ETH Lecture 401-0663-00L Numerical Methods for CSE

Numerical Methods for Computational Science and Engineering

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URL: http://www.sam.math.ethz.ch/~hiptmair/tmp/NumCSE/NumCSE16.pdf

VI. Approximation of Functions in 1D

Case study: Model reduction in circuit simulation

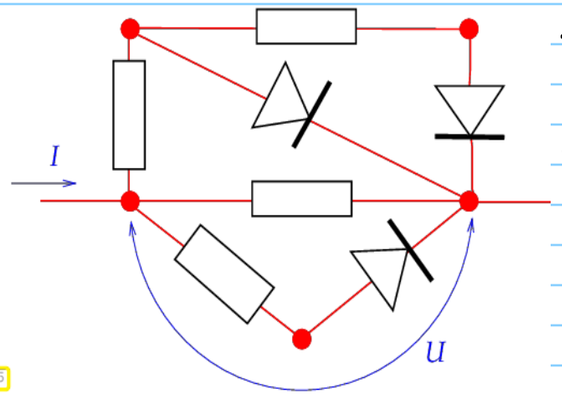


Fig. 205

◁ 2-port (non-linear) circuit element

$$I = I(U)$$

↳ available for any U through solving a system of equations (expensive!)

Goal: replace $I = I(U)$ with a function $U \to \tilde{I}(U)$ that allows fast point evaluation (also of derivative)

Approximation of functions: Generic view

Given: function $\mathbf{f} : D \subset \mathbb{R}^n \mapsto \mathbb{R}^d$ (often in procedural form **double** f(**double**), Rem. 5.1.4)

Goal: Find a "SIMPLE" function $\tilde{\mathbf{f}} : D \mapsto \mathbb{R}^d$ such that the approximation error $\mathbf{f} - \tilde{\mathbf{f}}$ is "SMALL"

a function

measured w.r.t. some norm

Here: $n = 1$

e.g. polynomials

- Supremum norm: $\|f\|_\infty = \max_{t \in D} |f(t)|$
- L^2-norm: $\|f\|_2^2 := \int_D |f(t)|^2 \, dt$

Policy: Approximation by interpolation

Mapping functions → functions

Interpolation scheme + sampling → approximation scheme : $A : C^0(I) \to V$

$$f : I \subset \mathbb{R} \to \mathbb{K} \xrightarrow{\text{sampling}} \overbrace{(t_i, y_i := f(t_i))_{i=0}^m}^{\text{data points}} \xrightarrow{\text{interpolation}} \tilde{f} := \mathsf{I}_T \mathbf{y} \quad (\tilde{f}(t_i) = y_i) .$$

free choice of nodes $t_i \in I$

New freedom: choice of nodes

②

6.1. Approximation by global Polynomials

Known: Approximation by Taylor polynomial

$$f \in C^{k+1} : f(t) \approx \underbrace{f(t_0) + f'(t_0)(t-t_0) + f''(t_0)\tfrac{1}{2}(t-t_0)^2 + \dots + f^{(k)}(t_0)\tfrac{1}{k!}(t-t_0)^k}_{\in \mathcal{P}_k}$$

smoothness requirement

4.1.1. Polynomial Approximation: Theory

① Uniform approximation = approx. in $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(I)}$

Theorem 6.1.6. Uniform approximation by polynomials

For $f \in C^0([0,1])$, define the n-th Bernstein approximant as — Bernstein polynomials

$$p_n(t) = \sum_{j=0}^{n} f(j/n)\binom{n}{j} t^j (1-t)^{n-j}, \quad p_n \in \mathcal{P}_n . \tag{6.1.7}$$

It satisfies $\|f - p_n\|_\infty \to 0$ for $n \to \infty$. If $f \in C^m([0,1])$, then even $\left\|f^{(k)} - p_n^{(k)}\right\|_\infty \to 0$ for $n \to \infty$ and all $0 \le k \le m$.

$\to$ $p_n \to f$ converges "slowly"

② Quantitative uniform best approximation

Theorem 6.1.15. L^∞ polynomial best approximation estimate

If $f \in C^r([-1,1])$ (r times continuously differentiable), $r \in \mathbb{N}$, then, for any polynomial degree $n \ge r$,

$$\underbrace{\inf_{p \in \mathcal{P}_n} \|f - p\|_{L^\infty([-1,1])}}_{} \le (1 + \pi^2/2)^r \frac{(n-r)!}{n!} \left\|f^{(r)}\right\|_{L^\infty([-1,1])} .$$

(norm of) best approximation error — smoothness requ.

- Asymptotics for $n \to \infty$: $O(n^{-r})$ *

* = algebraic convergence (in degree) [with rate r]

Example for asymptotic behavior of family $\{A_n\}$ of approximation schemes:

How does $\|f - A_n f\|$ behave as a function of $n \to \infty$

- Note: Smoothness of f governs rate r of alg. cvg.

$$f \in C^\infty \Rightarrow \forall r \in \mathbb{N} : \exists C = C(r)$$

$$\inf_{p \in \mathcal{P}_n} \|f - p\| \le C(r)\, n^{-r}$$

$\to \infty$ as $r \to \infty$ possible

③

Transformation to arbitrary intervals $[a,b]$:

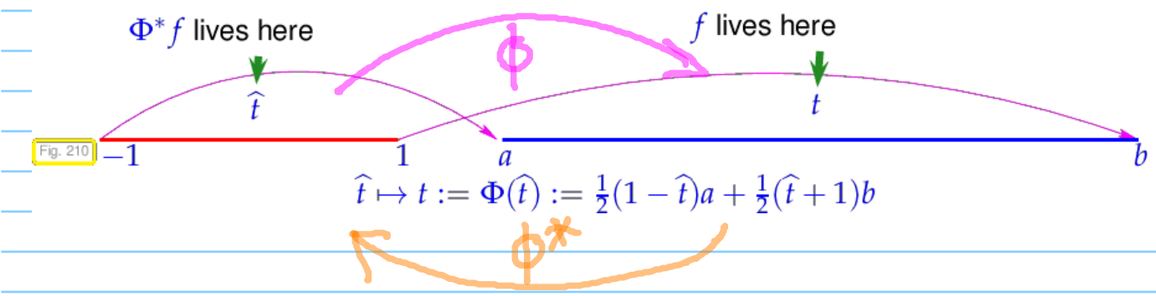
Tool: affine transformation $\Phi : [-1,1] \longrightarrow [a,b]$

$$\Phi(\hat{t}) = a + \tfrac{1}{2}(b-a)(\hat{t}+1) \quad , -1 \le \hat{t} \le 1$$

Pullback of functions: $(\Phi^* f)(\hat{t}) := f(\Phi(\hat{t}))$

for $f : [a,b] \longrightarrow \mathbb{R}$

$$\boxed{\Phi^* : C^0([a,b]) \to C^0([-1,1]) \quad , \quad \Phi^*(f)(\hat{t}) := f(\Phi(\hat{t})) \, , \quad -1 \le \hat{t} \le 1} \, . \tag{6.1.20}$$



$$\boxed{\Phi^* f := f \circ \Phi}$$

- $\Phi^* : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ bijective
- $\|\Phi^* f\|_{\infty,[-1,1]} = \|f\|_{\infty,[a,b]}$
- "$\Phi^* \leftrightarrow$ stretching/compressing"

$$(\Phi^* f)' = \Phi^*(f') \circ \Phi' = \tfrac{1}{2}(b-a)\, \Phi^*(f')$$

[chain rule]

$$(\Phi^* f)^{(r)} = \left(\frac{b-a}{2}\right)^r \Phi^*(f^{(r)}) \, . \tag{6.1.28}$$

Lemma 6.1.24 ▶

$$\left\|(\Phi^* f)^{(r)}\right\|_{L^\infty([-1,1])} = \left(\frac{b-a}{2}\right)^r \left\|f^{(r)}\right\|_{L^\infty([a,b])} \, , \quad f \in C^r([a,b]), r \in \mathbb{N}_0 \, . \tag{6.1.29}$$

$$\inf_{p \in \mathcal{P}_n} \|f - p\|_{\infty,[a,b]} = \inf_{p \in \mathcal{P}_n} \|\Phi^*(f-p)\|_{\infty,[-1,1]}$$

$[\Phi^*(\mathcal{P}_n) = \mathcal{P}_n]$ $$= \inf_{p \in \mathcal{P}_n} \|\Phi^* f - \cancel{\Phi^*} p\|_{\infty,[-1,1]}$$

[Thm 6.1.15] $$\le C n^{-r} \|(\Phi^* f)^{(r)}\|_{\infty,[-1,1]}$$

$$\le C n^{-r} \left(\frac{b-a}{2}\right)^r \|f^{(r)}\|_{\infty,[a,b]}$$

depends on length of interval

6.1.2. Error estimates for Polynomial Interpolation

Definition 6.1.32. Lagrangian (interpolation polynomial) approximation scheme

Given an interval $I \subset \mathbb{R}$, $n \in \mathbb{N}$, a node set $\mathcal{T} = \{t_0, \dots, t_n\} \subset I$, the Lagrangian (interpolation polynomial) approximation scheme $\mathsf{L}_{\mathcal{T}} : C^0(I) \to \mathcal{P}_n$ is defined by

$$\mathsf{L}_{\mathcal{T}}(f) := \mathsf{I}_{\mathcal{T}}(\mathbf{y}) \in \mathcal{P}_n \quad \text{with} \quad \mathbf{y} := (f(t_0), \dots, f(t_n))^T \in \mathbb{K}^{n+1}.$$

New: Consider families $\mathcal{T}_n = \{t_0^{(n)}, \dots, t_n^{(n)}\}$ of nodes, e.g.

equidistant nodes $t_j^{(n)} = a + \frac{b-a}{n} j$, $j = 0, \dots, n$

$\Rightarrow$ family of polynomial approx. schemes $\{A_n\}_{n \in \mathbb{N}}$

We study $\|f - A_n f\| \leq T(n)$ for $n \to \infty$

Ex:

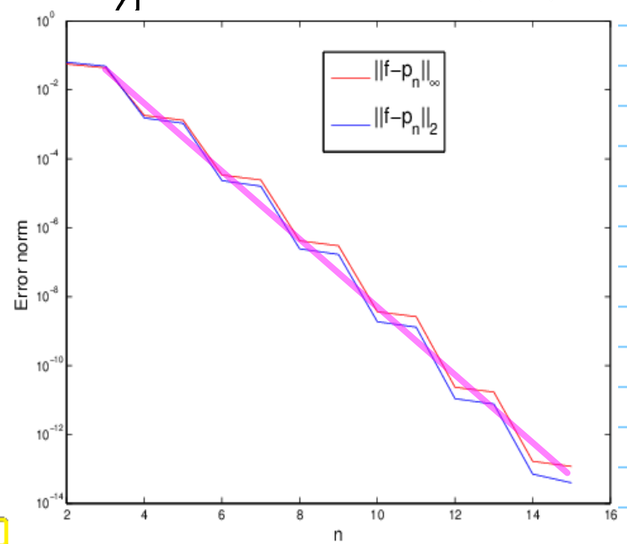


Fig. 211

$\triangleleft$ $f(t) = \sin(t)$

$I = [0, \pi]$

equid. nodes

[lin-log plot!]

$\varepsilon_n := \|f - \mathsf{L}_{\mathcal{T}_n} f\|$

Observed: $\log \varepsilon_n \approx C - Kn$, $K > 0$

$\varepsilon_n \approx e^C (e^{-K})^n$

$\Leftrightarrow$ $\varepsilon_n = O(q^n)$ for $n \to \infty$ for some $0 \leq q < 1$

$\hookrightarrow$ exponential cvg. (in degree n)

Algebraic convergence: $\|f - \mathsf{I}_{\mathcal{T}} f\| = O(n^{-p})$ (rate)

Exponential convergence: $\|f - \mathsf{I}_{\mathcal{T}} f\| = O(q^n)$

for $n \to \infty$ ("asymptotic!")

Ex: $f(t) = \frac{1}{1+t^2}$, $t \in \mathbb{R}$, $I = [-5, 5]$

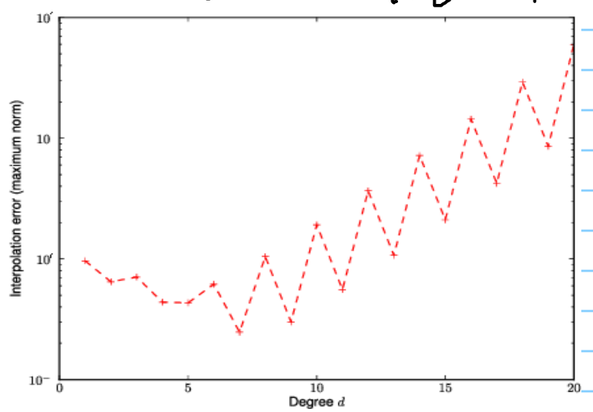


Fig. 213

"Equidistant interpolation"

Approximate $\|f - \mathsf{L}_{\mathcal{T}_n} f\|_\infty$ on $[-5, 5]$

5

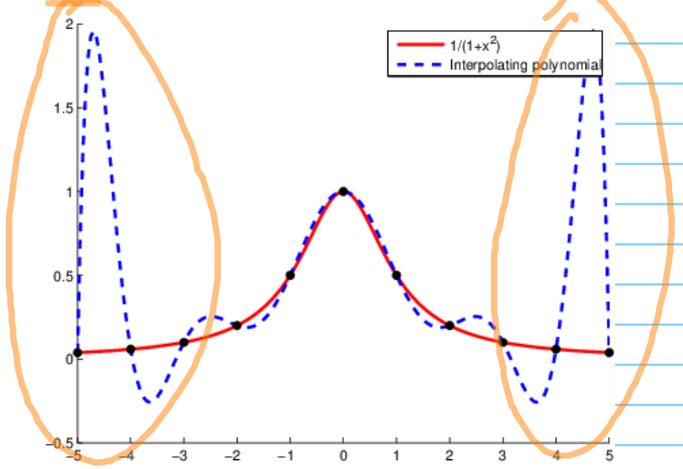


Fig. 212

Interpolating polynomial, $n = 10$

Theorem 6.1.44. Representation of interpolation error [?, Thm. 8.22], [?, Thm. 37.4]

We consider $f \in C^{n+1}(I)$ and the Lagrangian interpolation approximation scheme (→ Def. 6.1.32) for a node set $\mathcal{T} := \{t_0, \dots, t_n\} \subset I$. Then,

for every $t \in I$ there exists a $\tau_t \in]\min\{t, t_0, \dots, t_n\}, \max\{t, t_0, \dots, t_n\}[$ such that

smoothness requirement

$$\underbrace{f(t) - \mathsf{L}_{\mathcal{T}}(f)(t)}_{\text{interpolation error in } t} = \frac{f^{(n+1)}(\tau_t)}{(n+1)!} \cdot \prod_{j=0}^{n}(t - t_j) \,. \qquad (6.1.45)$$

$n \hat{=}$ degree

$=: w(t)$ [nodal polynomial]

Proof: fix $t \in I \setminus \mathcal{T}$

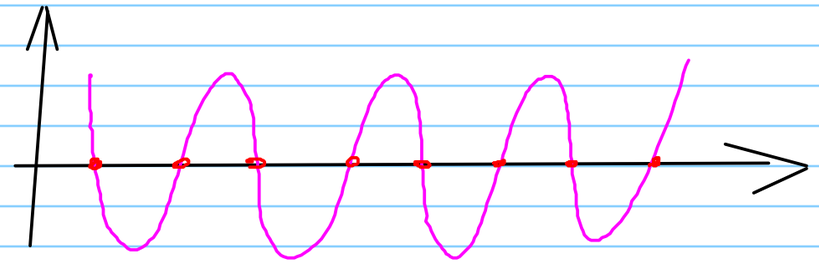
choose $c \in \mathbb{R}$ s.t. $f(t) - \mathsf{L}_{\mathcal{T}} f(t) - c\, w(t) = 0$

$$\varphi(x) := f(x) - \underbrace{\mathsf{L}_{\mathcal{T}} f(x)}_{\in \mathcal{P}_n} - c\, w(x) \in C^{n+1}$$

$\varphi(t_j) = 0$ by interpolation conditions

$\varphi(t) = 0$ by choice of c

↳ has $n+2$ distinct zeros



→ φ' has $n+1$ zeros

φ'' has n zeros

⋮

$\varphi^{(n+1)}$ has at least 1 zero τ_t

$\varphi^{(n+1)}(x) = f^{(n+1)}(x) - 0 - c\,(n+1)!$

$\varphi^{(n+1)}(\tau_t) = 0$ gives the formula

$$c = \frac{f^{(n+1)}(\tau_t)}{(n+1)!}$$

□

6

Take sup over $t \in [t_0, t_n]$ in (6.1.45)

Thm. 6.1.44 $\Rightarrow$ $$\boxed{\|f - \mathsf{L}_T f\|_{L^\infty(I)} \le \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \max_{t\in I} |(t-t_0)\cdot\cdots\cdot(t-t_n)|} \quad . \qquad (6.1.50)$$

- $f(t) = \sin(t)$ $\triangleright$ $\|f^{(n)}\|_\infty \le 1 \quad \forall n \in \mathbb{N}$

  $f \in C^\infty$

$\left\|f^{(k)}\right\|_{L^\infty(I)} \le 1, \quad \forall k \in \mathbb{N}_0$ $\Rightarrow$ $$\|f-p\|_{L^\infty(I)} \le \frac{1}{(1+n)!} \max_{t\in I} \underbrace{\left|(t-0)(t-\tfrac{\pi}{n})(t-\tfrac{2\pi}{n})\cdot\cdots\cdot(t-\pi)\right|}_{\text{Extremal for } t \approx \frac{\pi}{2n}} \le \frac{1}{n+1}\left(\frac{\pi}{n}\right)^{n+1} .$$

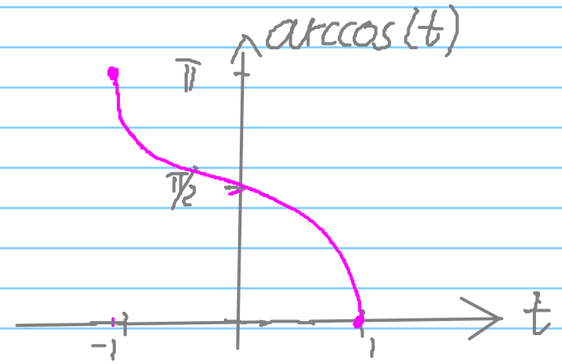
$I = [0,\pi]$

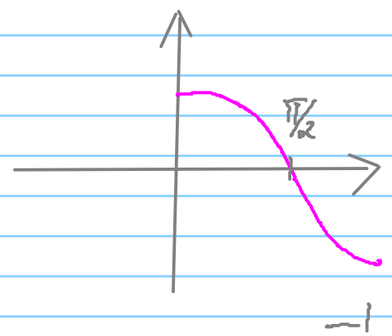
superexponential convergence

exp. cvg. $\|\text{error}\| = O(q^n)$

- $f(t) = \frac{1}{1+t^2}$ $\Rightarrow$ $\|f^{(n)}\|_\infty \sim 2^n n!$ for $n \to \infty$

$\to$ Exponential blow-up of bound in (6.1.50)





4.1.3. Chebychev Interpolation

Goal: n/f-independent a priori "optimal" choice of J

$\hookrightarrow$ beyond our control

Thm. 6.1.44 $\Rightarrow$ $$\boxed{\|f - \mathsf{L}_T f\|_{L^\infty(I)} \le \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \underbrace{\max_{t\in I} |(t-t_0)\cdot\cdots\cdot(t-t_n)|}_{= \|w\|_\infty}} \quad . \qquad (6.1.50)$$

$\leftarrow$ make this as small as possible

$w \in \mathcal{P}_{n+1}$ w/ leading coeff $= 1$

Optimal choice of interpolation nodes independent of interpoland

Idea: choose nodes $t_0, \dots, t_n$ such that $\|w\|_{L^\infty(I)}$ is minimal!

This is equivalent to finding a polynomial $q \in \mathcal{P}_{n+1}$

- with leading coefficient $= 1$,
- such that it minimizes the norm $\|q\|_{L^\infty(I)}$.

$\blacktriangleright$ Then choose nodes $t_0, \dots, t_n$ as zeros of q (caution: t_j must belong to I).

$\hookrightarrow$ fix q already!

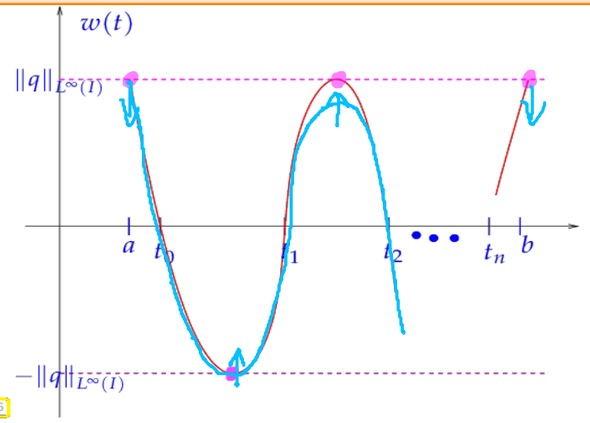


Fig. 215

Heuristic demands:

- t^* extremal point of q $\rightarrow$ $|q(t^*)| = \|q\|_{L^\infty(I)}$,
- q has $n+1$ zeros in I $(*)$,
- $|q(-1)| = |q(1)| = \|q\|_{L^\infty(I)}$.

- $|q| = \|q\|_\infty$ at all extrema

7 Solution polynomials:

Definition 6.1.75. Chebychev polynomials → [?, Ch. 32]

The nth Chebychev polynomial is $T_n(t) := \cos(n \arccos t)$, $\quad -1 \le t \le 1, n \in \mathbb{N}$.

▷ $|T_n(t)| \le 1 \quad \forall\, -1 \le t \le 1$

Theorem 6.1.76. 3-term recursion for Chebychev polynomials → [?, (32.2)]

The function T_n defined in Def. 6.1.75 satisfy the 3-term recursion

$$T_{n+1}(t) = 2t\,T_n(t) - T_{n-1}(t) \quad , \quad T_0 \equiv 1\,, \quad T_1(t) = t\,, \quad n \in \mathbb{N}\,. \tag{6.1.77}$$

▷ $T_n \in \mathcal{P}_n$ with leading coeff. 2^{n-1}

[$\cos(n+1)x = 2\cos nx \cdot \cos x - \cos(n-1)x$]

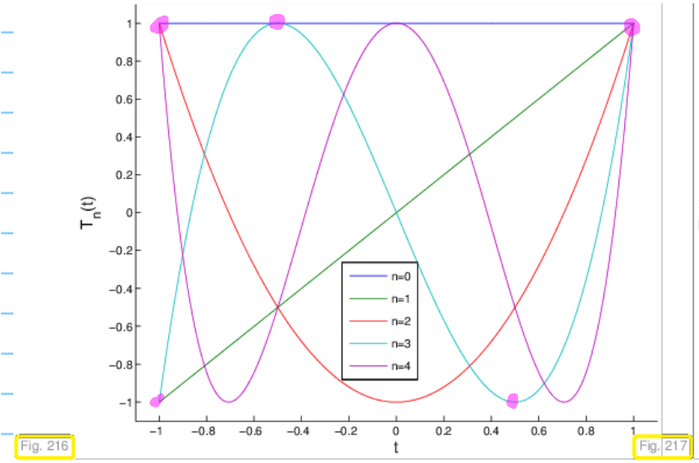


Fig. 216

Chebychev polynomials $T_0, \dots, T_4$

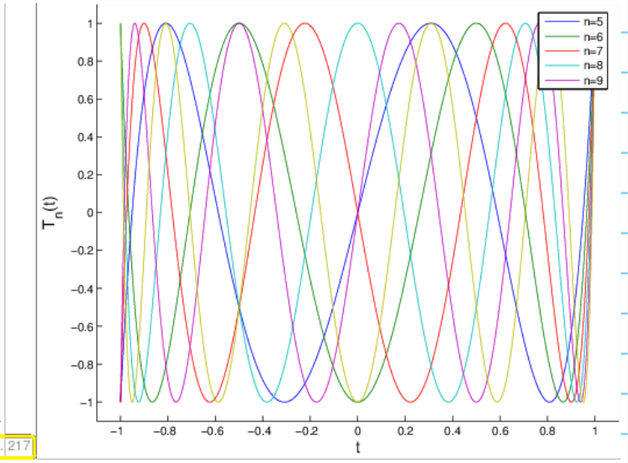


Fig. 217

Chebychev polynomials $T_5, \dots, T_9$

▷ Choose t_j s.t. $W := \frac{1}{2^n} T_{n+1}$

$\cos x = 0 \iff x \in (2\mathbb{N}+1)\pi/2$

▷ $t_j = \cos \frac{2j+1}{2n}\pi$ [Chebychev nodes]

$j = 0, \dots, n$

Optimal, because

Theorem 6.1.81. Minimax property of the Chebychev polynomials [?, Section 7.1.4.], [?, Thm. 32.2]

The polynomials T_n from Def. 6.1.75 minimize the supremum norm in the following sense:

$$\|T_n\|_{L^\infty([-1,1])} = \inf\{\|p\|_{L^\infty([-1,1])} : p \in \mathcal{P}_n, p(t) = 2^{n-1}t^n + \cdots\}\,, \quad \forall n \in \mathbb{N}\,.$$

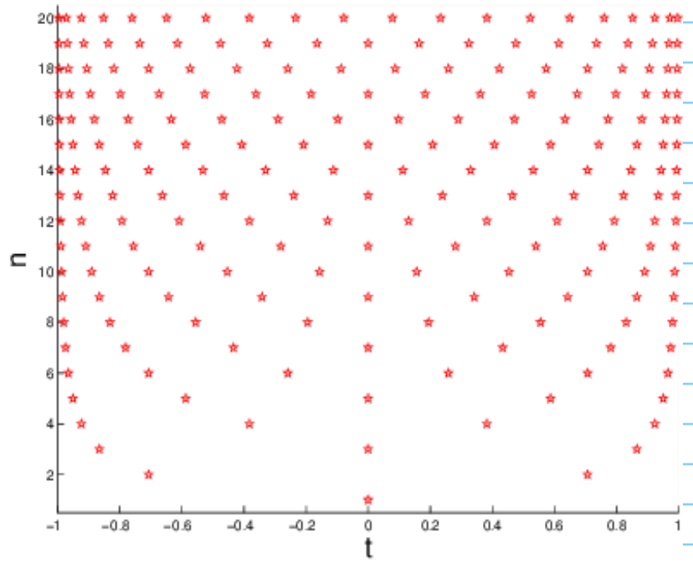


Fig. 218

△ Chebychev nodes cluster at endpoints

8 Example: $f(t) = \frac{1}{1+t^2}$ on $I = [-5,5]$

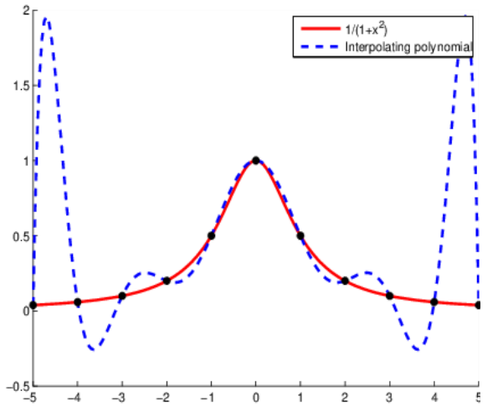


Fig. 220

Equidistant nodes

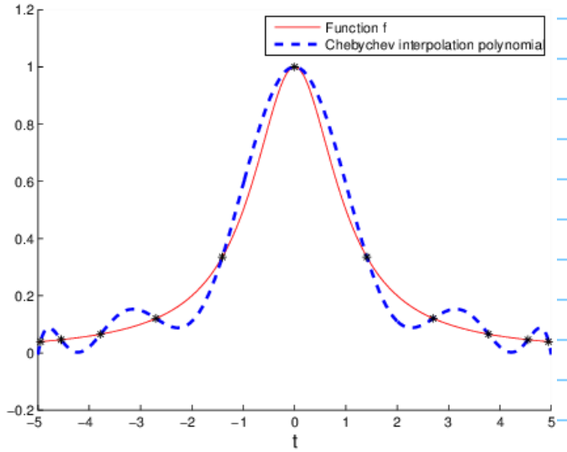


Fig. 221

Chebychev nodes

On $[a,b]$: transform nodes by $\hat{t} \to a + \frac{1}{2}(b-a)(\hat{t}+1)$

The Chebychev nodes in the interval $I = [a,b]$ are

$$t_k := a + \tfrac{1}{2}(b-a)\left(\cos\left(\frac{2k+1}{2(n+1)}\pi\right)+1\right), \quad k = 0,\dots,n. \tag{6.1.86}$$

For Chebychev interpolation

$$w = \frac{1}{2^n} T_n$$

$$\triangleright \ \|w\|_\infty = \frac{1}{2^n}$$

$$\triangleright \ \|f - I_T(f)\|_{L^\infty(I)} = \left\|\hat{f} - I_{\hat{T}}(\hat{f})\right\|_{L^\infty([-1,1])} \le \frac{2^{-n}}{(n+1)!}\left\|\frac{d^{n+1}\hat{f}}{d\hat{t}^{n+1}}\right\|_{L^\infty([-1,1])}$$

Refined estimate for Chebychev interpolation

$$\le \frac{2^{-2n-1}}{(n+1)!}|I|^{n+1}\left\|f^{(n+1)}\right\|_{L^\infty(I)}. \tag{6.1.87}$$

$\hookrightarrow |I| = |b-a|$ for $I = [a,b]$

Interpolation error estimates and Lebesgue constant

$I_J \hat{=}$ polynomial interpolation operator for $J = \{t_0,\dots,t_n\}$

$$\lambda_J := \sup_{x \in \mathbb{R}^n} \frac{\|I_J x\|_\infty}{\|x\|_\infty}$$

$[\ I_J : \mathbb{R}^{n+1} \to \mathcal{P}_n\ ]$

$L_J \hat{=}$ approximation scheme based on I_J

$[\ L_J : C^0(I) \to \mathcal{P}_n : L_J = I_J[f(t_0),\dots,f(t_n)]^T$

Note: $\|L_J g\|_\infty = \|I_J[g(t_0),\dots,g(t_n)]^T\|_\infty$

$\le \lambda_J \max_j |g(t_j)|$ (*)

$\le \lambda_J \|g\|_\infty$

$\|f - L_J f\|_\infty = \|(f-p) - (L_J f - L_J p)\|_\infty$, any $p \in \mathcal{P}_n$

interpolation error $= \|(f-p) - L_J(f-p)\|_\infty$ by linearity

Note: $L_J p = p$ $\le \|f-p\|_\infty + \|L_J(f-p)\|_\infty$

$\overset{(*)}{\le} (1+\lambda_J)\|f-p\|_\infty$

$$\|f - L_T f\|_{L^\infty(I)} \le (1+\lambda_T) \inf_{p \in \mathcal{P}_n} \|f-p\|_{L^\infty(I)} \quad \forall f \in C^0(I), \tag{6.1.61}$$

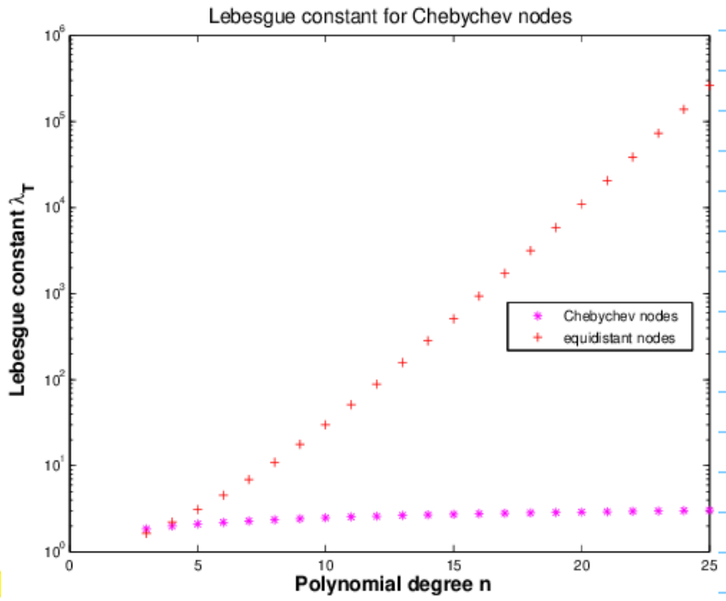


Fig. 173

Equidistant nodes

$\lambda_J \geq C e^{n/2}$

Chebychev nodes

$\lambda_T \leq \frac{2}{\pi} \log(1+n) + 1$

combine with Jackson estimates

Thm 6.1.15

$$\|f - \mathsf{L}_T f\|_{L^\infty([-1,1])} \leq (2/\pi \log(1+n) + 2)(1 + \pi^2/2)^r \frac{(n-r)!}{n!} \left\| f^{(r)} \right\|_{L^\infty([-1,1])} . \qquad (6.1.91)$$

▷ Applies to $f \in C^r$ ⇒ obtain n-asymptotics

$$\|f - \mathsf{I}_T(f)\|_{L^\infty(I)} = \left\| \widehat{f} - \mathsf{I}_{\widehat{T}}(\widehat{f}) \right\|_{L^\infty([-1,1])} \leq \frac{2^{-n}}{(n+1)!} \left\| \frac{\mathrm{d}^{n+1}\widehat{f}}{\mathrm{d}\widehat{t}^{n+1}} \right\|_{L^\infty([-1,1])}$$

$$\leq \frac{2^{-2n-1}}{(n+1)!} |I|^{n+1} \left\| f^{(n+1)} \right\|_{L^\infty(I)} . \qquad (6.1.87)$$

(usually) prevents us from extracting behavior for $n \to \infty$

Experiments: "n-cvg"

(i) $f(t) = \frac{1}{1+t^2}$, $I = [-5,5]$

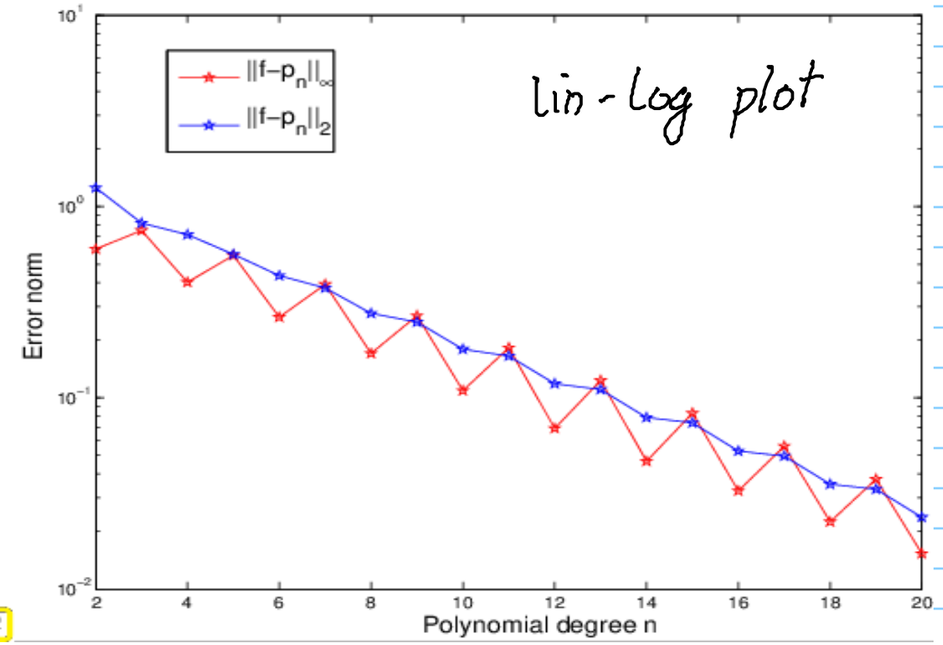


Fig. 222

⇒ exp. cvg.

("rather generally true" for $f \in C^\infty$)

(10) "Tent function"

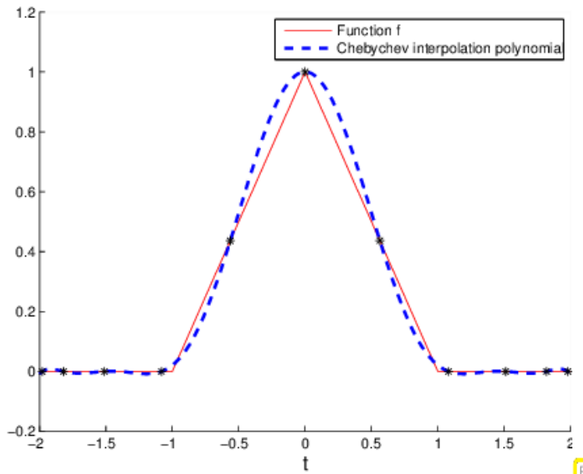


Fig. 223

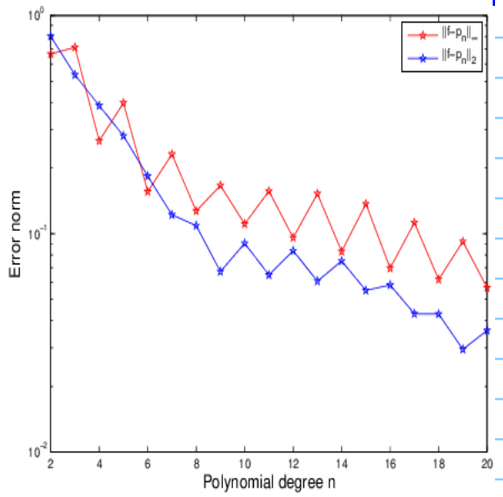


Fig. 224

still converges

6.1.3.3. Algorithms for Chebychev interpolation

C++11 code 6.1.99: Definition of class for Chebychev interpolation

```
class ChebInterp {
  private:
  // various internal data describing Chebychev interpolating
     polynomial p
  public:
    // Constructor taking function f and degree n as arguments
    template <typename Function>
      PolyInterp(const Function &f,unsigned int n);
    // Evaluation operator:  y_j = p(x_j),  j = 1,...,m  (m "large")
    void eval(const vector<double> &x,vector <double> &y) const;
};
```

Cheb. interp. polynomial

Chebychev expansion : $p(t) = \sum_{j=0}^{n} \alpha_j T_j(t)$

Fast evaluation : "Chebychev Horner scheme"

Idea : Use 3-term recursion

$$T_j(x) = 2x\,T_{j-1}(x) - T_{j-2}(x)$$

$$p(x) = \sum_{j=0}^{n-1} \alpha_j T_j(x) + \alpha_n T_n(x)$$

$$\overset{(6.1.77)}{=} \sum_{j=0}^{n-1} \alpha_j T_j(x) + \alpha_n (2xT_{n-1}(x) - T_{n-2}(x))$$

$$= \underbrace{\sum_{j=0}^{n-3} \alpha_j T_j(x) + (\alpha_{n-2} - \alpha_n) T_{n-2}(x) + (\alpha_{n-1} + 2x\alpha_n) T_{n-1}(x)}_{\text{another Cheb.-exp.} \;\rightarrow\; \text{recursion}} .$$

C++11 code 6.1.103: Recursive evaluation of Chebychev expansion (6.1.100)

```
// Recursive evaluation of a polynomial p = Σ_{j=1}^{n+1} a_j T_{j-1} at point x
// based on (6.1.102)
// IN : Vector of coefficients a
// evaluation point x
// OUT: Value at point x
double recclenshaw(const VectorXd& a, const double x) {
  const VectorXd::Index n = a.size() - 1;
  if      (n == 0) return a(0);              // Constant polynomial
  else if (n == 1) return (x*a(1) + a(0));   // Value α_1 * x + α_0
  else {
    VectorXd new_a(n);
    new_a << a.head(n - 2), a(n - 2) - a(n), a(n - 1)+ 2*x*a(n);
    return recclenshaw(new_a,x); // recursion
  }
}
```

$\Rightarrow$ Cost $= O(n)$

10

Computing α_j :

I.C. $\Leftrightarrow$ $p(t_k) = f(t_k)\,,\quad k = 0,\dots,n\,,$ for $t_k := \cos\left(\frac{2k+1}{2(n+1)}\pi\right)$. (6.1.106)

Chebychev nodes

$\hookrightarrow (n\times 1)\times(n+1)$ LSE

Trick : Periodize p : $T_n(t) = \cos(n\cos^{-1}(t))$

$$q(s) := p(\cos 2\pi s) = \sum_{j=0}^{n} \alpha_j T_j(\cos 2\pi s) \overset{\text{Def. 6.1.75}}{=} \sum_{j=0}^{n} \alpha_j \cos(2\pi j s)$$

$$= \sum_{j=0}^{n} \tfrac{1}{2}\alpha_j\left(\exp(2\pi\imath j s) + \exp(-2\pi\imath j s)\right) \quad [\text{ by } \cos z = \tfrac{1}{2}(e^{z} + e^{-z})\,]$$

(6.1.107)

$$= \sum_{j=-n}^{n+1} \beta_j \exp(-2\pi\imath j s)\,, \quad \text{with} \quad \beta_j := \begin{cases} 0 & \text{, for } j = n+1\,, \\ \frac{1}{2}\alpha_j & \text{, for } j = 1,\dots,n\,, \\ \alpha_0 & \text{, for } j = 0\,, \\ \frac{1}{2}\alpha_{n-j} & \text{, for } j = -n,\dots,-1\,. \end{cases}$$

[trigonometric polynomial]

I.C. for q $\Longrightarrow$ I.C. for q at nodes

$t = \cos(2\pi s)$ $\longrightarrow$

$t_k = \cos\left(\frac{2k+1}{4(n+1)}2\pi\right)$ $\longrightarrow$ $s_k = \frac{2k+1}{4(n+1)}$

[equidistant nodes]

▷ FFT-based alg. for comp. $\beta_j \Longrightarrow \alpha_j$

Cost $O(n\log n)$

MATLAB-code 6.1.111: Efficient computation of Chebychev expansion coefficient of Chebychev interpolant

```
// efficiently compute coefficients α_j in the Chebychev expansion
// p = Σ_{j=0}^{n} α_j T_j of p ∈ P_n based on values y_k,
// k = 0,...,n, in Chebychev nodes t_k, k = 0,...,n
// IN: values y_k passed in y
// OUT: coefficients α_j
VectorXd chebexp(const VectorXd& y) {
  const int n = y.size() - 1;          // degree of polynomial
  const std::complex<double> M_I(0, 1); // imaginary unit
  // create vector z, see (6.1.109)
  VectorXcd b(2*(n + 1));
  const std::complex<double> om = -M_I*(M_PI*n)/((double)(n+1));
  for (int j = 0; j <= n; ++j) {
    b(j) = std::exp(om*double(j))*y(j); // this cast to double is
      necessary!!
    b(2*n+1-j) = std::exp(om*double(2*n+1-j))*y(j);
  }

  // Solve linear system (6.1.110) with effort O(n log n)
  Eigen::FFT<double> fft;  // EIGEN's helper class for DFT
  VectorXcd c = fft.inv(b); // -> c = ifft(z), inverse fourier
    transform
  // recover β_j, see (6.1.110)
  VectorXd beta(c.size());
  const std::complex<double> sc = M_PI_2/(n + 1)*M_I;
  for (unsigned j = 0; j < c.size(); ++j)
    beta(j) = ( std::exp(sc*double(-n+j))*c[j] ).real();
  // recover α_j, see (6.1.107)
  VectorXd alpha = 2*beta.tail(n); alpha(0) = beta(n);
  return alpha;
}
```

6.5. Approximation by piecewise polynomials

For Chebychev interpolation:

$$\|f - \mathsf{I}_{\mathcal{T}}(f)\|_{L^\infty(I)} = \left\|\widehat{f} - \mathsf{I}_{\widehat{\mathcal{T}}}(\widehat{f})\right\|_{L^\infty([-1,1])} \leq \frac{2^{-n}}{(n+1)!}\left\|\frac{\mathrm{d}^{n+1}\widehat{f}}{\mathrm{d}\widehat{t}^{n+1}}\right\|_{L^\infty([-1,1])}$$

$$\leq \frac{2^{-2n-1}}{(n+1)!}|I|^{n+1}\left\|f^{(n+1)}\right\|_{L^\infty(I)} . \qquad (6.1.87)$$

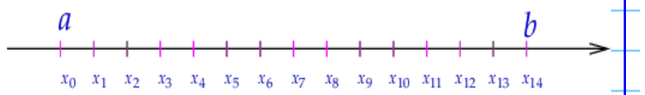
▷ reduce interpolation by shrinking the interval

Idea: piecewise interpolation on $I := [a,b]$

Tool: $\mathcal{M} = \{a = x_0 < x_1 < \dots < x_m = b\}$

Terminology:

- $x_j \hat{=}$ nodes of the mesh $\mathcal{M}$,
- $[x_{j-1}, x_j[\hat{=}$ intervals/cells of the mesh,
- $h_{\mathcal{M}} := \max_j |x_j - x_{j-1}| \hat{=}$ mesh width,
- If $x_j = a + jh \hat{=}$ equidistant (uniform) mesh with meshwidth $h > 0$



Then: (local) Lagrange interpolation on cells

General local Lagrange interpolation on a mesh

1. Choose local degree $n_j \in \mathbb{N}_0$ for each cell of the mesh, $j = 1, \dots, m$.
2. Choose set of *local* interpolation nodes

$$\mathcal{T}^j := \{t_0^j, \dots, t_{n_j}^j\} \subset I_j := [x_{j-1}, x_j] , \quad j = 1, \dots, m ,$$

for each mesh cell/grid interval I_j.

3. Define piecewise polynomial interpolant $s : [x_0, x_m] \to \mathbb{K}$:

$$s_j := s_{|I_j} \in \mathcal{P}_{n_j} \quad \text{and} \quad s_j(t_i^j) = f(t_i^j) \quad i = 0, \dots, n_j , \quad j = 1, \dots, m . \qquad (6.5.5)$$

Owing to Thm. 5.2.14, s_j is well defined.

p.w. linear intp.: $n_j = 1$, $t_0^j = x_{j-1}$, $t_1^j = x_j$

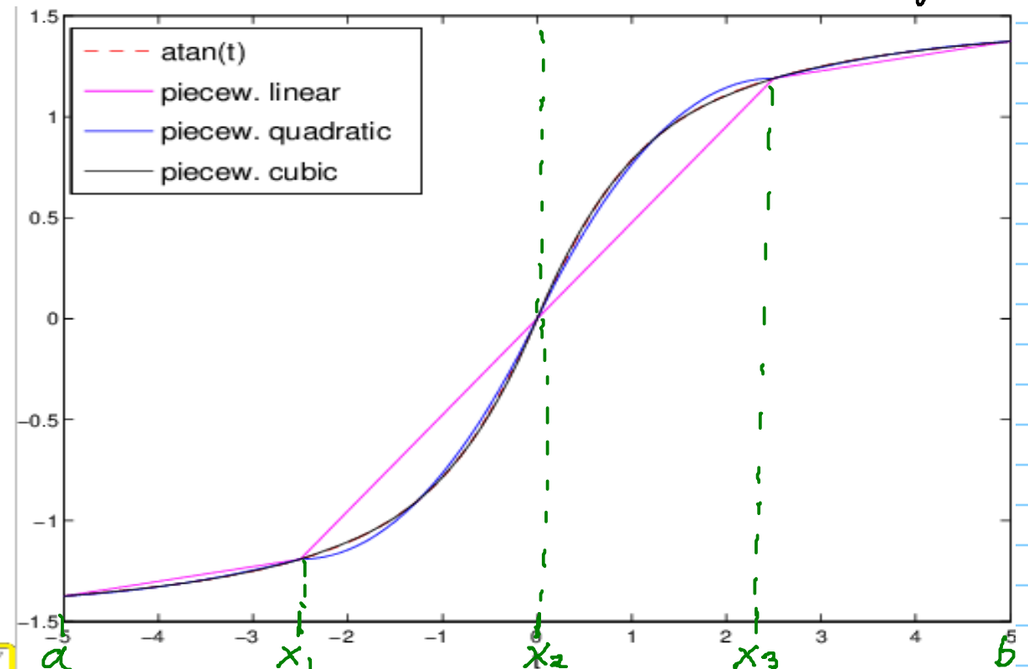


Fig. 247

p.w. quadr : $n_j = 2$: $t_0^j = x_{j-1}$

$t_1^j = \frac{1}{2}(x_{j-1} + x_j)$

$t_2^j = x_j$

C^0-p.w. interpolant guaranteed if all x_j's are interpolation nodes!

[No C^1 interpolant, though!]

Error estimates for special case $n_j = n$ (fixed):

Improve accuracy by $h_M \longrightarrow 0$ [h-convergence]

Example: $f(t) = \operatorname{atan}(t)$ on $[-5,5]$

Equidistant mesh with $10/h_M$ cells, equid int. nodes

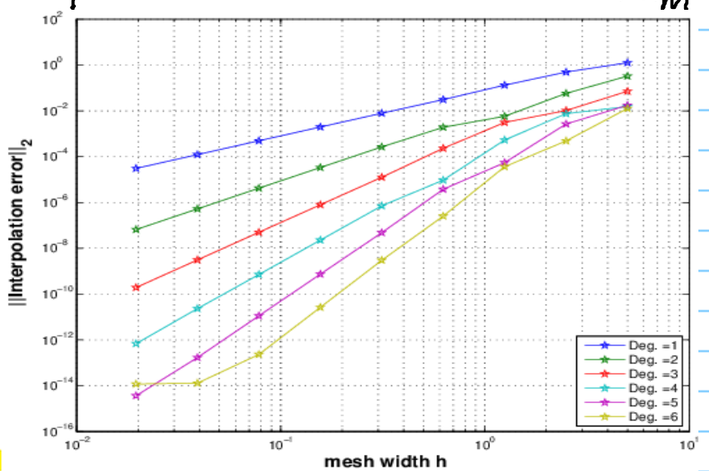


Fig. 248

log-log plot & linear error curves

$\triangleright$ alg. cvg. in h_M

Theory : Apply on $I := [x_{j-1}, x_j]$, $x_{j-1} \le t_j \le x_j$

Thm. 6.1.44 $\Rightarrow$ $$\boxed{\|f - \mathsf{L}_T f\|_{L^\infty(I)} \le \frac{\|f^{(n+1)}\|_{L^\infty(I)}}{(n+1)!} \underbrace{\max_{t\in I}|(t-t_0)\cdot\cdots\cdot(t-t_n)|}_{\le h_M^{n+1}}} . \qquad (6.1.50)$$

then max over all cells:

$$(6.1.50) \Rightarrow \|f - s\|_{L^\infty([x_0,x_m])} \le \frac{h_{\mathcal{M}}^{n+1}}{(n+1)!} \left\|f^{(n+1)}\right\|_{L^\infty([x_0,x_m])} , \qquad (6.5.12)$$

with mesh width $h_{\mathcal{M}} := \max\{|x_j - x_{j-1}| : j = 1,\dots,m\}$.

$\downarrow$

Alg. cvg. in h_M with rate $n+1$!

Advantage : (I) Poor smoothness of f still guarantees alg. cvg. in h_M

(II) Suitable also for merely piecewise smooth functions

(III) Locality $\Rightarrow$ update friendly

(IV) simple & cheap : cost $= O(m)$

Drawback : "Slow" algebraic convergence

p.w. quadratic $(n = 2)$