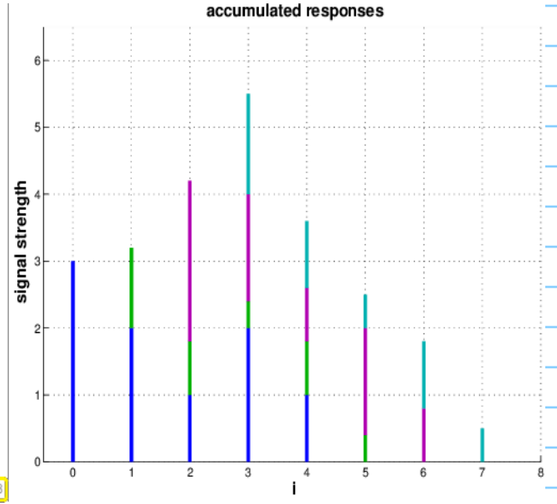


Fig. 138

For finite signals $(\ldots, 0, \ldots 0, x_0, \ldots, x_{n-1}, 0, \ldots) \sim \underline{x} \in \mathbb{R}^n$

LT-FIR channel ~ linear mapping $\mathbb{R}^n \to \mathbb{R}^{2n-1}$

↳ has a matrix representation

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{2n-2} \end{bmatrix} = \begin{bmatrix} h_0 & 0 & \cdots & 0 \\ h_1 & & & 0 \\ & \ddots & & \\ h_{n-1} & & & h_0 \\ 0 & & & h_1 \\ & \ddots & & \\ 0 & & 0 & h_{n-1} \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix} . \tag{4.1.18}$$

$$\Updownarrow$$

$$(F((x_j)_{j\in\mathbb{Z}}))_j = \sum_{k\in\mathbb{Z}} x_k h_{j-k} =: y_j$$

Definition 4.1.22. Discrete convolution

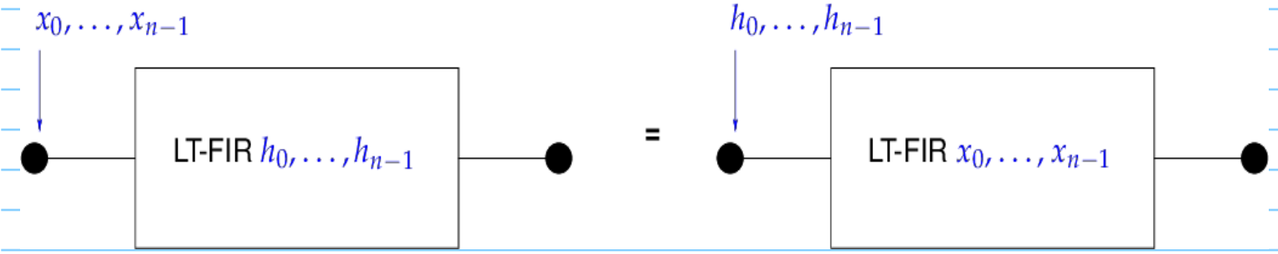
Given $\mathbf{x} = [x_0, \ldots, x_{n-1}]^\top \in \mathbb{K}^n$, $\mathbf{h} = [h_0, \ldots, h_{n-1}]^\top \in \mathbb{K}^n$ their discrete convolution is the vector $\mathbf{y} \in \mathbb{K}^{2n-1}$ with components

$$y_k = \sum_{j=0}^{n-1} h_{k-j} x_j , \quad k = 0, \ldots, 2n-2 \quad (h_j := 0 \text{ for } j < 0) . \tag{4.1.23}$$

Notation: $\underline{y} = \underline{h} * \underline{x} = \underline{x} * \underline{h}$

(4)

$$\sum_{k\in\mathbb{Z}} x_k h_{j-k} \underset{\ell:=j-k}{=} \sum_{\ell\in\mathbb{Z}} x_{j-\ell} h_\ell$$



$\hat{=}$ $\underline{x} \in \mathbb{R}^n : \underline{x} = [x_0, \ldots, x_{n-1}]^T$

Now: n-periodic input $(x_j)_{j\in\mathbb{Z}}$: $x_j = x_{j+n}$ $\forall j$

$\Rightarrow$ n-periodic output of LT-FIR $\hat{=}$ $\underline{y} \in \mathbb{R}^n$

LT-FIR in n-periodic setting : linear mapping $\mathbb{R}^n \to \mathbb{R}^n$

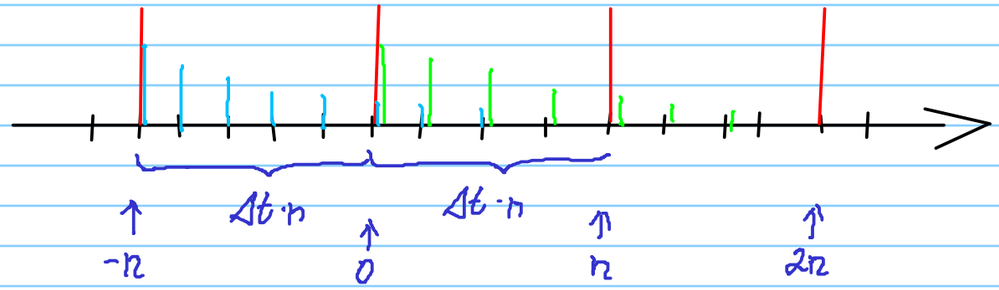
$$y_k = \sum_{\ell=0}^{n-1} p_{k-\ell} x_\ell \qquad [\underline{y} = \underline{p} *_n \underline{x}]$$

p : n-periodic impulse response

If $(h_j)_{j\in\mathbb{Z}} \hat{=}$ I.R. of LT-FIR, then

$$p_j = \sum_{k\in\mathbb{N}_0} h_{j+kn}, \quad j = 0, \ldots, n-1$$

Hint: $(p_j) \hat{=}$ output for



Definition 4.1.29. Discrete periodic convolution

The discrete periodic convolution of two n-periodic sequences $(p_k)_{k\in\mathbb{Z}}$, $(x_k)_{k\in\mathbb{Z}}$ yields the n-periodic sequence

$$(y_k) := (p_k) *_n (x_k) \quad , \quad y_k := \sum_{j=0}^{n-1} p_{k-j} x_j = \sum_{j=0}^{n-1} x_{k-j} p_j \, , \quad k \in \mathbb{Z} \, .$$

↑ defined by n-periodicity

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} p_0 & p_{n-1} & p_{n-2} & \cdots & \cdots & p_1 \\ p_1 & p_0 & p_{n-1} & & & \vdots \\ p_2 & p_1 & p_0 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & p_{n-1} \\ p_{n-1} & \cdots & & & p_1 & p_0 \end{bmatrix}}_{=:\mathbf{P}} \begin{bmatrix} x_0 \\ \vdots \\ x_{n-1} \end{bmatrix} . \qquad (4.1.27)$$

disc. per. conv = multiplication with circulant matrix

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Definition 4.1.34. Circulant matrix → [?, Sect. 54]

A matrix $\mathbf{C} = [c_{ij}]_{i,j=1}^{n} \in \mathbb{K}^{n,n}$ is circulant $\quad [\ C = \mathrm{circul}(p)\ ]$

$:\Leftrightarrow \quad \exists (p_k)_{k\in\mathbb{Z}}$ n-periodic sequence: $c_{ij} = p_{j-i}, 1 \le i,j \le n.$

4.2. Discrete Fourier Transform (DFT)

Experiment : EV of random circulant matrices

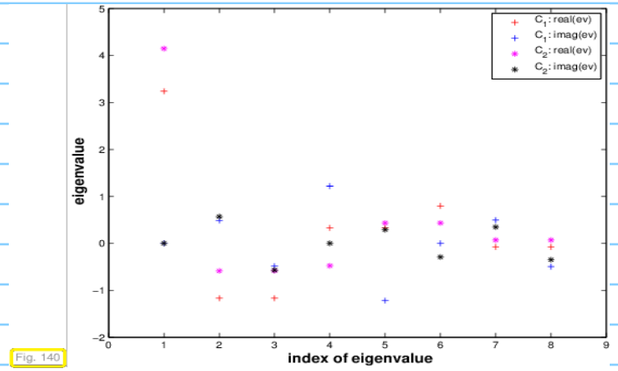


Fig. 140

[8×8 example]

↳ "random spectrum"

Reduction : discrete convolution ⇒ periodic d.c.

→ $\underline{z} = \underline{a} * \underline{b} = \underline{x} *_{2n-1} \underline{y}$

$\underline{a}, \underline{b} \in \mathbb{R}^n$

duration $2n-2$

Zero padding : $\underline{x} := \begin{bmatrix} \underline{a} \\ 0 \end{bmatrix} \in \mathbb{R}^{2n-1}$

$\underline{y} := \begin{bmatrix} \underline{b} \\ 0 \end{bmatrix} \in \mathbb{R}^{2n-1}$

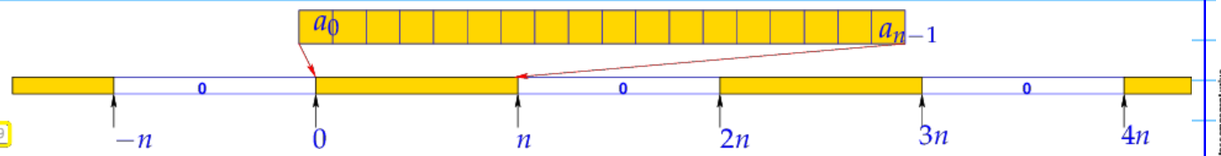
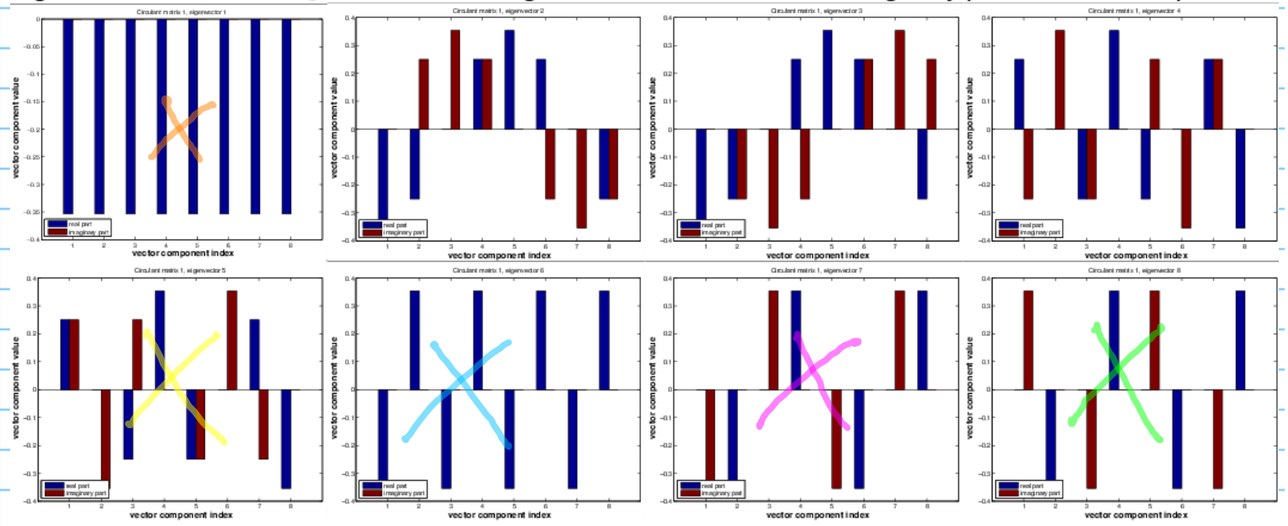
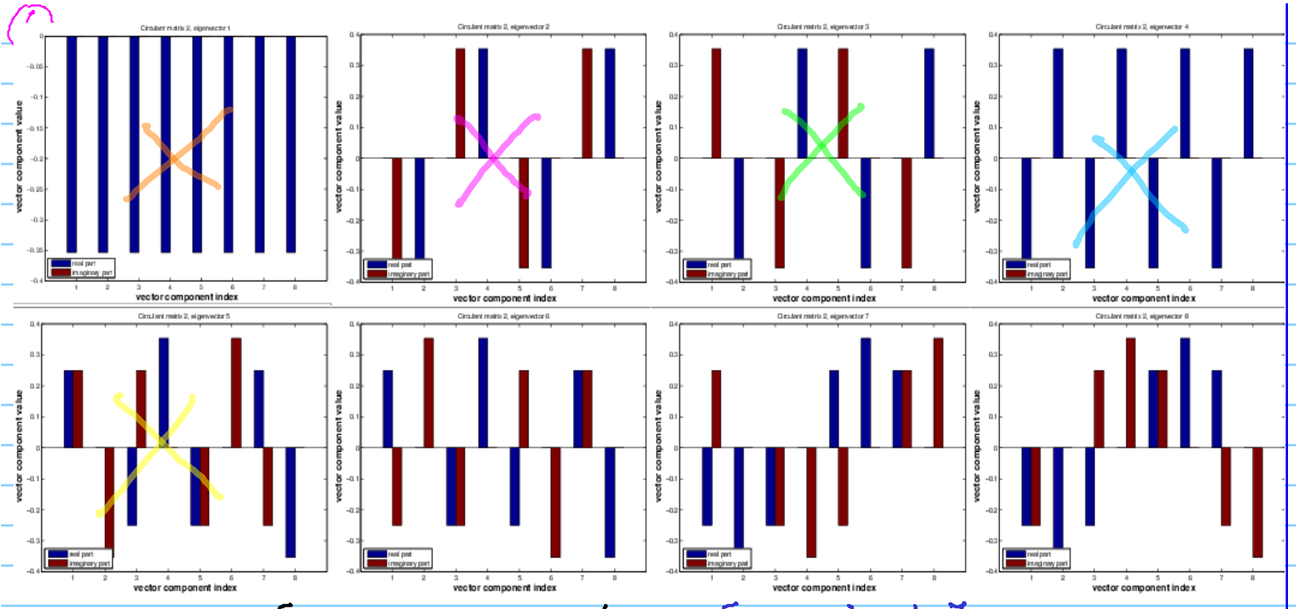


Fig. 139

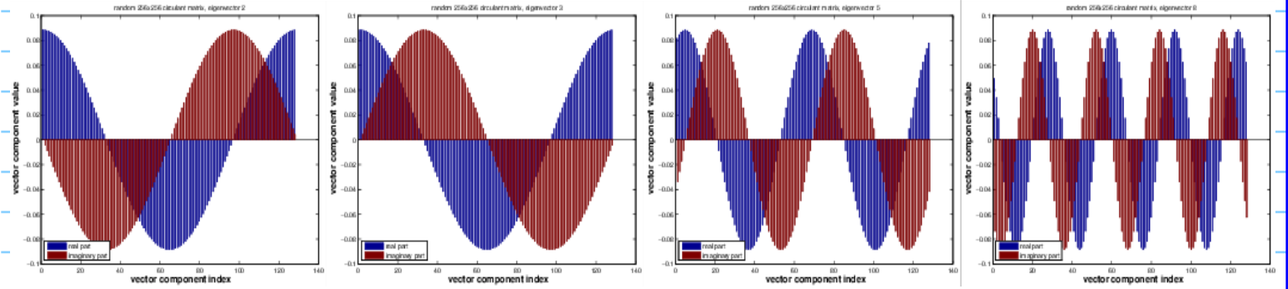
Eigenvectors of matrix $\mathbf{C}_1$, visualized through the size of the real and imaginary parts of their components.





▷ Same eigenvectors [complex!]

Eigenvectors of C = gallery('circul',(1:128)); :



$$\omega_n := e^{-\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) - i\sin\left(\frac{2\pi}{n}\right) \quad \text{[root unity]}$$

$$\omega_n^n = 1 \qquad (*)$$

$$\omega_n^{-\ell} = \overline{\omega}_n^{\ell} \quad \text{[complex conjugation]}$$

$$\omega_n^{jk} = (\omega_n^k)^j = (\omega_n^j)^k$$

$$\underline{v}_k = [\omega_n^{jk}]_{j=0}^{n-1} \in \mathbb{C}^n, \quad 0 \le k < n$$

$$C := \mathrm{circul}(\underline{u})$$

$$(C\underline{v}_k)_j = \sum_{\ell=0}^{n-1} u_\ell (\underline{v}_k)_{j-\ell} = (\underline{u} *_n \underline{v}_k)_j$$

$$= \sum_{\ell=0}^{n-1} u_\ell \, \omega_n^{k(j-\ell)}$$

$$= \omega_n^{kj} \underbrace{\sum_{\ell=0}^{n-1} u_\ell \, \omega_n^{-k\ell}}_{\text{eigenvalue } \lambda_k}$$

$\uparrow$ $(\underline{v}_k)_j$

▷ $C\underline{v}_k = \lambda_k \underline{v}_k$

Basis of eigenvectors for any circulant matrix

$$\{\mathbf{v}_0, \ldots, \mathbf{v}_{n-1}\} = \left\{ \begin{bmatrix} \omega_n^0 \\ \vdots \\ \omega_n^0 \end{bmatrix} \begin{bmatrix} \omega_n^0 \\ \omega_n^1 \\ \vdots \\ \omega_n^{n-1} \end{bmatrix} \cdots \begin{bmatrix} \omega_n^0 \\ \omega_n^{n-2} \\ \omega_n^{2(n-2)} \\ \vdots \\ \omega_n^{(n-1)(n-2)} \end{bmatrix} \begin{bmatrix} \omega_n^0 \\ \omega_n^{n-1} \\ \omega_n^{2(n-1)} \\ \vdots \\ \omega_n^{(n-1)^2} \end{bmatrix} \right\}. \qquad (4.2.11)$$

= trigonometric basis of $\mathbb{C}^n$

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$$\mathbf{F}_n = \begin{bmatrix} \omega_n^0 & \omega_n^0 & \cdots & \omega_n^0 \\ \omega_n^0 & \omega_n^1 & \cdots & \omega_n^{n-1} \\ \omega_n^0 & \omega_n^2 & \cdots & \omega_n^{2n-2} \\ \vdots & \vdots & & \vdots \\ \omega_n^0 & \omega_n^{n-1} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix} = \left[\omega_n^{lj}\right]_{l,j=0}^{n-1} \in \mathbb{C}^{n,n} . \qquad (4.2.13)$$

Fourier matrix

$$C F_n = F_n D \quad , D \hat{=} \text{diagonal}$$

$$(D)_{k,k} = \sum_{\ell=0}^{n-1} u_\ell \underbrace{\omega_n^{-k\ell}}_{(\overline{F}_n)_{k\ell}} \Rightarrow [(D)_{k,k}]_k = \overline{F}_n u$$

▷ Orthogonal eigenvector

$$F_n^H F_n = n \cdot I_n$$

Lemma 4.2.14. Properties of Fourier matrix

The scaled Fourier-matrix $\frac{1}{\sqrt{n}}\mathbf{F}_n$ is unitary (→ Def. 6.2.2) : $\mathbf{F}_n^{-1} = \frac{1}{n}\mathbf{F}_n^H = \frac{1}{n}\overline{\mathbf{F}}_n$.

Lemma 4.2.16. Diagonalization of circulant matrices (→ Def. 4.1.34)

For any circulant matrix $\mathbf{C} \in \mathbb{K}^{n,n}$, $c_{ij} = u_{i-j}$, $(u_k)_{k\in\mathbb{Z}}$ n-periodic sequence, holds true

$$\mathbf{C}\overline{\mathbf{F}}_n = \overline{\mathbf{F}}_n \operatorname{diag}(d_1,\ldots,d_n) \quad , \quad \mathbf{d} = \mathbf{F}_n[u_0,\ldots,u_{n-1}]^\top .$$

$$C = \frac{1}{n}\overline{F}_n \operatorname{diag}(d_1,..,d_n) F_n$$

Cx can be realized by multiplication with Fourier matrix

↳ periodic disc. conv.

The Fourier matrix :

$$V_k^H V_m = \sum_{\ell=0}^{n-1} \omega_n^{-k\ell} \omega_n^{m\ell} = \sum_{\ell=0}^{n-1} \omega^{\ell(m-k)} .$$

$V_k = [\omega_n^{jk}]_{j=0}^{n-1}$, $0 \le k, m < n$

$$= \sum_{\ell=0}^{n-1} \left[e^{\frac{2\pi i}{n}(m-k)}\right]^\ell = \begin{cases} n & m = k \\ \frac{1-\left[e^{\frac{2\pi i}{n}(m-k)}\right]^n}{1-e^{\frac{2\pi i}{n}(m-k)}} = 0 & m \neq k \end{cases} \quad (*)$$

$$\begin{bmatrix} \omega_n^0 & \omega_n^k & \cdots & \omega_n^{(n-1)k} \end{bmatrix} \cdot \begin{bmatrix} \omega_n^0 \\ \omega_n^m \\ \vdots \\ \omega_n^{(n-1)m} \end{bmatrix} = V_k^H \cdot V_m$$

Definition 4.2.18. Discrete Fourier transform (DFT)

The linear map $\mathcal{F}_n : \mathbb{C}^n \mapsto \mathbb{C}^n$, $\mathcal{F}_n(\mathbf{y}) := \mathbf{F}_n\mathbf{y}$, $\mathbf{y} \in \mathbb{C}^n$, is called discrete Fourier transform (DFT), i.e. for $\mathbf{c} := \mathcal{F}_n(\mathbf{y}) = \mathbf{F}_n\mathbf{y}$

$$c_k = \sum_{j=0}^{n-1} y_j \omega_n^{kj} \quad , \quad k = 0, \ldots, n-1 \, . \tag{4.2.19}$$

[Lemma 4.2.14]

$$\underline{c} = \mathbf{F}_n \underline{y} \iff \underline{y} = \frac{1}{n}\overline{\mathbf{F}}_n \underline{c}$$

Inverse DFT : $y_k = \frac{1}{n}\sum_{j=0}^{n-1} c_j \omega_n^{-kj}$

C++11 code 4.2.22: Demo: discrete Fourier transform in EIGEN ⇢ GITLAB

```
int main() {
  using Comp = complex<double>;
  const VectorXcd::Index n = 5;
  VectorXcd y(n),c(n),x(n);
  y << Comp(1,0),Comp(2,1),Comp(3,2),Comp(4,3),Comp(5,4);
  FFT<double> fft; // DFT transform object
  c = fft.fwd(y);  // DTF of y, see Def. 4.2.18
  x = fft.inv(c);  // inverse DFT of c, see (4.2.20)

  cout << "y = " << y.transpose() << endl
       << "c = " << c.transpose() << endl
       << "x = " << x.transpose() << endl;
  return 0;
}
```

4.2.1. Discrete Convolutions via DFT

- periodic conv. $\iff$ multiplication w/ circulant matrix

Conclusion (from $\overline{\mathbf{F}}_n = n\mathbf{F}_n^{-1}$): $\mathbf{C} = \mathbf{F}_n^{-1}\operatorname{diag}(d_1, \ldots, d_n)\mathbf{F}_n$. (4.2.17)

$C = \operatorname{circ}(u)$, $d := \mathbf{F}_n u$

C++11 code 4.2.25: Discrete periodic convolution: DFT implementation ⇢ GITLAB

```
VectorXcd pconvfft(const VectorXcd& u, const VectorXcd& x) {
  Eigen::FFT<double> fft;
  VectorXcd tmp = ( fft.fwd(u) ).cwiseProduct( fft.fwd(x) );
  return fft.inv(tmp);
}
```

- general disc. conv. : by zero padding

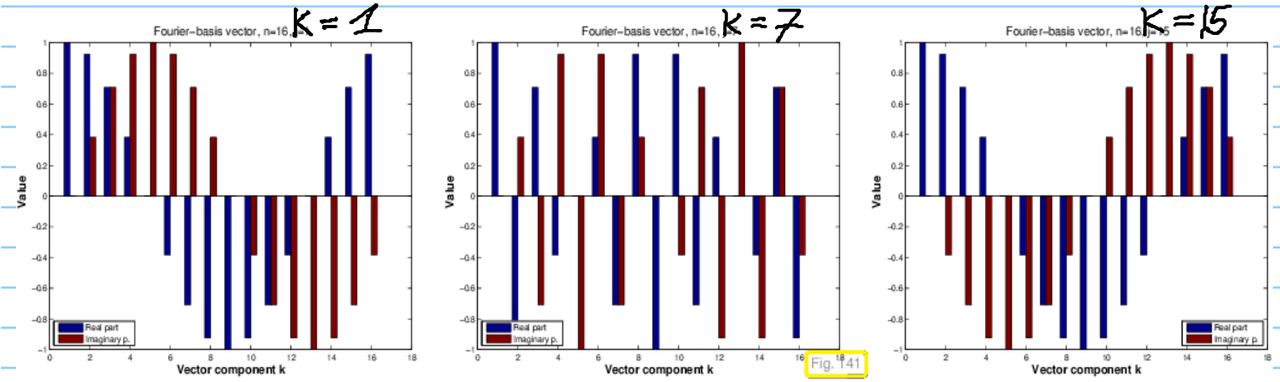
C++11 code 4.2.26: Implementation of discrete convolution (→ Def. 4.1.22) based on periodic discrete convolution ⇢ GITLAB

```
VectorXcd myconv(const VectorXcd& h, const VectorXcd& x) {
  const long n = h.size();
  // Zero padding, cf. (4.1.37)
  VectorXcd hp(2*n - 1), xp(2*n - 1);
  hp << h, VectorXcd::Zero(n - 1);
  xp << x, VectorXcd::Zero(n - 1);
  // Periodic discrete convolution of length 2n-1, ??
  return pconvfft(hp, xp);
}
```

4.2.2. Frequency Filtering via DFT

Trigonometric basis vector $\longleftrightarrow$ oscillation

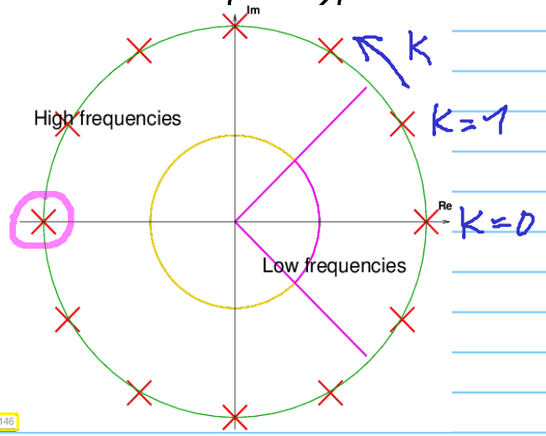
$\hookrightarrow$ $y_k = \left[\omega_n^{kj}\right]_{j=0}^{n-1} = \left[\cos\left(2\pi \frac{kj}{n}\right)\right]_{j=0}^{n-1} + i\left[\sin\left(\frac{2\pi}{n} kj\right)\right]_{j=0}^{n-1}$



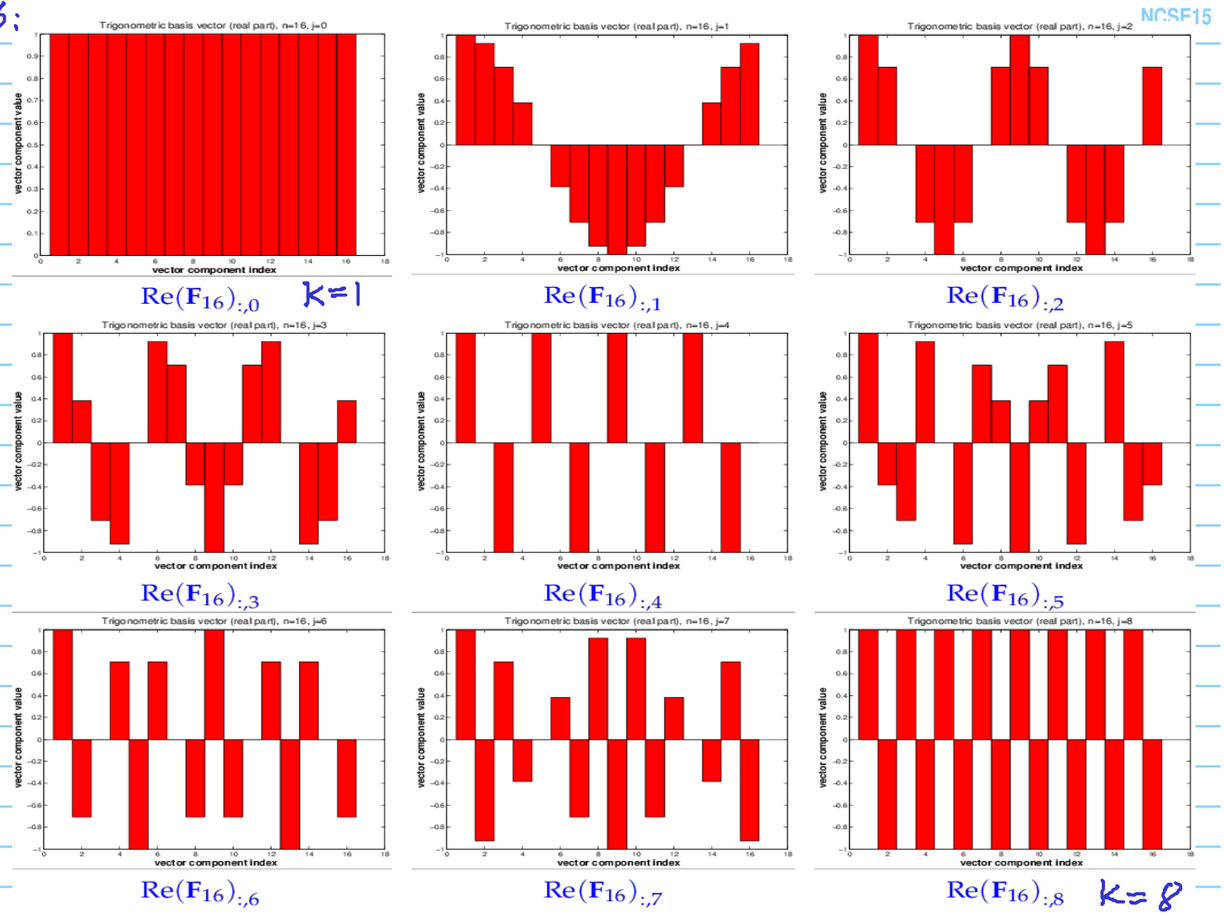
"slow oscillation/low frequency" "fast oscillation/high frequency" "slow oscillation/low frequency"

$n = 16$

DFT $\hat{=}$ Additive decomposition of a signal into frequency contribution



$n = 16$:



$\mathrm{Re}(\mathbf{F}_{16})_{:,0}$ $k=1$ $\mathrm{Re}(\mathbf{F}_{16})_{:,1}$ $\mathrm{Re}(\mathbf{F}_{16})_{:,2}$

$\mathrm{Re}(\mathbf{F}_{16})_{:,3}$ $\mathrm{Re}(\mathbf{F}_{16})_{:,4}$ $\mathrm{Re}(\mathbf{F}_{16})_{:,5}$

$\mathrm{Re}(\mathbf{F}_{16})_{:,6}$ $\mathrm{Re}(\mathbf{F}_{16})_{:,7}$ $\mathrm{Re}(\mathbf{F}_{16})_{:,8}$ $k=8$

C++11-code 4.2.33: DFT-based frequency filtering ➙ GITLAB

```
void freqfilter(const VectorXd& y, int k,
                VectorXd& low, VectorXd& high) {
  const VectorXd::Index n = y.size();
  if (n%2 != 0)
    throw std::runtime_error("Even vector length required!");
  const VectorXd::Index m = y.size()/2;

  Eigen::FFT<double> fft; // DFT helper object
  VectorXcd c = fft.fwd(y); // Perform DFT of input vector

  VectorXcd clow = c;
  // Set high frequency coefficients to zero, Fig. 146
  for (int j = -k; j <= +k; ++j) clow(m+j) = 0;
  // (Complementary) vector of high frequency coefficients
  VectorXcd chigh = c - clow;

  // Recover filtered time-domain signals
  low = fft.inv(clow).real();
  high = fft.inv(chigh).real();
}
```


Frequency identification

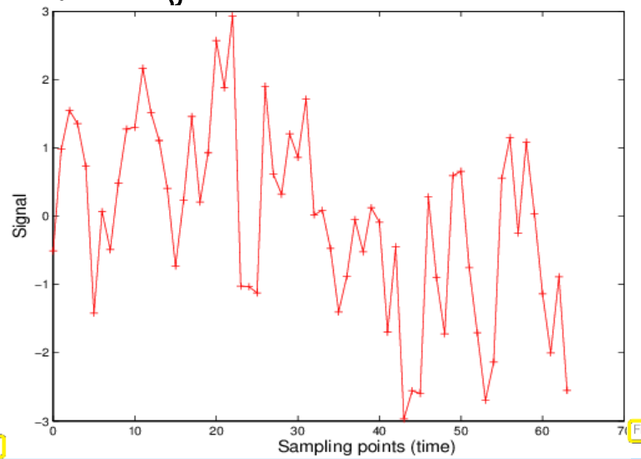


Fig. 142

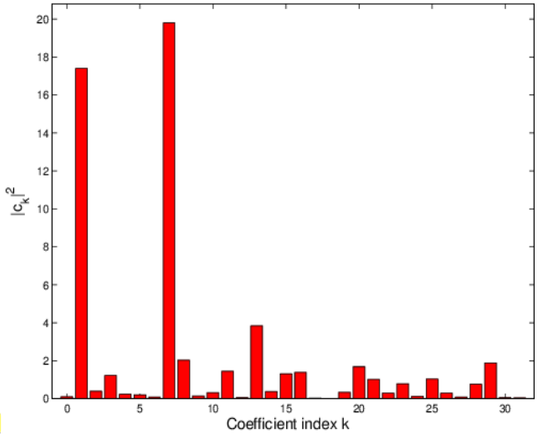


Fig. 143

Google: 'Vorlesungsverzeichnis'

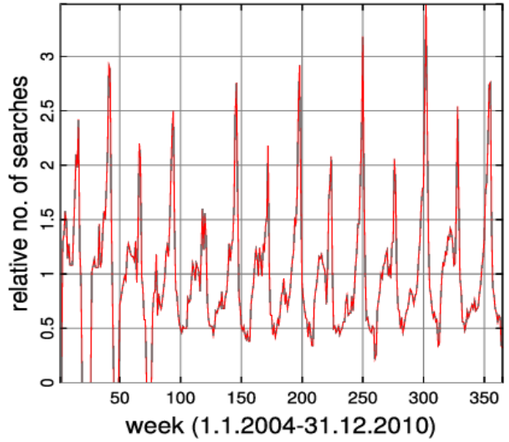


Fig. 144

Fourier spectrum

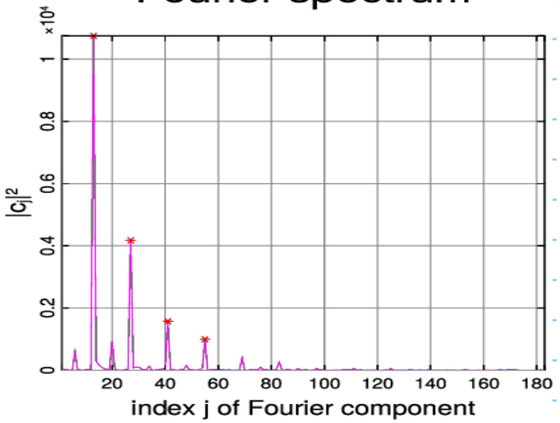


Fig. 145

4.2.4. 2D DFT

2D trigonometric basis of $\mathbb{C}^{m,n}$

$$\{V_{\nu,\mu}\} := \{\underbrace{[\omega_m^{\ell\nu}]_{\ell=0}^{m-1}\,([\omega_n^{k\mu}]_{k=0}^{n-1})^T}_{\in \mathbb{C}^{m,n}}\}_{\substack{\nu=0,\dots,m-1\\ \mu=0,\dots,n-1}}$$

Application: Image processing [Deblurring]

"pixel crosstalk" ↳ $P \in \mathbb{R}^{m,n}$

$$c_{\ell,j} = \sum_{k=-L}^{L}\sum_{q=-L}^{L} s_{k,q}\,(P)_{\ell+k,j+q}\,, \quad L \ll m,n$$

$s_{k,q}$: point spread function (PSF)

[periodic cont. of P assumed]

$$C = \mathcal{B}(P)$$

↳ linear blurring operator $\mathcal{B}: \mathbb{C}^{m,n} \to \mathbb{C}^{m,n}$

$$\Big(\mathcal{B}\big(\underbrace{\omega_m^{\nu k}\omega_n^{\mu q}}_{V_{\mu\nu}}\big)_{k,q\in\mathbb{Z}}\Big)_{l,j} = \sum_{k=-L}^{L}\sum_{q=-L}^{L} s_{k,q}\,\omega_m^{\nu(l+k)}\omega_n^{\mu(j+q)} = \underbrace{\omega_m^{\nu l}\omega_n^{\mu j}}_{=(V_{\mu\nu})_{l,j}}\underbrace{\sum_{k=-L}^{L}\sum_{q=-L}^{L} s_{k,q}\,\omega_m^{\nu k}\omega_n^{\mu q}}_{\text{eigenvalue } \lambda_{\mu\nu}}.$$

$V_{\mu\nu}$ ← eigenvector

▷ $\mathcal{B}(V_{\mu\nu}) = \lambda_{\mu\nu} V_{\mu\nu}$

Expand P into 2D trigonometric basis

$$(P)_{\ell,k} = \underbrace{\sum_{\nu=0}^{m-1}\sum_{\mu=0}^{n-1} (Y)_{\nu,\mu}\,\omega_m^{\ell\nu}\omega_n^{\mu k}}_{\text{2x DFT}}$$

$$(C)_{k_1,k_2} = \sum_{j_1=0}^{m-1}\sum_{j_2=0}^{n-1} y_{j_1,j_2}\,\omega_m^{j_1k_1}\omega_n^{j_2k_2} = \sum_{j_1=0}^{m-1}\omega_m^{j_1k_1}\left(\sum_{j_2=0}^{n-1}\omega_n^{j_2k_2}y_{j_1,j_2}\right), \quad 0\le k_1<m,\ 0\le k_2<n.$$

$$(C)_{k_1,k_2} = \sum_{j_1=0}^{m-1}\big(\mathbf{F}_n(\mathbf{Y})_{j_1,:}\big)_{k_2}\omega_m^{j_1k_1} \quad\blacktriangleright\quad \boxed{\mathbf{C} = \mathbf{F}_m(\mathbf{F}_n\mathbf{Y}^\top)^\top = \mathbf{F}_m\mathbf{Y}\mathbf{F}_n}\,. \tag{4.2.46}$$

$$\mathbf{C} = \sum_{j_1=0}^{m-1}\sum_{j_2=0}^{n-1} y_{j_1,j_2}(\mathbf{F}_m)_{:,j_1}(\mathbf{F}_n)_{:,j_2}^\top \;\Rightarrow\; \boxed{\mathbf{Y} = \mathbf{F}_m^{-1}\mathbf{C}\mathbf{F}_n^{-1} = \frac{1}{mn}\overline{\mathbf{F}}_m\mathbf{C}\overline{\mathbf{F}}_n}\,. \tag{4.2.47}$$

Fourier basis coefficients ↑ ↑ pixel image

C++11 code 4.2.48: Two-dimensional discrete Fourier transform ↦ GITLAB

```
template <typename Scalar>
void fft2(Eigen::MatrixXcd &C, const Eigen::MatrixBase<Scalar> &Y) {
  using idx_t = Eigen::MatrixXcd::Index;
  const idx_t m = Y.rows(),n=Y.cols();
  C.resize(m,n);
  Eigen::MatrixXcd tmp(m,n);

  Eigen::FFT<double> fft; // Helper class for DFT
  // Transform rows of matrix Y
  for (idx_t k=0;k<m;k++) {
    Eigen::VectorXcd tv(Y.row(k));
    tmp.row(k) = fft.fwd(tv).transpose();
  }

  // Transform columns of temporary matrix
  for (idx_t k=0;k<n;k++) {
    Eigen::VectorXcd tv(tmp.col(k));
    C.col(k) = fft.fwd(tv);
  }
}
```

↳ DFT-based deblurring: $\Leftrightarrow$ $\mathcal{B}^{-1}$

(i) fft2: $\mathcal{P} \to Y$

(ii) Y. cwise Quotient $([\lambda_{pq}])$

(iii) inverse fft2: (4.2.47)

Estimating PSF:

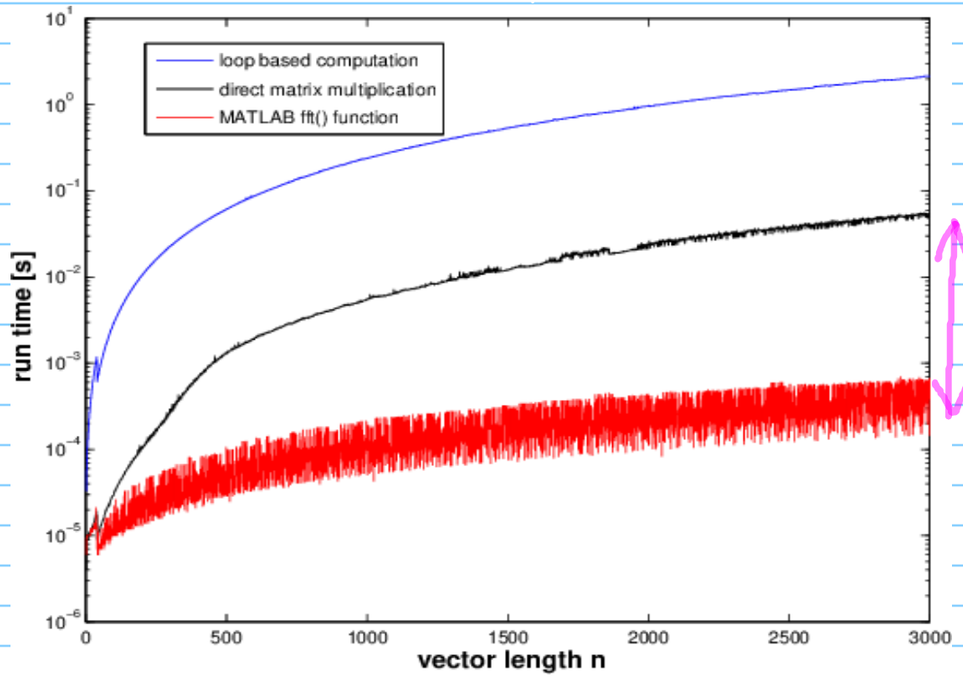
$P_i \hat{=}$ test images

$C_i \hat{=}$ blurred images $\quad i = 1, \ldots, m$

$$[S_{k,q}] := \operatorname*{argmin}_{[r_{k,q}]} \sum_{i=1}^{m} \| \mathcal{B}([r_{k,q}]; P_i) - C_i \|_F^2$$

4.3. Fast Fourier Transform (FFT)

Naive DFT: $\underline{c} = F_n \underline{y}$: cost $= O(n^2)$



DFT for $n = 2m$: [Gauss 1805]

$$c_k = \sum_{j=0}^{n-1} y_j \omega_n^{kj} \qquad [\omega^{kj} = e^{\frac{2\pi i}{n} kj}]$$

$$= \sum_{j=0}^{m-1} y_{2j} \omega_m^{kj} + \sum_{j=0}^{m-1} y_{2j+1} \omega_{2m}^{k(2j+1)}$$

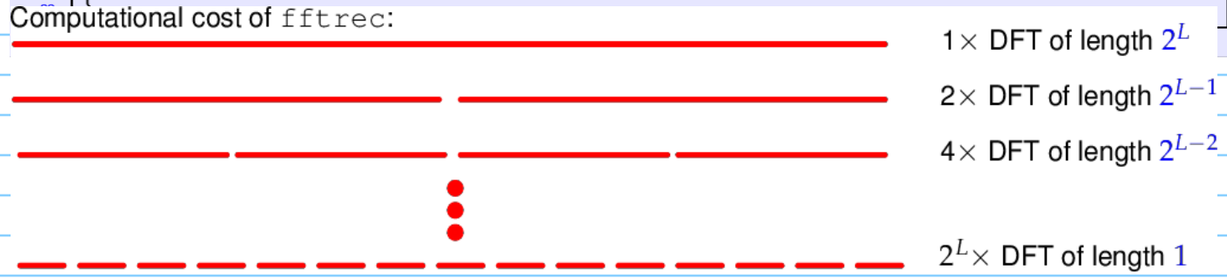
$$= \underbrace{\sum_{j=0}^{m-1} y_{2j} \omega_m^{kj}}_{\text{DFT of length } m} + \omega_n^k \underbrace{\sum_{j=0}^{m-1} y_{2j+1} \omega_m^{kj}}_{\text{DFT of length } m}$$

$n = 2^L \Rightarrow$ recursion

C++11 code 4.3.5: Recursive FFT → GITLAB

```
// Recursive DFT for vectors of length n = 2^L
VectorXcd fftrec(const VectorXcd& y) {
  using idx_t = VectorXcd::Index;
  using comp = std::complex<double>;
  // Nothing to do for DFT of length 1
  const idx_t n = y.size();
  if (n == 1) return y;
  if (n % 2 != 0) throw std::runtime_error("size(y) must be even!");
  const Eigen::Map<const
      Eigen::Matrix<comp,Eigen::Dynamic,Eigen::Dynamic,Eigen::RowMajor>>
    Y(y.data(),n/2,2); // for selecting even and off compoents
  const VectorXcd c1 = fftrec(Y.col(0)),c2 = fftrec(Y.col(1));
  const comp omega = std::exp(-2*M_PI/n*comp(0,1)); // root of unity
                                                    ω_n
  comp s(1.0,0.0);
  VectorXcd c(n);
  // Scaling of DFT of odd components plus periodic copying
  for (long k = 0; k < n; ++k) {
    c(k) = c1(k%(n/2)) + c2(k%(n/2))*s;
    s *= omega;
  }
  return c;
}
```

Computational cost of `fftrec`:



▷ Cost $= O(Ln) = O(\log_2 n \cdot n)$

$\ll O(n^2)$ for $n \gg 1$

Rationale for reduction to DFT !

Asymptotic complexity of discrete periodic convolution, see Code 4.2.25:

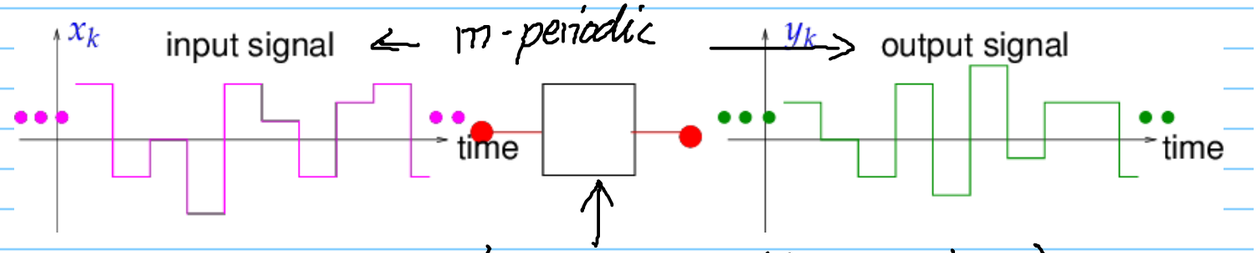
Cost(`pconvfft(u,x)`, $\mathbf{u},\mathbf{x} \in \mathbb{C}^n$) = $O(n \log n)$.

Asymptotic complexity of discrete convolution, see Code 4.2.26:

Cost(`myconv(h,x)`, $\mathbf{h},\mathbf{x} \in \mathbb{C}^n$) = $O(n \log n)$.

4.5. Toeplitz Matrix Technique

Ex: Parameter estimation for LT-FIR channel



Impulse response $(h_0, \ldots, h_{n-1})$, $m \geq n$

$\underline{y} = \underline{h} *_m \underline{x} \hat{=}$ m equ.

↳ overdetermined LSE for $\underline{h}$

$\Leftrightarrow$ least squares problem

$$\|\mathbf{Ah} - \mathbf{y}\|_2 = \left\| \begin{bmatrix} x_0 & x_{-1} & \cdots & & \cdots & x_{1-n} \\ x_1 & x_0 & x_{-1} & & & \vdots \\ \vdots & x_1 & x_0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & x_{-1} \\ x_{n-1} & & & & x_1 & x_0 \\ x_n & x_{n-1} & & & & x_1 \\ \vdots & & & & & \vdots \\ x_{m-1} & \cdots & & & \cdots & x_{m-n} \end{bmatrix} \begin{bmatrix} h_0 \\ \vdots \\ \underline{x} \\ \vdots \\ h_{n-1} \end{bmatrix} - \begin{bmatrix} y_0 \\ \vdots \\ \underline{b} \\ \vdots \\ y_{m-1} \end{bmatrix} \right\|_2 \to \min .$$

A

$$(A)_{ij} = x_{i-j}$$

Normal equ.: $A^T A x = A^T b$

$$(A^T A)_{ij} = \sum_{k=0}^{m-1} x_{i-k}\, x_{j-k} =: z_{i-j} : (z_j)$$

for m-periodic

sum over one period → index shifting

Toeplitz matrix ("data space")

Definition 4.5.8. Toeplitz matrix

$\mathbf{T} = (t_{ij})_{i,j=1}^{n} \in \mathbb{K}^{m,n}$ is a Toeplitz matrix, if there is a vector $\mathbf{u} = [u_{-m+1}, \ldots, u_{n-1}] \in \mathbb{K}^{m+n-1}$ such that $t_{ij} = u_{j-i}$, $1 \le i \le m$, $1 \le j \le n$.

$$\mathbf{T} = \begin{bmatrix} u_0 & u_1 & \cdots & & \cdots & u_{n-1} \\ u_{-1} & u_0 & u_1 & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & u_1 \\ u_{1-m} & \cdots & & \cdots & u_{-1} & u_0 \end{bmatrix}$$

Efficient alg. for $y = Tx$

Idea [m = n]:

$$\mathbf{C} = \left[\begin{array}{cccccc|ccccc} u_0 & u_1 & \cdots & & \cdots & u_{n-1} & 0 & u_{1-n} & \cdots & & u_{-1} \\ u_{-1} & u_0 & u_1 & & & \vdots & u_{n-1} & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & & & & & \ddots & \\ \vdots & & & \ddots & \ddots & u_1 & \vdots & & \ddots & \ddots & u_{1-n} \\ u_{1-n} & \cdots & & \cdots & u_{-1} & u_0 & u_1 & & & u_{n-1} & 0 \\ \hline 0 & u_{1-n} & \cdots & & \cdots & u_{-1} & u_0 & u_1 & \cdots & & u_{n-1} \\ u_{n-1} & 0 & \ddots & & & \vdots & u_{-1} & u_0 & u_1 & & \vdots \\ \vdots & \ddots & \ddots & & & & \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & & & \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & u_{1-n} & & \vdots & & & \ddots & u_1 \\ u_1 & & & u_{n-1} & 0 & & u_{1-n} & \cdots & & u_{-1} & u_0 \end{array}\right]$$

A circulant matrix: $C = \begin{bmatrix} T & * \\ * & * \end{bmatrix}$

$$C \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Tx \\ * \end{bmatrix}$$

$\triangleright$ Cost(Tx) = Cost(Cx) = $O(n \log n)$