Lecture 4: Periodic and quasi-periodic Green's functions

Habib Ammari

Department of Mathematics, ETH Zürich

Periodic and quasi-periodic Green's functions

- Periodic and quasi-periodic Green's functions:
  - Periodic Green's functions for gratings;
  - Periodic, and quasi-periodic Green's functions;
  - Periodic and quasi-periodic layer potentials for the Laplacian and the Helmholtz operator.
- Applications:
  - Diffractive gratings;
  - Photonic and phononic crystals;
  - Metasurfaces;
  - Metamaterials.

Periodic and quasi-periodic Green's functions

- $G_\sharp$: Periodic Green's function for the one-dimensional grating in $\mathbb{R}^2$:
  - $G_\sharp : \mathbb{R}^2 \to \mathbb{C}$:

$$\Delta G_\sharp(x) = \sum_{n\in\mathbb{Z}} \delta_0(x + (n,0)).$$

  - Explicit formula: $x = (x_1, x_2)$,

$$G_\sharp(x) = \frac{1}{4\pi} \ln \big(\sinh^2(\pi x_2) + \sin^2(\pi x_1) \big).$$

Periodic and quasi-periodic Green's functions

- Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} \delta_0(x_1+n) = \sum_{n\in\mathbb{Z}} e^{i2\pi n x_1}.$$

- $\Rightarrow$

$$\begin{aligned}\Delta G_\sharp(x) &= \sum_{n\in\mathbb{Z}} \delta_0(x+(n,0)) \\ &= \sum_{n\in\mathbb{Z}} \delta_0(x_2)\delta_0(x_1+n) \\ &= \sum_{n\in\mathbb{Z}} \delta_0(x_2) e^{i2\pi n x_1}.\end{aligned}$$

- $G_\sharp$: periodic in x_1 of period 1 $\Rightarrow$

$$G_\sharp(x) = \sum_{n\in\mathbb{Z}} \beta_n(x_2) e^{i2\pi n x_1}.$$

$$\Delta G_\sharp(x) = \sum_{n\in\mathbb{Z}} (\beta_n^{''}(x_2) + (i2\pi n)^2 \beta_n(x_2)) e^{i2\pi n x_1}.$$

Periodic and quasi-periodic Green's functions

- ODE:

$$\beta_n''(x_2) + (i2\pi n)^2 \beta_n(x_2) = \delta_0(x_2).$$

- Solution:

$$\begin{aligned} \beta_0(x_2) &= \frac{1}{2}|x_2| + c, \\ \beta_n(x_2) &= \frac{-1}{4\pi|n|} e^{-2\pi|n||x_2|}, \quad n \neq 0; \end{aligned}$$

c: constant.

- Define $c := -\frac{\ln(2)}{2\pi}$ and use the summation identity:

$$\sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{2\pi n} e^{-2\pi n|x_2|} \cos(2\pi n x_1) = \frac{1}{2}|x_2| - \frac{\ln(2)}{2\pi}$$

$$- \frac{1}{4\pi} \ln\left(\sinh^2(\pi x_2) + \sin^2(\pi x_1)\right).$$

Periodic and quasi-periodic Green's functions

- $$
\begin{aligned}
G_\sharp(x) &= \frac{1}{2}|x_2| + c - \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{4\pi|n|} e^{-2\pi|n||x_2|} e^{i2\pi n x_1} \\
&= \frac{1}{2}|x_2| + c - \sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{2\pi n} e^{-2\pi n|x_2|} \cos(2\pi n x_1) \\
&= \frac{1}{4\pi} \ln\left(\sinh^2(\pi x_2) + \sin^2(\pi x_1)\right).
\end{aligned}
$$

- Taylor expansion of $G_\sharp$:

$$G_\sharp(x) = \frac{\ln|x|}{2\pi} + R(x);$$

- R: smooth function s.t.

$$R(x) = \frac{1}{4\pi}\ln(1 + O(x_2^2 - x_1^2)).$$

Periodic and quasi-periodic Green's functions

- $G_\sharp(x,y) := G_\sharp(x-y)$; $\Omega \Subset \big(-\frac{1}{2},\frac{1}{2}\big) \times \mathbb{R}$: bounded smooth domain;
- One-dimensional periodic single-layer potential and periodic Neumann–Poincaré operator:

$$\begin{aligned} \mathcal{S}_{\Omega,\sharp} : H^{-\frac{1}{2}}(\partial\Omega) &\longrightarrow H^1_{\mathrm{loc}}(\mathbb{R}^2), H^{\frac{1}{2}}(\partial\Omega) \\ \varphi &\longmapsto \mathcal{S}_{\Omega,\sharp}[\varphi](x) = \int_{\partial\Omega} G_\sharp(x,y)\varphi(y)d\sigma(y) \end{aligned}$$

for $x \in \mathbb{R}^2$ (or $x \in \partial\Omega$);

$$\begin{aligned} \mathcal{K}^*_{\Omega,\sharp} : H^{-\frac{1}{2}}(\partial\Omega) &\longrightarrow H^{-\frac{1}{2}}(\partial\Omega) \\ \varphi &\longmapsto \mathcal{K}^*_{\Omega,\sharp}[\varphi](x) = \int_{\partial\Omega} \frac{\partial G_\sharp(x,y)}{\partial\nu(x)}\varphi(y)d\sigma(y) \end{aligned}$$

for $x \in \partial\Omega$.

Periodic and quasi-periodic Green's functions

- Symmetrization of the periodic Neumann–Poincaré operator $\mathcal{K}^*_{\Omega,\sharp}$:
  - For any $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\mathcal{S}_{\Omega,\sharp}[\varphi]$: harmonic in Ω and in $\big(-\dfrac{1}{2}, \dfrac{1}{2}\big) \times \mathbb{R} \setminus \overline{\Omega}$;
  - Trace formula: For any $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)$,
$$(-\frac{1}{2}I + \mathcal{K}^*_{\Omega,\sharp})[\varphi] = \frac{\partial \mathcal{S}_{\Omega,\sharp}[\varphi]}{\partial \nu}\Big|_{-};$$
  - Calderón identity: $\mathcal{K}_{\Omega,\sharp}\mathcal{S}_{\Omega,\sharp} = \mathcal{S}_{\Omega,\sharp}\mathcal{K}^*_{\Omega,\sharp}$; $\mathcal{K}_{\Omega,\sharp}$: L^2-adjoint of $\mathcal{K}^*_{\Omega,\sharp}$;
  - $\mathcal{K}^*_{\Omega,\sharp} : H_0^{-\frac{1}{2}}(\partial\Omega) \to H_0^{-\frac{1}{2}}(\partial\Omega)$: compact self-adjoint equipped with the inner product:
$$\langle u, v\rangle_{\mathcal{H}_0^*} = -\langle \mathcal{S}_{\Omega,\sharp}[v], u\rangle_{\frac{1}{2},-\frac{1}{2}}$$
  - (λ_j, φ_j), $j = 1, 2, \ldots$: eigenvalue and normalized eigenfunction pair of $\mathcal{K}^*_{\Omega,\sharp}$ in $\mathcal{H}_0^*(\partial\Omega)$; $\lambda_j \in (-\frac{1}{2}, \frac{1}{2})$ and $\lambda_j \to 0$ as $j \to \infty$.

Periodic and quasi-periodic Green's functions

- Periodic Green's function:
  - Effective medium properties of subwavelength resonators;
  - Periodic transmission problem for the Laplace operator.
- $Y = (-1/2, 1/2)^d$: unit cell; $\overline{D} \subset Y$.
- Periodic transmission problem: for $p = 1, \ldots, d$,

$$\begin{cases} \nabla \cdot \Big(1 + (k-1)\chi(D)\Big)\nabla u_p = 0 \quad \text{in } Y\,, \\ u_p - x_p \quad \text{periodic (in each direction) with period } 1\,, \\ \displaystyle\int_Y u_p \, dx = 0\,. \end{cases}$$

- Representation formula for u_p.

Periodic and quasi-periodic Green's functions

- Lattice sum representation of the periodic Green's function:

$$G_\sharp(x) = - \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \frac{e^{i 2\pi n \cdot x}}{4\pi^2 |n|^2}.$$

- In the sense of distributions:

$$\Delta G_\sharp(x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} e^{i 2\pi n \cdot x} = \sum_{n \in \mathbb{Z}^d} e^{i 2\pi n \cdot x} - 1\,;$$

- $G_\sharp$ has mean zero: $\int_Y G_\sharp = 0$.
- Poisson's summation formula:

$$\sum_{n \in \mathbb{Z}^d} e^{i 2\pi n \cdot x} = \sum_{n \in \mathbb{Z}^d} \delta_0(x - n)\,,$$

- $\Rightarrow$

$$\Delta G_\sharp(x) = \sum_{n \in \mathbb{Z}^d} \delta_0(x - n) - 1\,.$$

Periodic and quasi-periodic Green's functions

- There exists a smooth function $R_d(x)$ in the unit cell Y s.t.

$$G_\sharp(x) = \begin{cases} \dfrac{1}{2\pi} \ln|x| + R_2(x), & d = 2\,, \\ \dfrac{1}{(2-d)\omega_d} \dfrac{1}{|x|^{d-2}} + R_d(x), & d \geq 3\,. \end{cases}$$

- Taylor's formula expansion of $R_d(x)$ at 0 for $d \geq 2$:

$$R_d(x) = R_d(0) - \frac{1}{2d}(x_1^2 + \ldots + x_d^2) + O(|x|^4)\,.$$

Periodic and quasi-periodic Green's functions

- Periodic single-layer potential of $\phi \in L^2_0(\partial\Omega)$:

$$\mathcal{S}^0_{\Omega,\sharp}[\phi](x) := \int_{\partial\Omega} G_\sharp(x-y)\phi(y)\, d\sigma(y), \quad x \in \mathbb{R}^2 .$$

- Behaviors at the boundary: $\phi \in L^2_0(\partial\Omega)$,

$$\frac{\partial}{\partial\nu}\mathcal{S}^0_{\Omega,\sharp}[\phi]\Big|_\pm (x) = (\pm\frac{1}{2}I + (\mathcal{K}^0_{\Omega,\sharp})^*)[\phi](x) \text{ on } \partial\Omega\,;$$

- $(\mathcal{K}^0_{\Omega,\sharp})^* : L^2_0(\partial\Omega) \to L^2_0(\partial\Omega)$:

$$(\mathcal{K}^0_{\Omega,\sharp})^*[\phi](x) = \text{p.v.} \int_{\partial\Omega} \frac{\partial}{\partial\nu(x)} G_\sharp(x-y)\phi(y)\, d\sigma(y), \quad x \in \partial D\,.$$

- If $\phi \in L^2_0(\partial\Omega)$, then $\mathcal{S}^0_{\Omega,\sharp}[\phi]$: harmonic in Ω and $Y \setminus \overline{\Omega}$.
- If $|\lambda| \geq \frac{1}{2}$, then $\lambda I - (\mathcal{K}^0_{\Omega,\sharp})^*$: invertible on $L^2_0(\partial\Omega)$.

Periodic and quasi-periodic Green's functions

- Representation formula for the solution of the periodic transmission problem: $u_p, \ p = 1, \ldots, d$,

$$u_p(x) = x_p + C_p + \mathcal{S}^0_{\Omega,\sharp}\Big(\frac{k+1}{2(k-1)} I - (\mathcal{K}^0_{\Omega,\sharp})^*\Big)^{-1}[\nu_p](x) \quad \text{in } Y\,;$$

- C_p: constant and ν_p: p-component of the outward unit normal ν to $\partial\Omega$.

Periodic and quasi-periodic Green's functions

- $u_p, p = 1, \ldots, d$, satisfies

$$\begin{cases} \Delta u_p = 0 \quad \text{in } \Omega \cup (Y \setminus \overline{\Omega}), \\ u_p|_+ - u_p|_- = 0 \quad \text{on } \partial\Omega, \\ \displaystyle \frac{\partial u_p}{\partial \nu}\Big|_+ - k \frac{\partial u_p}{\partial \nu}\Big|_- = 0 \quad \text{on } \partial\Omega, \\ u_p - x_p \ \text{ periodic with period } 1, \\ \displaystyle \int_Y u_p \, dx = 0. \end{cases}$$

- Define $V_p(x) = \mathcal{S}^0_{\Omega,\sharp}((\frac{k+1}{2(k-1)} I - (\mathcal{K}^0_{\Omega,\sharp})^*)^{-1}[\nu_p](x) \quad \text{in } Y.$

$$\begin{cases} \Delta V_p = 0 \ \text{ in } \ \Omega \cup (Y \setminus \overline{D}), \\ V_p|_+ - V_p|_- = 0 \ \text{ on } \ \partial\Omega, \\ \displaystyle \frac{\partial V_p}{\partial \nu}|_+ - k \frac{\partial V_p}{\partial \nu}|_- = (k-1)\nu_p \ \text{ on } \partial\Omega, \\ V_p \text{ periodic with period } 1. \end{cases}$$

Periodic and quasi-periodic Green's functions

- Choose C_p s.t. $\int_Y u_p \, dx = 0$.
- General periodic lattice in two dimensions:
  - $r_n = n_1 a^{(1)} + n_2 a^{(2)}, n = (n_1, n_2) \in \mathbb{Z}^2$.
  - $a^{(1)}$ and $a^{(2)}$ determine the unit cell $Y := \{ s a^{(1)} + t a^{(2)}, s, t \in (-1/2, 1/2) \}$ of the array.
  - Reciprocal vector of r_n: $k_n \cdot a^{(i)} = n_i, i = 1, 2$.
  - Periodic Green's function of the Laplacian:

$$\begin{cases} \Delta G_\sharp^a = \displaystyle\sum_{n \in \mathbb{Z}^2} \delta_0(x - r_n) - \frac{1}{|Y|}, \\ G_\sharp^a(x + r_n) = G_\sharp^a(x), \quad \forall \, n \in \mathbb{Z}^2. \end{cases}$$

Periodic and quasi-periodic Green's functions

- Rotate and scale the given lattice in order to satisfy $a^{(1)} = (1,0)$ and $a^{(2)} = (a, b)$ with $b > 0 \Rightarrow$

$$r_n = n_1(1,0) + n_2(a,b), \quad k_n = n_1(1, -\frac{a}{b}) + n_2(0, \frac{1}{b}), \quad n = (n_1, n_2) \in \mathbb{Z}^2.$$

- Lattice sum representation of $G_\sharp^a$:

$$G_\sharp^a(x) = -\sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{i2\pi(n_1 x_1 + (-\frac{a}{b} n_1 + \frac{1}{b} n_2) x_2)}}{4\pi^2 \left(n_1^2 + (-\frac{a}{b} n_1 + \frac{1}{b} n_2)^2\right)^2}.$$

Periodic and quasi-periodic Green's functions

- Quasi-periodic Green's functions:
  - For $\alpha \in (0, 2\pi)^d$, a function u: α-quasi-periodic if $e^{-i\alpha\cdot x}u$: periodic.
  - Lattice sum representation of quasi-periodic Green's function:

$$G_\alpha(x) = -\sum_{n\in\mathbb{Z}^d} \frac{e^{i(2\pi n+\alpha)\cdot x}}{|2\pi n + \alpha|^2}, \quad \alpha \in (0, 2\pi)^d.$$

  - $e^{-i\alpha\cdot x}G_\alpha(x)$: periodic in $\mathbb{R}^d$.

$$\Delta G_\alpha(x) = \sum_{n\in\mathbb{Z}^d} \delta_0(x-n)e^{i\alpha\cdot n} \quad \text{in } \mathbb{R}^d,$$

$$\left(\Delta + i\alpha\cdot\nabla - |\alpha|^2\right)\left(e^{-i\alpha\cdot x}G_\alpha(x)\right) = \sum_{n\in\mathbb{Z}^d} \delta_0(x-n) \quad \text{in } \mathbb{R}^d.$$

Periodic and quasi-periodic Green's functions

- $\mathcal{S}^0_{\Omega,\alpha}, \mathcal{D}^0_{\Omega,\alpha}$, and $(\mathcal{K}^0_{\Omega,\alpha})^*$: α-quasi-periodic single- and double-layer potentials and the α-quasi-periodic Neumann–Poincaré operator associated with G_α.
- $\alpha \in (0, 2\pi)^2$; $(\mathcal{K}^0_{\Omega,\alpha})^* : H_0^{-\frac{1}{2}}(\partial\Omega) \to H_0^{-\frac{1}{2}}(\partial\Omega)$: compact self-adjoint equipped with the following inner product

$$\langle u, v\rangle_{\mathcal{H}_0^*} = -\langle \mathcal{S}^0_{\Omega,\alpha}[v], u\rangle_{\frac{1}{2},-\frac{1}{2}}$$

- $(\lambda_{j,\alpha}, \varphi_{j,\alpha})$, $j = 1, 2, \ldots$: eigenvalue and normalized eigenfunction pair of $(\mathcal{K}^0_{\Omega,\alpha})^*$ in $\mathcal{H}_0^*(\partial\Omega)$, then $\lambda_{j,\alpha} \in (-\frac{1}{2}, \frac{1}{2})$ and $\lambda_{j,\alpha} \to 0$ as $j \to \infty$.
- Ewald's method: computing periodic and quasi-periodic Green's functions (series slowly converge).

Periodic and quasi-periodic Green's functions

- Quasi-periodic layer potentials for the Helmholtz equation:
  - α: quasi-momentum variable in the Brillouin zone $B = [0, 2\pi)^2$.
  - Two-dimensional quasi-periodic Green's function $G^{\alpha,\omega}$:

$$(\Delta + \omega^2) G^{\alpha,\omega}(x, y) = \sum_{n \in \mathbb{Z}^2} \delta_0(x - y - n) e^{in \cdot \alpha}.$$

  - If $\omega \neq |2\pi n + \alpha|, \forall\, n \in \mathbb{Z}^2$, Poisson's summation formula:

$$\sum_{n \in \mathbb{Z}^2} e^{i(2\pi n + \alpha) \cdot x} = \sum_{n \in \mathbb{Z}^2} \delta_0(x - n) e^{in \cdot \alpha}.$$

  - $\Rightarrow$ $G^{\alpha,\omega}$ can be represented as a sum of augmented plane waves over the reciprocal lattice:

$$G^{\alpha,\omega}(x, y) = \sum_{n \in \mathbb{Z}^2} \frac{e^{i(2\pi n + \alpha) \cdot (x - y)}}{\omega^2 - |2\pi n + \alpha|^2}.$$

Periodic and quasi-periodic Green's functions

- Representation of $G^{\alpha,\omega}$ as a sum of images:

$$G^{\alpha,\omega}(x,y) = -\frac{i}{4}\sum_{n\in\mathbb{Z}^2} H_0^{(1)}(\omega|x-n-y|)e^{in\cdot\alpha};$$

- $H_0^{(1)}$: Hankel function of the first kind of order 0.
- Series in the spatial representation of $G^{\alpha,\omega}$ converges uniformly for x, y in compact sets of $\mathbb{R}^2$ and $\omega \neq |2\pi n + \alpha|$ for all $n \in \mathbb{Z}^2$.
- $H_0^{(1)}(z) = (2i/\pi)\ln z + O(1)$ as $z \to 0 \Rightarrow G^{\alpha,\omega}(x,y) - (1/2\pi)\ln|x-y|$: smooth for all $x, y \in Y$.
- Disadvantage of the spectral representation of the Green's function: singularity as $|x-y| \to 0$ is not explicit.

Periodic and quasi-periodic Green's functions

- Assumption: $\omega \neq |2\pi n + \alpha|$ for all $n \in \mathbb{Z}^2$.
- D: bounded smooth domain in $\mathbb{R}^2$; ν: unit outward normal to ∂D.
- For $\omega > 0$; $\mathcal{S}^{\alpha,\omega}$ and $\mathcal{D}^{\alpha,\omega}$: quasi-periodic single- and double-layer potentials. associated with $G^{\alpha,\omega}$ on D;
- Given density $\varphi \in L^2(\partial D)$,

$$\mathcal{S}^{\alpha,\omega}[\varphi](x) = \int_{\partial D} G_\omega^\alpha(x,y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^2,$$

$$\mathcal{D}^{\alpha,\omega}[\varphi](x) = \int_{\partial D} \frac{\partial G_\omega^\alpha(x,y)}{\partial \nu(y)}\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D.$$

- $\mathcal{S}^{\alpha,\omega}[\varphi]$ and $\mathcal{D}^{\alpha,\omega}[\varphi]$ satisfy $(\Delta + \omega^2)\mathcal{S}^{\alpha,\omega}[\varphi] = (\Delta + \omega^2)\mathcal{D}^{\alpha,\omega}[\varphi] = 0$ in D and $Y \setminus \overline{D}$.
- $\mathcal{S}^{\alpha,\omega}[\varphi]$ and $\mathcal{D}^{\alpha,\omega}[\varphi]$: α-quasi-periodic.

Periodic and quasi-periodic Green's functions

- Jump relations: $\varphi \in L^2(\partial D)$,

$$\frac{\partial (\mathcal{S}^{\alpha,\omega}[\varphi])}{\partial \nu}\Big|_{\pm}(x) = \Big(\pm \frac{1}{2} I + (\mathcal{K}^{-\alpha,\omega})^* \Big)[\varphi](x) \quad \text{a.e. } x \in \partial D,$$

$$(\mathcal{D}^{\alpha,\omega}[\varphi])\Big|_{\pm}(x) = \Big(\mp \frac{1}{2} I + \mathcal{K}^{\alpha,\omega} \Big)[\varphi](x) \quad \text{a.e. } x \in \partial D,$$

- $\mathcal{K}^{\alpha,\omega}$:

$$\mathcal{K}^{\alpha,\omega}[\varphi](x) = \text{p.v.} \int_{\partial D} \frac{\partial G^{\alpha,\omega}(x,y)}{\partial \nu(y)} \varphi(y)\, d\sigma(y)$$

- $(\mathcal{K}^{-\alpha,\omega})^*$: L^2-adjoint operator of $\mathcal{K}^{-\alpha,\omega}$,

$$(\mathcal{K}^{-\alpha,\omega})^*[\varphi](x) = \text{p.v.} \int_{\partial D} \frac{\partial G^{\alpha,\omega}(x,y)}{\partial \nu(x)} \varphi(y)\, d\sigma(y).$$

- $\mathcal{K}^{\alpha,\omega}$ and $(\mathcal{K}^{-\alpha,\omega})^*$: compact on $L^2(\partial D)$;
- $G^{\alpha,\omega}(x,y) - (1/2\pi) \ln |x-y|$: smooth for all x, y.

Periodic and quasi-periodic Green's functions

- Assumption: $\alpha \neq 0$ and ω^2: neither an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition on ∂D nor in $Y \setminus \overline{D}$ with the Dirichlet boundary condition on ∂D and the α-quasi-periodic condition on ∂Y.
- $\mathcal{S}^{\alpha,\omega} : L^2(\partial D) \to H^1(\partial D)$: invertible.

Periodic and quasi-periodic Green's functions

- Suppose that $\phi \in L^2(\partial D)$ satisfies $\mathcal{S}^{\alpha,\omega}[\phi] = 0$ on ∂D.
- Then $u = \mathcal{S}^{\alpha,\omega}[\phi]$ satisfies $(\Delta + \omega^2)u = 0$ in D and in $Y \setminus \overline{D}$.
- ω^2: neither an eigenvalue of $-\Delta$ in D with the Dirichlet boundary condition nor in $Y \setminus \overline{D}$ with the Dirichlet boundary condition on ∂D and the quasi-periodic condition on ∂Y $\Rightarrow$ $u = 0$ in Y.
- $\phi = \partial u/\partial\nu|_+ - \partial u/\partial\nu|_- = 0$.

Periodic and quasi-periodic Green's functions

- Define

$$G^{\alpha,0}(x,y) := G_\alpha(x-y) = -\sum_{n\in\mathbb{Z}^2} \frac{e^{i(2\pi n+\alpha)\cdot(x-y)}}{|2\pi n+\alpha|^2} \quad \text{for } \alpha \neq 0.$$

- For $\alpha = 0$:

$$G^{0,0}(x,y) := G_\sharp(x-y) = -\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{e^{i2\pi n\cdot(x-y)}}{4\pi^2|n|^2},$$

- $G^{0,0}(x,y)$ satisfies

$$\Delta_x G^{0,0}(x,y) = \delta_y - 1 \quad \text{in } Y$$

with periodic Dirichlet boundary conditions on ∂Y.

Periodic and quasi-periodic Green's functions

- As $\omega \to 0$, $G^{\alpha,\omega}$ can be decomposed as

$$G^{\alpha,\omega}(x,y) = G^{\alpha,0}(x,y) - \sum_{l=1}^{+\infty} \underbrace{\omega^{2l} \sum_{n \in \mathbb{Z}^2} \frac{e^{i(2\pi n+\alpha)\cdot(x-y)}}{|2\pi n + \alpha|^{2(l+1)}}}_{:= -G_l^{\alpha,\omega}(x,y)},$$

  for $\alpha \neq 0$;
- For $\alpha = 0$:

$$G^{0,\omega}(x,y) = \frac{1}{\omega^2} + G^{0,0}(x,y) - \sum_{l=1}^{+\infty} \underbrace{\omega^{2l} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{i2\pi n\cdot(x-y)}}{(4\pi^2)^{l+1}|n|^{2(l+1)}}}_{:= -G_l^{0,\omega}(x,y)}.$$

Periodic and quasi-periodic Green's functions

- $\mathcal{S}_l^{\alpha,\omega}$ and $(\mathcal{K}_l^{-\alpha,\omega})^*$, for $l \geq 0$ and $\alpha \in [0, 2\pi)^2$, layer potentials associated with the kernel $G_l^{\alpha,\omega}(x, y)$:

$$\mathcal{S}^{\alpha,\omega} = \mathcal{S}^{\alpha,0} + \sum_{l=1}^{+\infty} \mathcal{S}_l^{\alpha,\omega} \quad \text{and} \quad (\mathcal{K}^{\alpha,\omega})^* = (\mathcal{K}^{\alpha,0})^* + \sum_{l=1}^{+\infty} (\mathcal{K}_l^{-\alpha,\omega})^*.$$

- $(1/2)\, I + (\mathcal{K}^{-\alpha,0})^* : L^2(\partial D) \rightarrow L^2(\partial D)$: invertible.

Periodic and quasi-periodic Green's functions

- u and v: α-quasi-periodic smooth functions $\Rightarrow$

$$\int_{\partial Y} \frac{\partial u}{\partial \nu} \overline{v} \, d\sigma = 0.$$

$\Leftarrow$

$$\int_{\partial Y} \frac{\partial u}{\partial \nu} \overline{v} = \int_{\partial Y} \left[\frac{\partial (u e^{-i\alpha \cdot x})}{\partial \nu} + i\alpha \cdot \nu u e^{-i\alpha \cdot x} \right] \overline{e^{-i\alpha \cdot x} v}.$$

Periodic and quasi-periodic Green's functions

- $\phi \in L^2(\partial D)$ satisfy $((1/2)\,I + (\mathcal{K}^{-\alpha,0})^*)[\phi] = 0$ on ∂D.
- If $\alpha = 0$, then $\int_{\partial D} \phi = 0$.
- For $x \in D$

$$\mathcal{D}^{0,0}[1](x) = -\int_{Y \setminus \overline{D}} \Delta_y G^{0,0}(x,y)dy = |Y \setminus \overline{D}|,$$

- $|\ \ |$: volume.
- $$(\frac{1}{2}I + \mathcal{K}^{0,0})[1] = |Y \setminus \overline{D}| \quad \text{on } \partial D.$$

$$|Y \setminus \overline{D}| \int_{\partial D} \phi \, d\sigma = \int_{\partial D} (\frac{1}{2}I + \mathcal{K}^{0,0})[1]\,\phi \, d\sigma = \int_{\partial D} (\frac{1}{2}I + (\mathcal{K}^{0,0})^*)[\phi]\, d\sigma = 0.$$

Periodic and quasi-periodic Green's functions

- For any $\alpha \in [0, 2\pi)^2$, $u = \mathcal{S}^{\alpha,0}[\phi]$ is α-quasi-periodic and satisfies $\Delta u = 0$ in $Y \setminus \overline{D}$ with

$$\left.\frac{\partial u}{\partial \nu}\right|_+ = (\frac{1}{2} I + (\mathcal{K}^{-\alpha,0})^*)[\phi] = 0 \quad \text{on } \partial D.$$

- $$\int_{Y \setminus \overline{D}} |\nabla u|^2 = \int_{\partial Y} \frac{\partial u}{\partial \nu} \overline{u} - \int_{\partial D} \left.\frac{\partial u}{\partial \nu}\right|_+ \overline{u} = 0.$$

- u: constant in $Y \setminus \overline{D}$ and hence in $D \Rightarrow$

$$\phi = \left.\frac{\partial u}{\partial \nu}\right|_+ - \left.\frac{\partial u}{\partial \nu}\right|_- = 0.$$