ETH Lecture 401-0663-00L Numerical Methods for CSE

Numerical Methods for Computational Science and Engineering

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URL: http://www.sam.math.ethz.ch/~hiptmair/tmp/NumCSE/NumCSE16.pdf

VII. Numerical Quadrature

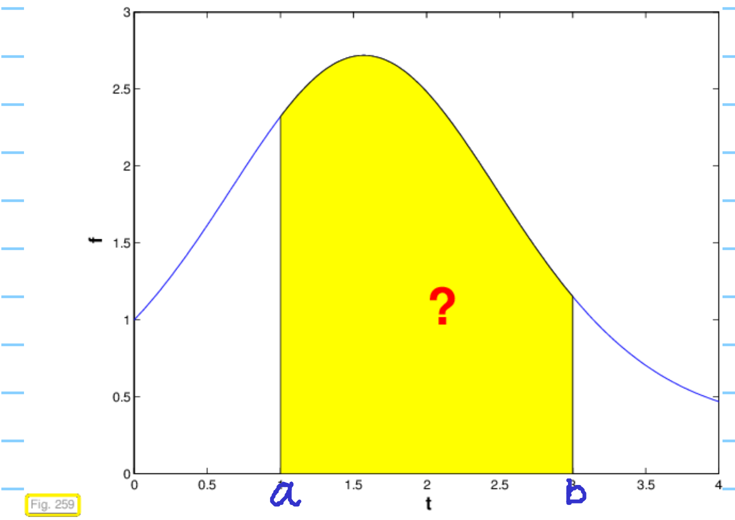


Fig. 259

Approximate $\int_a^b f(t)\,dt$

only point evaluations possible

Application:

assume time-harmonic periodic excitation with period $T > 0$.

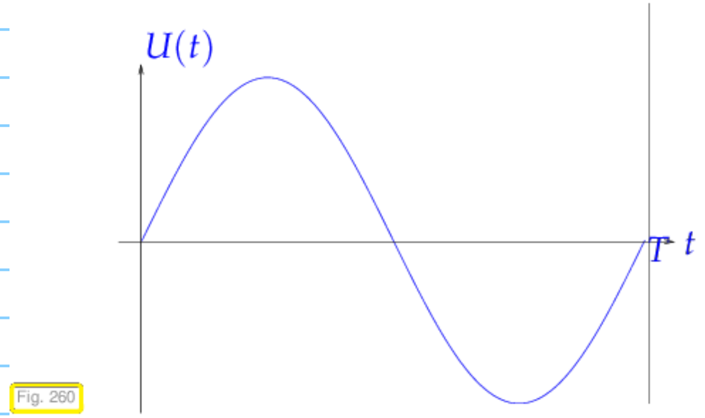


Fig. 260

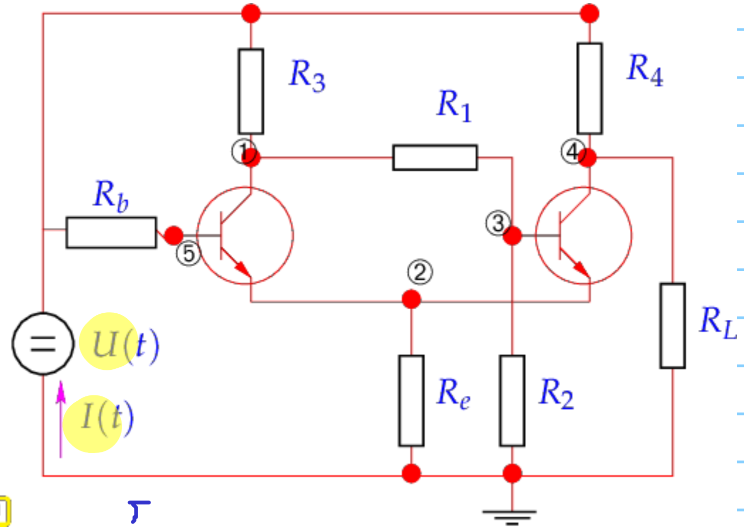


Fig. 261

Total heat generation: $W = \int_0^T U(t) I(U(t))\,dt$

output of "black box" function

5.1. Quadrature formulas

Definition 7.1.1. Quadrature formula/quadrature rule

An n-point quadrature formula/quadrature rule on $[a,b]$ provides an approximation of the value of an integral through a *weighted sum* of point values of the integrand:

$$\int_a^b f(t)\,\mathrm{d}t \approx Q(f) := \sum_{j=1}^{n} w_j f(c_j) . \tag{7.1.2}$$

weight

$\in [a,b]$ $\hat{=}$ nodes

cost(Q.F.) = # f-evaluation = n

(2)

Given: Q.F. $(\hat{w}_j, \hat{c}_j)_{j=1}^{n}$ on $[-1,1]$ (reference interval)

. $\Phi : [-1,1] \to [a,b]$, $\Phi(\tau) = a + \frac{1}{2}(b-a)(\tau+1)$

$$\triangleright \int_a^b f(t)\,dt = \int_{-1}^{1} f(\Phi(\tau))\left|\frac{d\Phi}{d\tau}(\tau)\right| d\tau = \tfrac{1}{2}(b-a)\int_{-1}^{1} f(\Phi(\tau))\,d\tau$$

$$\approx \underbrace{\tfrac{1}{2}(b-a)\sum_{j=1}^{n}\hat{w}_j f(\Phi(\hat{c}_j))}_{\text{Q.F. on } [a,b]} \qquad \text{Apply Q.F.}$$

Idea: transformation formula for integrals

$$\int_a^b f(t)\,\mathrm{d}t = \tfrac{1}{2}(b-a)\int_{-1}^{1}\hat{f}(\tau)\,\mathrm{d}\tau\,, \qquad (7.1.5)$$
$$\hat{f}(\tau) := f(\tfrac{1}{2}(1-\tau)a + \tfrac{1}{2}(\tau+1)b)\,.$$

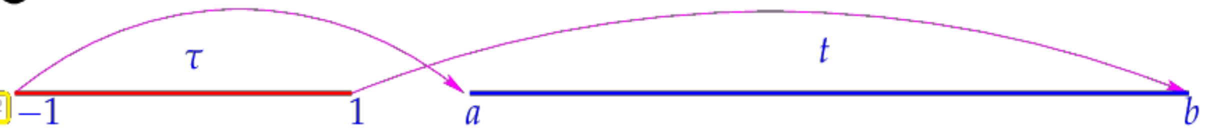


Fig. 262

$\tau \mapsto t := \Phi(\tau)t := \frac{1}{2}(1-\tau)a + \frac{1}{2}(\tau+1)b$

▶ quadrature formula for general interval $[a,b]$, $a,b \in \mathbb{R}$:

$$\int_a^b f(t)\,\mathrm{d}t \approx \tfrac{1}{2}(b-a)\sum_{j=1}^{n}\hat{w}_j\hat{f}(\hat{c}_j) = \sum_{j=1}^{n} w_j f(c_j) \quad \text{with} \quad \begin{aligned} c_j &= \tfrac{1}{2}(1-\hat{c}_j)a + \tfrac{1}{2}(1+\hat{c}_j)b\,,\\ w_j &= \tfrac{1}{2}(b-a)\hat{w}_j\,.\end{aligned}$$

In codes: tabulated quadrature formula

```
struct QuadTab {
  template <typename VecType>
    static void getrule(int n,VecType &c,VecType &w,
                        double a=-1.0,double b=1.0);
}
```

Quadrature by interpolation:

Interpolation scheme: $I_{\mathcal{T}} : \begin{cases} \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \longrightarrow V \\ ([t_j],[y_j])_{j=0}^{n} \longrightarrow I_{\mathcal{T}}[y] \end{cases}$ (V: space of simple fcts)

$\mathcal{T} \subset [a,b]$

$$\int_a^b f(t)\,dt \approx \int_a^b (I_{\mathcal{T}}[f(t_j)]_{j=0}^{n})(\tau)\,d\tau$$

If $I_{\mathcal{T}}$ is linear

$$(*) \qquad = \sum_{j=0}^{n} \underset{\text{nodes}}{f(t_j)} \underbrace{\int_a^b \overbrace{I_{\mathcal{T}}(e_j)}^{\text{cardinal function}}(\tau)\,d\tau}_{\text{weight } w_j}$$

e_j: j-th unit vector

weight w_j ↳ depend only on $I_{\mathcal{T}}$

Quadrature error :

$$E_n(f) = \left| \int_a^b f(t)\,dt - Q_n(f) \right|$$

Our focus : n-dependence of $E_n(f)$ for families of Q.F. (for $n \to \infty$)

↳ "Asymptotic convergence"

If Q.F. induced by $I_{\mathcal{T}}$

$$E_n(f) = \left| \int_a^b (f - I_{\mathcal{T}}[f(t_j)]_j)(\tau)\,d\tau \right|$$

$$\leq |b-a| \cdot \| f - I_{\mathcal{T}}[f(t_j)]_j \|_{\infty,[a,b]}$$

↑ interpolation error

[→ Chapter 6 !]

5.2. Polynomial Quadrature Formulas

→ Now : $I_{\mathcal{T}} \hat{=}$ Lagrange interpolation

Idea: replace integrand f with $p_{n-1} := \mathsf{I}_{\mathcal{T}} \in \mathcal{P}_{n-1}$ = polynomial Lagrange interpolant of f (→ Cor. 5.2.15) for given node set $\mathcal{T} := \{t_0, \dots, t_{n-1}\} \subset [a,b]$

$$\blacktriangleright \quad \int_a^b f(t)\,\mathrm{d}t \approx Q_n(f) := \int_a^b p_{n-1}(t)\,\mathrm{d}t\,. \tag{7.2.1}$$

$$\sum_{j=0}^{n} f(t_j) \underbrace{\int_a^b I_{\mathcal{T}}(e_j)(\tau)\,d\tau}$$

If $I_{\mathcal{T}} \hat{=}$ Lagrange interpolation :

$$I_{\mathcal{T}}(e_j) = L_j \quad : \text{Lagrange pols.}$$

Examples :

$n=1$:

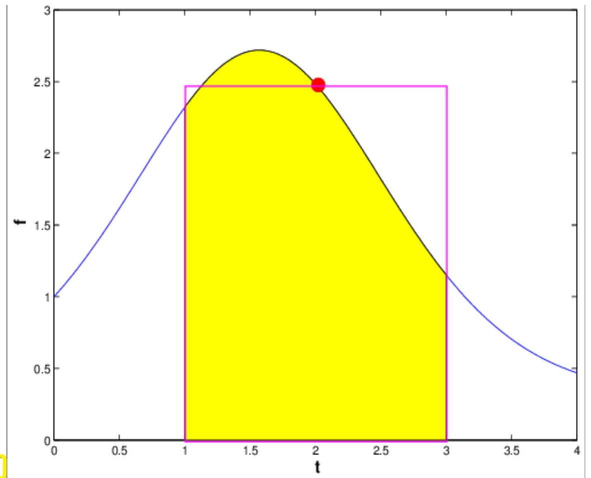


Fig. 263

The midpoint rule is (7.2.2) for $n = 1$ and $t_0 = \frac{1}{2}(a+b)$. It leads to the 1-point quadrature formula

$$\int_a^b f(t)\,\mathrm{d}t \approx Q_{\mathrm{mp}}(f) = (b-a) f(\tfrac{1}{2}(a+b))\,.$$

"midpoint"

◁ the area under the graph of f is approximated by the area of a rectangle.

$n=2$:

trapezoidal rule:

```
> trapez := newtoncotes(1);
```

$$\widehat{Q}_{\mathrm{trp}}(f) := \frac{1}{2}(f(0)+f(1)) \qquad (7.2.5)$$

$$\left(\int_a^b f(t)\,\mathrm{d}t \approx \frac{b-a}{2}(f(a)+f(b)) \right)$$

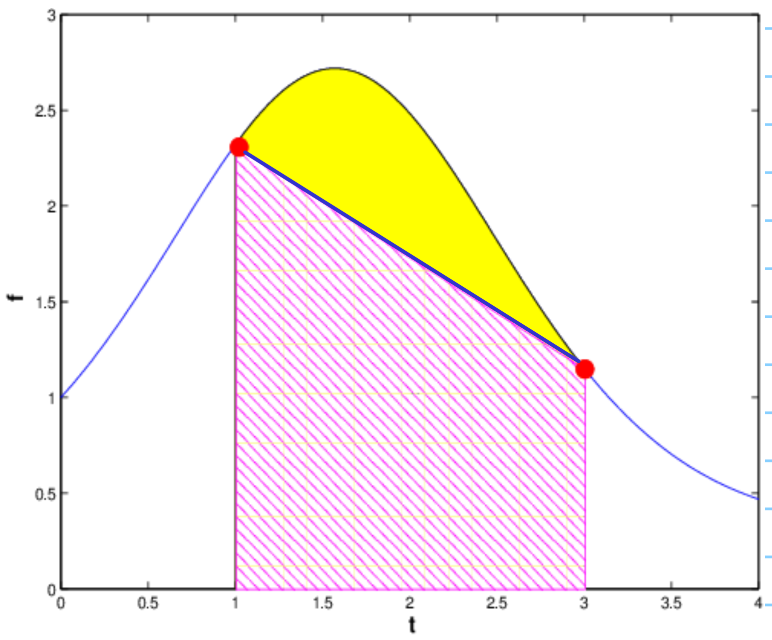


Fig. 264

Newton-Cotes Q.F.: equidistant nodes $t_j = a + \frac{j}{n-1}(b-a)$, $j=0,\dots,n-1$

↳ unstable for large n!

Better: Chebychev nodes → Clenshaw-Curtis Q.F.

5.3. Gauss Quadrature

Gauging "quality" of Q.F.

Definition 7.3.1. Order of a quadrature rule

The order of quadrature rule $Q_n : C^0([a,b]) \to \mathbb{R}$ is defined as

$$\mathrm{order}(Q_n) := \max\{m \in \mathbb{N}_0 : \; Q_n(p) = \int_a^b p(t)\,\mathrm{d}t \quad \forall p \in \mathcal{P}_m\} + 1 , \qquad (7.3.2)$$

that is, as the maximal degree $+1$ of polynomials for which the quadrature rule is guaranteed to be exact.

▷ Polynomial Q.F. with n points ⇒ order n

Theorem 7.3.5. Sufficient order conditions for quadrature rules

An n-point quadrature rule on $[a,b]$ (→ Def. 7.1.1)

$$Q_n(f) := \sum_{j=1}^{n} w_j f(t_j) , \quad f \in C^0([a,b]) ,$$

with nodes $t_j \in [a,b]$ and weights $w_j \in \mathbb{R}$, $j=1,\dots,n$, has order $\geq n$, if and only if

$$w_j = \int_a^b L_{j-1}(t)\,dt , \quad j=1,\dots,n ,$$

where L_k, $k=0,\dots,n-1$, is the k-th Lagrange polynomial (5.2.11) associated with the ordered node set $\{t_1,t_2,\dots,t_n\}$.

$L_{j-1}(t_\ell) = \delta_{j,\ell} \Rightarrow \int_a^b L_{j-1}(t)\,dt = Q_n(L_{j-1})$

$L_{j-1} \in \mathcal{P}_{n-1}$

$= w_j$

5

Order $> n$ for n-point Q.F. ?

possible ?

Theorem 7.3.12. Maximal order of n-point quadrature rule

The maximal order of an n-point quadrature rule is $2n$.

Proof (indirect): Assume $Q(p) = \int_a^b p(t)\,dt \quad \forall p \in \mathcal{P}_{2n}$

$p(t) := \prod_{j=1}^{n} (t-c_j)^2 \geq 0 \Rightarrow \int_a^b p(t)\,dt > 0$

$p \in \mathcal{P}_{2n}$

but $Q(p) = \sum_{j=1}^{n} w_j p(c_j) = 0$ $\square$

Example: 2-point rule of order 4

$$Q_n(p) = \int_a^b p(t)\,\mathrm{d}t \;\forall p \in \mathcal{P}_3 \quad \Leftrightarrow \quad Q_n(\{t \mapsto t^q\}) = \frac{1}{q+1}(b^{q+1} - a^{q+1}), \quad q = 0,1,2,3.$$

Enough to check exactness for basis of $\mathcal{P}_3$

4 equations for weights w_j and nodes c_j, $j = 1,2$ $\quad (a = -1, b = 1)$, *cf.* Rem. 7.3.6

non-linear system of equ.

$$\int_{-1}^{1} 1\,\mathrm{d}t = 2 = 1w_1 + 1w_2, \quad \int_{-1}^{1} t\,\mathrm{d}t = 0 = c_1 w_1 + c_2 w_2$$
$$\int_{-1}^{1} t^2\,\mathrm{d}t = \frac{2}{3} = c_1^2 w_1 + c_2^2 w_2, \quad \int_{-1}^{1} t^3\,\mathrm{d}t = 0 = c_1^3 w_1 + c_2^3 w_2. \tag{7.3.14}$$

➤ weights & nodes: $\{w_2 = 1, w_1 = 1, c_1 = 1/3\sqrt{3}, c_2 = -1/3\sqrt{3}\}$

▶ quadrature formula (order 4): $\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$ (7.3.15)

Optimist's **assumption**: $\exists$ family of n-point quadrature formulas on $[-1,1]$

$$Q_n(f) := \sum_{j=1}^{n} w_j^n f(c_j^n) \approx \int_{-1}^{1} f(t)\,\mathrm{d}t, \quad w_j \in \mathbb{R}, n \in \mathbb{N},$$

of order $2n$ $\Leftrightarrow$ exact for polynomials $\in \mathcal{P}_{2n-1}$. (7.3.17)

Define $\bar{P}_n(t) := (t - c_1^n) \cdot \dots \cdot (t - c_n^n), \quad t \in \mathbb{R} \quad \Rightarrow \quad \bar{P}_n \in \mathcal{P}_n$. $\bar{P}_n(c_j^n) = 0$

leading coeff $= 1$

$q \in \mathcal{P}_{n-1}$: $\int_{-1}^{1} \underbrace{q \bar{P}_n}_{\in \mathcal{P}_{2n-1}} dt \underset{\text{order } 2n}{=} \sum_{j=1}^{n} w_j (q\bar{P}_n)(c_j) = 0$

" $\bar{P}_n \perp \mathcal{P}_{n-1}$ " w.r.t. inner product $(f,g) \to \int_{-1}^{1} (fg)(t)\,dt$

$\bar{P}_n$ is defined by n coefficients

$\int_{-1}^{1} (q \bar{P}_n)(t)\,dt = 0 \quad \forall q \in \mathcal{P}_{n-1} \Rightarrow n = \dim \mathcal{P}_{n-1}$ conditions

$\Rightarrow \bar{P}_n \in \mathcal{P}_n$ unique

The nodes of an n-point quadrature formula of order $2n$, if it exists, must coincide with the zeros of the polynomial $\bar{P}_n$ defined recursively by (7.3.20).

(6)

Theorem 7.3.21. Existence of n-point quadrature formulas of order $2n$

Let $\{\bar{P}_n\}_{n\in\mathbb{N}_0}$ be a family of non-zero polynomials that satisfies

- $\bar{P}_n \in \mathcal{P}_n$,
- $\int_{-1}^{1} q(t)\bar{P}_n(t)\,\mathrm{d}t = 0$ *for all* $q \in \mathcal{P}_{n-1}$ $\quad$ ($L^2([-1,1])$-*orthogonality*),
- *The set* $\{c_j^n\}_{j=1}^m$, $m \le n$, *of real zeros of* $\bar{P}_n$ *is contained in* $[-1,1]$.

Then the quadrature rule (→ Def. 7.1.1) $\quad Q_n(f) := \sum_{j=1}^{m} w_j^n f(c_j^n)$

with weights chosen according to Thm. 7.3.5 provides a quadrature formula of order $2n$ *on* $[-1,1]$.

Proof: $p \in \mathcal{P}_{2n-1}$: $p = h\bar{P}_n + r$, $h, r \in \mathcal{P}_{n-1}$

["Polynomial division"]

$$\int_{-1}^{1} p(t)\,dt = \underbrace{\int_{-1}^{1} h(t)\bar{P}_n(t)\,dt}_{=0} + \int_{-1}^{1} r(t)\,dt =$$

$$= \sum_{j=1}^{n} W_j^n \underbrace{\left(r(c_j^n) + \overbrace{h\bar{P}_n(c_j^n)}^{=0}\right)}_{=p(c_j^n)}$$

$$= Q_n(p)$$

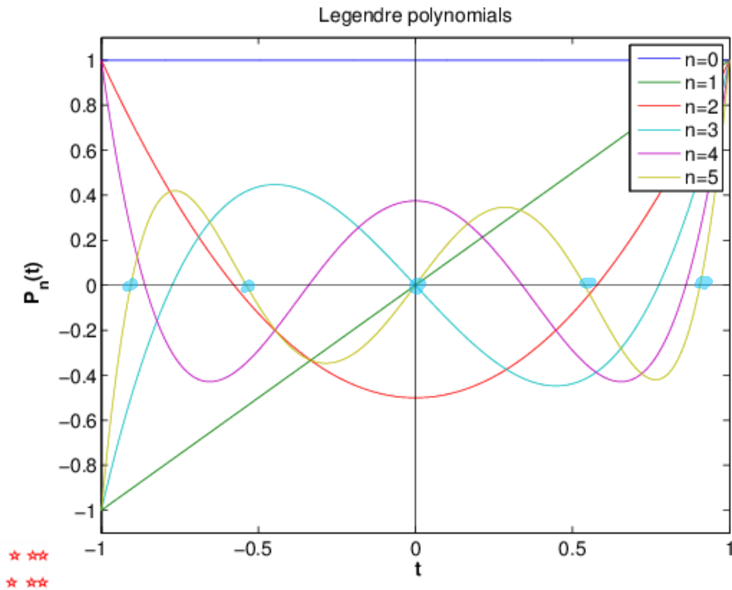
□

The polynomials $\bar{P}_n$:

Definition 7.3.26. Legendre polynomials

The n-th Legendre polynomial P_n is defined by

- $P_n \in \mathcal{P}_n$,
- $\int_{-1}^{1} P_n(t)q(t)\,\mathrm{d}t = 0 \;\forall q \in \mathcal{P}_{n-1}$,
- $P_n(1) = 1$.



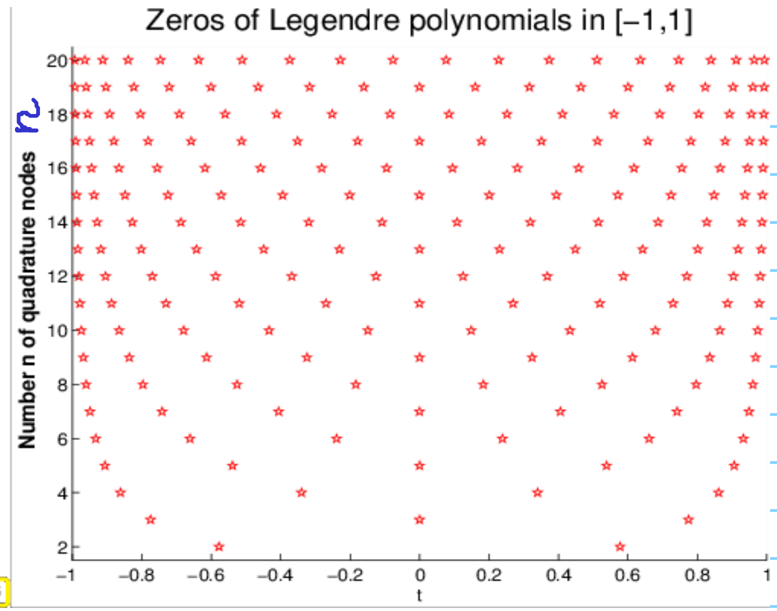


Fig. 266

◁ Obviously:

Lemma 7.3.27. Zeros of Legendre polynomials

P_n has n distinct zeros in $]-1,1[$.

Zeros of Legendre polynomials = Gauss points

nodes of unique n-pt Q.F. of order 2n !

Proof: Assume : P_n has only $k < n$ zeros x_j in $]-1,1[$

$q(t) := \pm \prod_{j=1}^{k}(t - x_j) \Rightarrow P_n q \geq 0$ (*)

$q \in \mathcal{P}_k$ $\quad\Downarrow$

$$\int_{-1}^{1} (P_n q)(t)\,dt > 0 \quad ⚡$$

(*) Wherever P_n changes sign, also q changes sign.

□

NCSE15

Obviously ▷

Lemma 7.3.30. Positivity of Gauss-Legendre quadrature weights

The weights of the Gauss-Legendre quadrature formulas are positive.

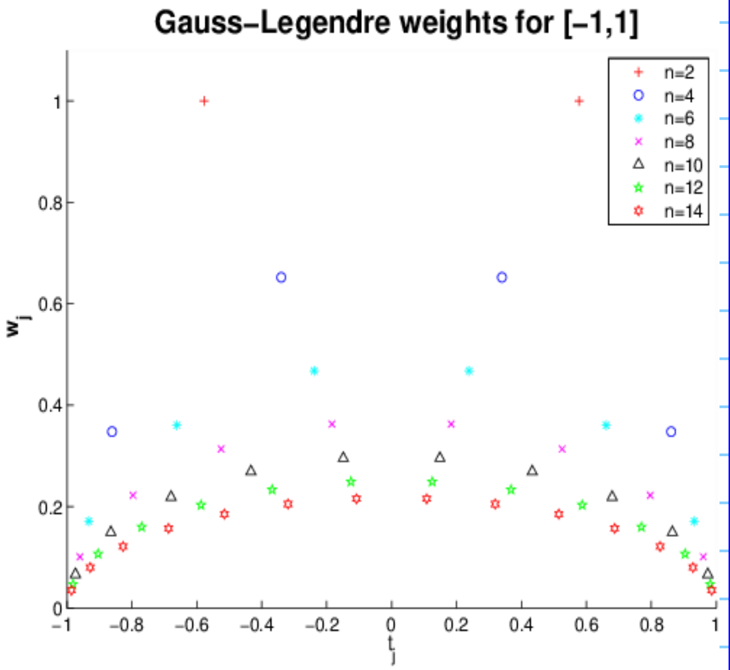


Fig. 267

Proof: $\{t_j\}_{j=1}^n \hat{=}$ Gauss nodes,

$$q(t) := \prod_{\substack{j=1\\ j\neq k}}^{n} (t-t_j)^2 \quad , \quad k \in \{1,\dots,n\}$$

$$q \in \mathcal{P}_{2n-2} : \quad Q_n(q) = \sum_{j=1}^{n} w_j\, q(t_j) = \underbrace{\int_{-1}^{1} q(t)\,dt}_{>0}$$

$$\| \quad w_k (t_k - t_j)^2$$

Order $2n$! $\square$

$\ulcorner$ If Q.F. has order $q \geq 1$ $\Rightarrow$ $Q_n(1) = \int_a^b 1\,dt = b-a$

$$\| \quad \sum_{j=1}^{n} w_j \cdot 1 \quad \lrcorner$$

Quadrature error $\Longleftrightarrow$ best approximation error

Q_n : Q.F. of order $q \geq 1$ $\Rightarrow$ $E_n \hat{=}$ quadrature error

$$E_n(f) = E_n(f-p) \quad \forall p \in \mathcal{P}_{q-1}$$

$$f \in C^0([a,b]) \quad = \left| \int_a^b (f-p)(t)\,dt - \sum_{j=1}^{n} w_j (f-p)(c_j) \right|$$

$$\leq |b-a|\, \|f-p\|_{\infty,[a,b]} + \sum_{j=1}^{n} |w_j|\, |f-p|(c_j)$$

$$\leq \left(|b-a| + \sum_{j=1}^{n} |w_j| \right) \|f-p\|_{\infty}$$

$$\text{If } w_j \geq 0 \Rightarrow \sum_{j=1}^{n} |w_j| = b-a$$

$$\leq 2(b-a)\, \|f-p\|_{\infty} \quad \forall p \in \mathcal{P}_{q-1}$$

Theorem 7.3.39. Quadrature error estimate for quadrature rules with positive weights

For every n-point quadrature rule Q_n as in (7.1.2) of order $q \in \mathbb{N}$ with weights $w_j \geq 0$, $j = 1,\dots,n$ the quadrature error satisfies

$$E_n(f) := \left| \int_a^b f(t)\,dt - Q_n(f) \right| \leq 2|b-a| \underbrace{\inf_{p \in \mathcal{P}_{q-1}} \|f-p\|_{L^\infty([a,b])}}_{\text{best approximation error}} \quad \forall f \in C^0([a,b]) . \qquad (7.3.40)$$

$\rightarrow$ applies to Gauss quadrature, Clenshaw-Curtis

Now apply polynomial best approximation estimates from Ch. 6:

$$f \in C^r([a,b]),\ r \in \mathbb{N} \Rightarrow E_n(f) = O(n^{-r})$$

[if $q \geq r$] for $n \to \infty$

order of Q.F.

$\hat{=}$ alg. cvg.

$$f \in C^\infty([a,b]) \Rightarrow E_n(f) = O(q^n)$$

("analytic") $0 \leq q < 1$

for $n \to \infty$

$\hat{=}$ exp. cvg.

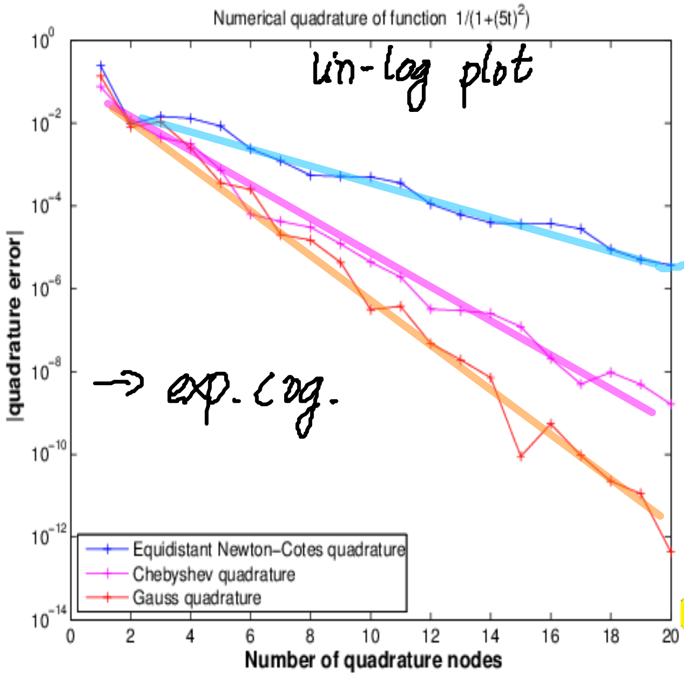


Fig. 268

quadrature error, $f_1(t) := \frac{1}{1+(5t)^2}$ on $[0,1]$

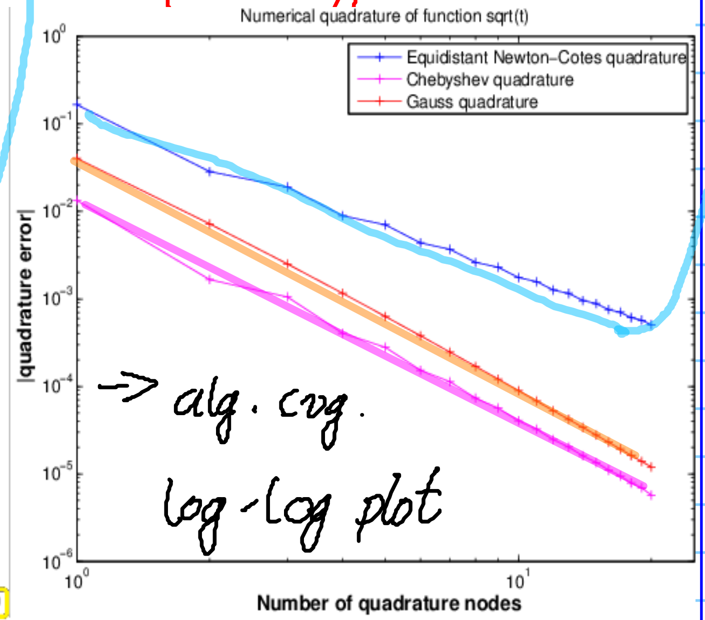


Fig. 269

quadrature error, $f_2(t) := \sqrt{t}$ on $[0,1]$

"Smoothing" integrands by transformation

$$\int_0^b \sqrt{t}\, f(t)\, dt \quad \text{with } f \in C^\infty([0,b])$$

↳ non-smooth integrand : slow alg. cvg. of quad. error

Idea : substitution : $s = \sqrt{t} \Rightarrow \frac{ds}{dt} = \frac{1}{2\sqrt{t}} \Rightarrow dt = 2s\,ds$

$$\int_0^b \sqrt{t}\, f(t)\, dt = \int_0^{\sqrt{b}} \underbrace{s\, f(s^2)\, 2s}_{\in C^\infty([0,\sqrt{b}])\,!}\, ds$$

↳ exp. cvg. of Gauss quadrature

Remark : How to read asymptotic estimates

$E_n(f) = O(n^{-r})$ for $n \to \infty$; $\quad E_n(f) \approx C n^{-r} \quad (*)$

[assume sharp estimate]

Note : $C > 0$ unknown

$(*) \Rightarrow$ no information for concrete n !

(*) tells us how much additional work (= additional f-evaluations) is needed to reduce the error by factor $\rho > 1$.

$$E_{n_{old}}(f) : E_{n_{new}}(f) = \rho$$

(*) $\Rightarrow$
$$\frac{n_{old}^{-r}}{n_{new}^{-r}} = \rho$$

$$n_{new} = n_{old} \cdot \rho^{1/r}$$

The larger r, the less the additional effort to gain a fixed error reduction!

5.4. Composite quadrature

↳ Q.F. induced by p.w. polynomial intp.

Other perspective: On mesh $\mathcal{M} = \{a = x_0 < x_1 < \dots < x_m = b\}$

$$\int_a^b f(t)\,dt = \sum_{j=1}^{m} \underbrace{\int_{x_{j-1}}^{x_j} f(t)\,dt}_{\text{apply } n_j\text{-pt local Q.F. on } [x_{j-1},x_j]}$$

General construction of composite quadrature rules

Idea:
- Partition integration domain $[a,b]$ by a mesh/grid ($\rightarrow$ Section 6.5)
$$\mathcal{M} := \{a = x_0 < x_1 < \ldots < x_m = b\}$$
- Apply quadrature formulas from Section 7.2, Section 7.3 locally on mesh intervals $I_j := [x_{j-1}, x_j]$, $j = 1, \ldots, m$, and sum up.

▶ composite quadrature rule

$$\#\text{f-eval} = \sum_{j=1}^{m} n_j$$

Examples :

Composite trapezoidal rule, *cf.* (7.2.5)

$$\int_a^b f(t)\mathrm{d}t = \tfrac{1}{2}(x_1 - x_0)f(a) + \sum_{j=1}^{m-1} \tfrac{1}{2}(x_{j+1} - x_{j-1})f(x_j) + \tfrac{1}{2}(x_m - x_{m-1})f(b)\ . \tag{7.4.4}$$

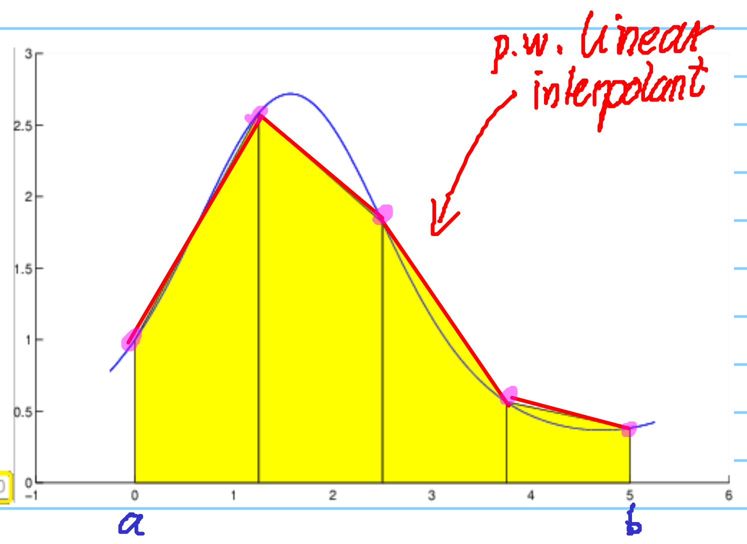


Fig. 270

Composite Simpson rule, *cf.* (7.2.6)

$$\int_a^b f(t)\mathrm{d}t = \tfrac{1}{6}(x_1 - x_0)f(a) + \sum_{j=1}^{m-1} \tfrac{1}{6}(x_{j+1} - x_{j-1})f(x_j) + \sum_{j=1}^{m} \tfrac{2}{3}(x_j - x_{j-1})f(\tfrac{1}{2}(x_j + x_{j-1})) + \tfrac{1}{6}(x_m - x_{m-1})f(b)\ . \tag{7.4.5}$$

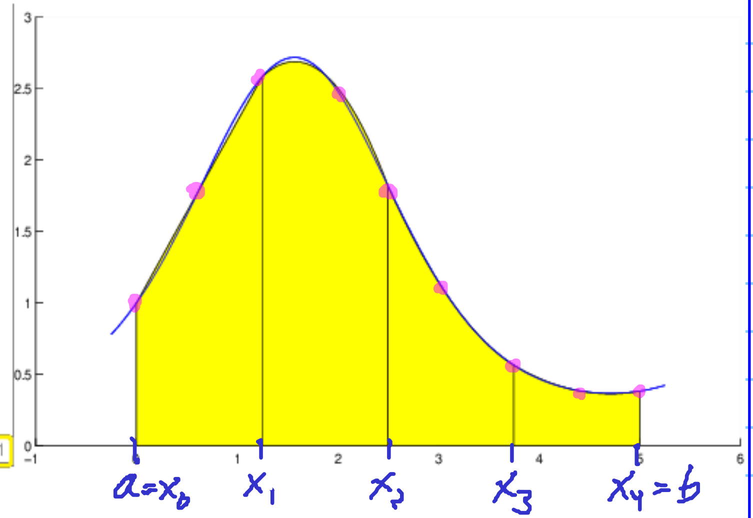


Fig. 271

Error estimates for comp. quad. by summing local quadrature errors

For pol. local Q.F. of order q_ℓ on $[x_{\ell-1}, x_\ell]$

$$\left|\int_{x_{j-1}}^{x_j} f(t)\,\mathrm{d}t - Q^j_{n_j}(f)\right| \overset{(*)}{\leq} C\,\overbrace{|x_j - x_{j-1}|}^{h_j}{}^{\min\{r,q_j\}+1} \left\|f^{(\min\{r,q_j\})}\right\|_{L^\infty([x_{j-1},x_j])}\ . \tag{7.2.10}$$

independent of j for $f \in C^r([a,b])$

$$\sum \underset{(r \text{ big})}{\Rightarrow} \quad E_n(f) \leq C \sum_{j=1}^{m} h_j^{q_j+1} \|f^{(q_j+1)}\|_{\infty,[x_{\ell-1},x_\ell]}$$

(*) From polynomial error estimates for interpolation

$$(f-p)(t) = \frac{1}{(q+1)!} f^{(q+1)}(\xi) \prod_{j=0}^{q} (t - t_j)$$

Now assume : $q = q_j \quad \forall j$

$$E_n(f) \leq C h_M^q \|f^{(q+1)}\|_{\infty,[a,b]} \underbrace{\sum_j h_j}_{= b-a}$$

$$h_M = \max_j h_j$$

$$\left|\int_{x_0}^{x_m} f(t)\,\mathrm{d}t - Q(f)\right| \le C\, h_M^{\min\{q,r\}}\, |b-a| \left\| f^{(\min\{q,r\})} \right\|_{L^\infty([a,b])}, \tag{7.4.10}$$

fixed q $\Rightarrow$ alg. cvg. in meshwidth h_M *

for r big ($\Leftrightarrow$ smooth f) : rate $= q$ [= order of local Q.F.]

* also called h-convergence

[reduction of quadrature error by using finer and finer meshes]

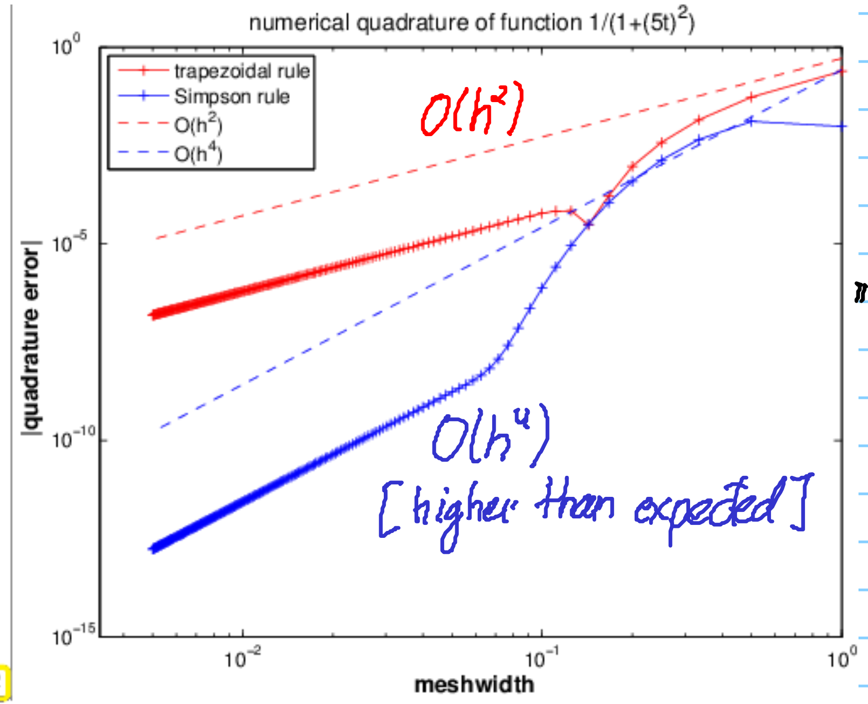


Fig. 272

quadrature error, $f_1(t) := \frac{1}{1+(5t)^2}$ on $[0,1]$

◁ equidistant mesh

log-log plot

"Linear" error graphs $\hat{=}$ alg. cvg.

(local order q)

Comparison : Comp. Quad. $\Longleftrightarrow$ Gauss quad.

$f \in C^r([a,b])$: $\underbrace{O(n^{-\min\{r,q\}}) \qquad O(n^{-r})}$

G.Q. is at least as good as comp. quad.

achieves best possible rate

$f \in C^\infty([a,b])$: $O(n^{-q})$ $\qquad$ $O(q^n)$ with $0 \le q < 1$

$\Uparrow$ much faster