Mathematical and Computational Methods in Photonics

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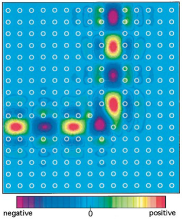
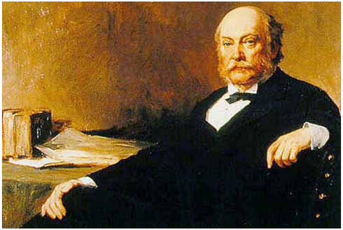
Mathematics for photonics

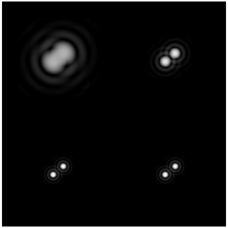
- Control, manipulate, reshape, guide, focus electromagnetic waves at sub-wavelength length scales (beyond the resolution limit).
- Direct, inverse, and optimal design problems for electromagnetic wave propagation in complex and resonant media.
- Build mathematical frameworks and develop effective numerical algorithms for photonic applications.
- Partial differential equations, spectral analysis, integral equations, computational techniques, and multi-scale analysis.



Resonances for plasmonic nanoparticles

- Key to super-resolution: push the resolution limit by reducing the focal spot size; confine light to a length scale significantly smaller than half the wavelength.
- Resolution: smallest detail that can be resolved.



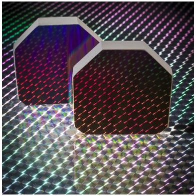


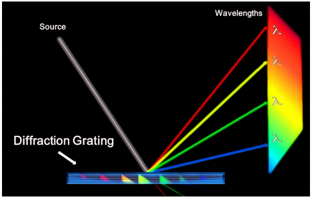
Mathematics for photonics

- Mathematical and computational tools:
  - Diffraction gratings;
  - Photonic crystals;
  - Plasmonic resonant nanoparticles;
  - Metamaterials and metasurfaces.

Mathematics for photonics

- Diffraction gratings:
  - Scattering by periodic structures: dominated by diffraction; small features of the structure → small number of propagating modes (other modes are evanescent).
  - Spectroscopic, telecommunications and laser applications.
  - Design problem: grating profile that give rise to a specified diffraction pattern.



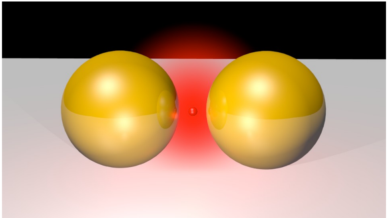


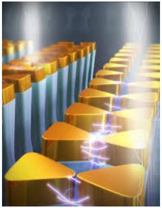
Mathematics for photonics

- Photonic crystals (also known as photonic band-gap materials):
  - Periodic dielectric structures that have a band gap that forbids propagation of a certain frequency range of light.
  - Band gap calculations: high-contrast materials, periodicity of the same order as the wavelength; efficient numerical schemes.
  - Control light and produce effects that are impossible with conventional optics.
  - Resonant cavities: making point defects in a photonic crystal $\rightarrow$ light can be localized, trapped in the defect. The frequency, symmetry, and other properties of the defect mode can be easily tuned to anything desired.

Mathematics for photonics

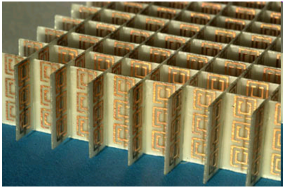
- Plasmonic nanoparticles:
  - Sub-wavelength resonance: quasi-static regime.
  - Scattering and absorption enhancement.
  - Super-resolution: single particle imaging.
  - Nanoantenna, concentrate light at sub-wavelength scale.

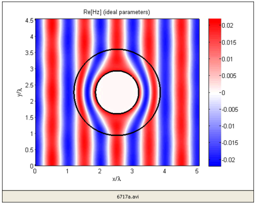




Mathematics for photonics

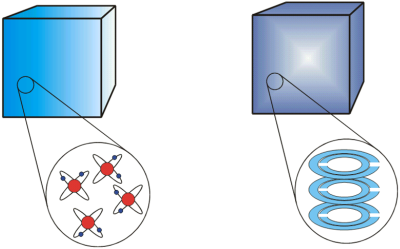
- Metamaterials and metasurfaces:
  - Negative material parameters.
  - Electromagnetic invisibility and cloaking: make a target invisible when probed by electromagnetic waves:
    - Interior cloaking: scattering cancellations techniques.
    - Exterior cloaking by anomalous resonances.
  - Sub-wavelength band gap materials: microstructure periodicity smaller than the wavelength.





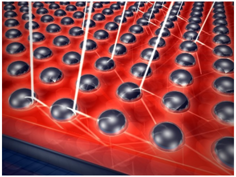
Mathematics for photonics

- Metamaterials and metasurfaces:
  - Microstructured materials.
  - Building block microstructure: sub-wavelength resonator.



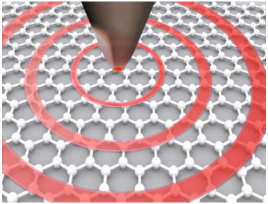
Mathematics for photonics

- Effective medium theory:
  - High contrast materials: for some range of frequencies.
  - Super-resolution and super-focusing of electromagnetic waves.
- Unify the mathematical theory of super-resolution, photonic bandgap materials, metamaterials, and cloaking.



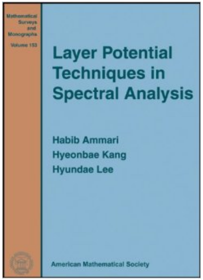
Mathematics for photonics

- Near-field optics:
  - Interaction between the plasmonic probe and the sample.
  - Super-resolution imaging of the sample.
  - Mechanism $\rightarrow$ quantitative imaging.



Mathematics for photonics

- Spectral analysis and integral equation formulations.
- Green's functions (free space, periodic, quasi-periodic, ...) $\rightarrow$ eigenvalue problems reduced to characteristic value problems (nonlinear eigenvalue problems).
- Gohberg-Sigal theory:
  - Generalization of Rouché theorem for operator valued function.
  - Sensitivity analysis (change in the shape, material parameters, environment, . . .) of diffraction pattern, band gaps, resonance for plasmonic nanoparticles, . . .



Mathematics for photonics

- 2014 Kavli Prize in Nanoscience (Norwegian Academy of Science & Letters): T.W. Ebbesen, S.W. Hell, and J.B. Pendry.
- "for their transformative contributions to the field of nano-optics that have broken long-held beliefs about the limitations of the resolution limits of optical microscopy and imaging.
  - "for the discovery of the extraordinary transmission of light through sub-wavelength apertures.
  - "for ground-breaking developments that have led to fluorescence microscopy with nanometre scale resolution, opening up nanoscale imaging to biological applications.
  - "for developing the theory underlying new optical nanoscale materials with unprecedented properties, such as the negative index of refraction, allowing for the formation of perfect lenses.



Mathematics for photonics

- Phononics:
  - Sound /light.
  - Elasticity equations/ Maxwell's equations.
  - Sub-wavelength resonances: Helmholtz resonator, Minnaert bubble/ plasmonic nanoparticle.
- Similar physical mechanisms and mathematical and computational frameworks to those in photonics:
  - Scattering enhancement by sub-wavelength acoustic resonators.
  - Phononic crystals.
  - Acoustic metamaterials and metasurfaces, sub-wavelength phononic band gap materials.
  - High contrast acoustic materials, super-resolution and super-focusing for acoustic waves.

Mathematics for photonics

- Gohberg-Sigal theory:
  - Argument principle: $V \subset \mathbb{C}$: bounded domain with smooth boundary ∂V positively oriented; $f(z)$: meromorphic function in a neighborhood of $\overline{V}$; P and N: the number of poles and zeros of f in V, counted with their multiplicities. If f has no poles and never vanishes on ∂V, then

$$\frac{1}{2\pi i} \int_{\partial V} \frac{f'(z)}{f(z)}\, dz = N - P.$$

  - Rouché's theorem: $f(z)$ and $g(z)$: holomorphic in a neighborhood of $\overline{V}$. If $|f(z)| > |g(z)|$ for all $z \in \partial V$, then f and $f+g$ have the same number of zeros in V.

Mathematics for photonics

- $\mathcal{L}(\mathcal{B}, \mathcal{B}')$: linear bounded operators from $\mathcal{B}$ into $\mathcal{B}'$ (Banach spaces).
- $\mathfrak{U}(z_0)$: set of all operator-valued functions in $\mathcal{L}(\mathcal{B}, \mathcal{B}')$ which are holomorphic in some neighborhood of z_0, except possibly at z_0.
- z_0 characteristic value of $A(z) \in \mathfrak{U}(z_0)$ if there exists a vector-valued function $\phi(z)$ with values in $\mathcal{B}$ such that
  - $\phi(z)$: holomorphic at z_0 and $\phi(z_0) \neq 0$,
  - $A(z)\phi(z)$: holomorphic at z_0 and vanishes at this point.
  - $\phi(z)$: root function of $A(z)$ associated with the characteristic value z_0.

Mathematics for photonics

- Generalized argument principle:

$$\mathcal{M}(A(z);\partial V) = \frac{1}{2\pi i}\mathrm{tr}\int_{\partial V} A^{-1}(z)\frac{d}{dz}A(z)dz.$$

- $\mathcal{M}(A(z);\partial V)$: number of characteristic values of $A(z)$ in V, counted with their multiplicities, minus the number of poles of $A(z)$ in V, counted with their multiplicities.
- Generalized Rouché's theorem :

$$\mathcal{M}(A(z);\partial V) = \mathcal{M}(A(z)+S(z);\partial V).$$

- $S(z)$: finitely meromorphic in V and continuous on ∂V s.t.

$$\|A^{-1}(z)S(z)\|_{\mathcal{L}(\mathcal{B},\mathcal{B})} < 1, \quad z \in \partial V.$$

- Finitely meromorphic operator: coefficients of the principal part of its Laurent expansion are operators of finite rank.

Mathematics for photonics

- $0 = \mu_1 < \mu_2 \leq \ldots$: eigenvalues of $-\Delta$ in Ω with Neumann conditions,

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

- $(u_j)_{j\geq 1}$: orthonormal basis of $L^2(\Omega)$ of normalized eigenvectors.
- $\omega = \sqrt{\mu}$; $\mathcal{S}_\Omega^\omega, \mathcal{D}_\Omega^\omega, \mathcal{K}_\Omega^\omega$: single- and double-layer potentials and Neumann-Poincaré operator associated with the outgoing fundamental solution $G_\omega(x,z)$ to the Helmholtz operator $\Delta + \omega^2$:

$$G_\omega(x,z) := \begin{cases} -\frac{i}{4} H_0^{(1)}(\omega|x-z|), & d = 2, \\ -\dfrac{e^{i|x-z|}}{4\pi|x-z|}, & d = 3. \end{cases}$$

- $H_0^{(1)}$: Hankel function of the first kind of order 0.

Mathematics for photonics

- Sommerfeld radiation condition: $|x| \to +\infty$,

$$\frac{x}{|x|} \cdot \nabla G_\omega(x,z) - i\omega G_\omega(x,z) = \begin{cases} O(|x|^{-3/2}), & d=2, \\ O(|x|^{-2}), & d=3. \end{cases}$$

- Layer potentials: $\varphi \in L^2(\partial\Omega)$,

$$\mathcal{S}_\Omega^\omega[\varphi](x) = \int_{\partial\Omega} G_\omega(x,y)\varphi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d,$$
$$\mathcal{D}_\Omega^\omega[\varphi](x) = \int_{\partial\Omega} \frac{\partial G_\omega(x,y)}{\partial\nu(y)}\varphi(y)\, d\sigma(y)\ , \quad x \in \mathbb{R}^d \setminus \partial\Omega,$$
$$\mathcal{K}_\Omega^\omega[\varphi](x) = \text{p.v.} \int_{\partial\Omega} \frac{\partial G_\omega(x,y)}{\partial\nu(y)}\varphi(y)\, d\sigma(y).$$

- $\sqrt{\mu_j}$: characteristic value of $\omega \mapsto (1/2)I - \mathcal{K}_\Omega^\omega$.
- Muller's method: compute zeros of $\omega \mapsto 1/(((1/2)I - \mathcal{K}_\Omega^\omega)^{-1}[\varphi], \psi)$ for fixed φ and ψ.

Mathematics for photonics

- D conductive particle inside Ω, $D = \varepsilon B + z$; $k \neq 1$: conductivity parameter; ε: characteristic size; d: space dimension.
- Characteristic values of the operator-valued function $\mathcal{A}_\varepsilon(\omega)$:

$$\omega \mapsto \mathcal{A}_\varepsilon(\omega) := \begin{pmatrix} \frac{1}{2}I - \mathcal{K}_\Omega^\omega & -\mathcal{S}_D^\omega & 0 \\ \mathcal{D}_\Omega^\omega & \mathcal{S}_D^\omega & -\mathcal{S}_D^{\frac{\omega}{\sqrt{k}}} \\ \varepsilon \frac{\partial}{\partial \nu} \mathcal{D}_\Omega^\omega & \varepsilon(\frac{1}{2}I + (\mathcal{K}_D^\omega)^*) & -\varepsilon k(-\frac{1}{2}I + (\mathcal{K}_D^{\frac{\omega}{\sqrt{k}}})^*) \end{pmatrix}.$$

- Generalized argument principle:

$$\omega_\varepsilon - \omega_0 = \frac{1}{2\pi i} \operatorname{tr} \int_{\partial V_{\delta_0}} (\omega - \omega_0) \mathcal{A}_\varepsilon(\omega)^{-1} \frac{d}{d\omega} \mathcal{A}_\varepsilon(\omega) d\omega.$$

Mathematics for photonics

- Eigenvalue expansion:

$$\mu_j^\varepsilon - \mu_j = \varepsilon^d \nabla u_j(z) \cdot M \nabla u_j(z) + o(\epsilon^d).$$

- Polarization tensor $M = (m_{ll'})$:

$$m_{ll'} = (k-1) \int_{\partial B} \psi_l \frac{\partial x_{l'}}{\partial \nu} \, d\sigma.$$

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi(B)) \nabla \psi_l = 0 & \text{in } \mathbb{R}^d, \\ \psi_l(x) - x_l = O(|x|^{1-d}) & \text{as } |x| \to +\infty. \end{cases}$$

- Eigenfunction expansion in Ω:

$$u_j^\varepsilon(x) = u_j(z) + \varepsilon \sum_{l=1}^{d} \partial_l u_j(z) \psi_l \left(\frac{x-z}{\varepsilon} \right) + o(\varepsilon).$$

- u_j^ε: normalized eigenfunction associated with μ_j^ε.

Mathematics for photonics

- Photonic crystals:
  - Floquet transform:

$$\mathcal{U}[f](x,\alpha) = \sum_{n \in \mathbb{Z}^d} f(x-n) e^{i\alpha\cdot n}.$$

  - $f(x)$: function decaying sufficiently fast.
  - $\mathcal{U}$: analogue of the Fourier transform for the periodic case.
  - $\alpha \in$ Brillouin zone $\mathbb{R}^d/(2\pi\mathbb{Z}^d)$: quasi-momentum (analogue of the dual variable in the Fourier transform).
  - Expansion of a periodic operator L in $L^2(\mathbb{R}^d)$ into a direct integral of operators:

$$L = \int_{\mathbb{R}^d/(2\pi\mathbb{Z}^d)}^{\oplus} L(\alpha)\, d\alpha.$$

  - $L(\alpha)[f] = \mathcal{U}[L[f]]$.

Mathematics for photonics

- Spectral theorem for a self-adjoint operator:

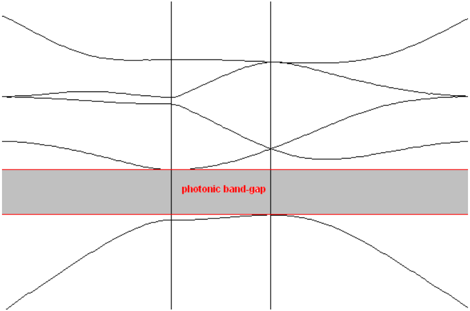
$$\sigma(L) = \bigcup_{\alpha \in \mathbb{R}^d/(2\pi\mathbb{Z}^d)} \sigma(L(\alpha)),$$

- $\sigma(L)$: spectrum of L.
- L: elliptic $\rightarrow$ $L(\alpha)$: compact resolvents $\rightarrow$ discrete spectra $(\mu_l(\alpha))_l$,

$$\sigma(L) = \left[\min_{\alpha} \mu_l(\alpha), \max_{\alpha} \mu_l(\alpha) \right].$$

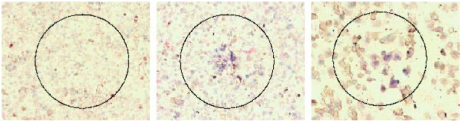
Mathematics for photonics

- Gohberg-Sigal theory:
  - Sensitivity analysis of band gaps with respect to changes of the coefficients of L.
  - Analysis of photonic crystal cavities: defect mode inside the band gap.



Resonances for plasmonic nanoparticles

- Gold nano-particles: accumulate selectively in tumor cells; bio-compatible; reduced toxicity.
- Detection: localized enhancement in radiation dose (strong scattering).
- Ablation: localized damage (strong absorption).
- Functionalization: targeted drugs.



M.A. El-Sayed et al.

Resonances for plasmonic nanoparticles

- Mechanisms of scattering and absorption enhancements and supreresolution using plasmonic nanoparticles.
- Spectral properties of Neumann-Poincaré operator.

Resonances for plasmonic nanoparticles

- D: nanoparticle in $\mathbb{R}^d, d = 2, 3$; $\mathcal{C}^{1,\alpha}$ boundary ∂D, $\alpha > 0$.
- $\varepsilon_c(\omega)$: complex permittivity of D; $\varepsilon_m > 0$: permittivity of the background medium;
- Permittivity contrast: $\lambda(\omega) = (\varepsilon_c(\omega) + \varepsilon_m)/(2(\varepsilon_c(\omega) - \varepsilon_m))$.
- Causality $\Rightarrow$ Kramer-Krönig relations (Hilbert transform), $\varepsilon_c(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$:

$$\varepsilon'(\omega) - \varepsilon_\infty = -\frac{2}{\pi}\text{p.v.} \int_0^{+\infty} \frac{s\varepsilon''(s)}{s^2 - \omega^2} ds,$$

$$\varepsilon''(\omega) = \frac{2\omega}{\pi}\text{p.v.} \int_0^{+\infty} \frac{\varepsilon'(s) - \varepsilon_\infty}{s^2 - \omega^2} ds.$$

- Drude model for the dielectric permittivity $\varepsilon_c(\omega)$:

$$\varepsilon_c(\omega) = \varepsilon_\infty(1 - \frac{\omega_p^2}{\omega^2 + i\tau\omega}), \qquad \varepsilon'(\omega) \leq 0 \quad \text{for} \quad \omega \leq \omega_p.$$

ω_p, τ: positive constants.

Resonances for plasmonic nanoparticles

- Fundamental solution to the Laplacian:

$$G(x) := \begin{cases} \dfrac{1}{2\pi} \ln |x| \,, & d = 2, \\ -\dfrac{1}{4\pi} |x|^{2-d} \,, & d = 3; \end{cases}$$

- Single-layer potential:

$$\mathcal{S}_D[\varphi](x) := \int_{\partial D} G(x-y)\varphi(y)\, ds(y), \quad x \in \mathbb{R}^d.$$

- Neumann-Poincaré operator $\mathcal{K}_D^*$:

$$\mathcal{K}_D^*[\varphi](x) := \int_{\partial D} \frac{\partial G}{\partial \nu(x)}(x-y)\varphi(y)\, ds(y)\,, \quad x \in \partial D.$$

  ν: normal to ∂D.

- $\mathcal{K}_D^*$: compact operator on $L^2(\partial D)$,

$$\frac{|\langle x-y, \nu(x)\rangle|}{|x-y|^d} \leq \frac{C}{|x-y|^{d-1-\alpha}}, \quad x, y \in \partial D.$$

- Spectrum of $\mathcal{K}_D^*$ lies in $(-\frac{1}{2}, \frac{1}{2}]$ (Kellog).

Resonances for plasmonic nanoparticles

- $\mathcal{K}_D^*$ self-adjoint on $L^2(\partial D)$ if and only if D is a disk or a ball.
- Symmetrization technique for Neumann-Poincaré operator $\mathcal{K}_D^*$:
  - Calderón's identity: $\mathcal{K}_D \mathcal{S}_D = \mathcal{S}_D \mathcal{K}_D^*$;
  - In three dimensions, $\mathcal{K}_D^*$: self-adjoint in the Hilbert space $\mathcal{H}^*(\partial D) = H^{-\frac{1}{2}}(\partial D)$ equipped with

$$(u, v)_{\mathcal{H}^*} = -(u, \mathcal{S}_D[v])_{-\frac{1}{2},\frac{1}{2}}$$

    $(\cdot, \cdot)_{-\frac{1}{2},\frac{1}{2}}$: duality pairing between $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$.
  - In two dimensions: $\exists ! \widetilde{\varphi}_0$ s.t. $\mathcal{S}_D[\widetilde{\varphi}_0] =$ constant on ∂D and $(\widetilde{\varphi_0}, 1)_{-\frac{1}{2},\frac{1}{2}} = 1$. $\mathcal{S}_D \to \widetilde{\mathcal{S}}_D$:

$$\widetilde{\mathcal{S}}_D[\varphi] = \begin{cases} \mathcal{S}_D[\varphi] & \text{if } (\varphi, 1)_{-\frac{1}{2},\frac{1}{2}} = 0, \\ -1 & \text{if } \varphi = \widetilde{\varphi}_0. \end{cases}$$

Resonances for plasmonic nanoparticles

- Symmetrization technique for Neumann-Poincaré operator $\mathcal{K}_D^*$:
  - Spectrum $\sigma(\mathcal{K}_D^*)$ discrete in $]-1/2, 1/2[$;
  - Ellipse: $\pm\frac{1}{2}(\frac{a-b}{a+b})^j$, elliptic harmonics ($a, b$: long and short axis).
  - Ball: $\frac{1}{2(2j+1)}$, spherical harmonics.
  - Twin property in two dimensions;
  - (λ_j, φ_j), $j = 0, 1, 2, \ldots$: eigenvalue and normalized eigenfunction pair of $\mathcal{K}_D^*$ in $\mathcal{H}^*(\partial D)$; $\lambda_j \in (-\frac{1}{2}, \frac{1}{2}]$ and $\lambda_j \to 0$ as $j \to \infty$;
  - φ_0: eigenfunction associated to $1/2$ ($\widetilde{\varphi}_0$ multiple of φ_0);
  - Spectral decomposition formula in $H^{-1/2}(\partial D)$,

$$\mathcal{K}_D^*[\psi] = \sum_{j=0}^{\infty} \lambda_j (\psi, \varphi_j)_{\mathcal{H}^*} \varphi_j.$$

Resonances for plasmonic nanoparticles

- u^i: incident plane wave; Helmholtz equation:

$$\begin{cases} \nabla\cdot\Big(\varepsilon_m\chi(\mathbb{R}^d\setminus\bar{D})+\varepsilon_c(\omega)\chi(\overline{D})\Big)\nabla u+\omega^2 u=0, \\ u^s := u-u^i \text{ satisfies the outgoing radiation condition.} \end{cases}$$

- Uniform small volume expansion with respect to the contrast: $D = z+\delta B$, $\delta\to 0$, $|x-z|\gg 2\pi/k_m$,

$$u^s = -M(\lambda(\omega),D)\nabla_z G_{k_m}(x-z)\cdot\nabla u^i(z)+O\Big(\frac{\delta^{d+1}}{\mathrm{dist}(\lambda(\omega),\sigma(\mathcal{K}_D^*))}\Big).$$

- G_{k_m}: outgoing fundamental solution to $\Delta+k_m^2$; $k_m:=\omega/\sqrt{\varepsilon_m}$;
- Polarization tensor:

$$M(\lambda(\omega),D) := \int_{\partial D} x(\lambda(\omega)I-\mathcal{K}_D^*)^{-1}[\nu](x)\,ds(x).$$

- Scaling and translation properties: $M(\lambda(\omega),z+\delta B)=\delta^d M(\lambda(\omega),B)$.

Resonances for plasmonic nanoparticles

Representation by equivalent ellipses and ellipsoids:

- Nanoparticle's permittivity: $\varepsilon_c(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$.
- $\varepsilon'(\omega) > 0$ and $\varepsilon''(\omega) = 0$: canonical representation; equivalent ellipse or ellipsoid with the same polarization tensor.
- Plasmonic nanoparticles: non Hermitian case.
- $\Im M(\lambda(\omega), D)$: equivalent frequency depending ellipse or ellipsoid with the same imaginary part of the polarization tensor.

Resonances for plasmonic nanoparticles

- Spectral decomposition: (l, m)-entry

$$M_{l,m}(\lambda(\omega), D) = \sum_{j=1}^{\infty} \frac{(\nu_m, \varphi_j)_{\mathcal{H}^*}(\nu_l, \varphi_j)_{\mathcal{H}^*}}{(1/2 - \lambda_j)(\lambda(\omega) - \lambda_j)}.$$

- $(\nu_m, \varphi_0)_{\mathcal{H}^*} = 0$; φ_0: eigenfunction of $\mathcal{K}_D^*$ associated to $1/2$.
- Quasi-static far-field approximation: $\delta \to 0$,

$$u^s = -\delta^d M(\lambda(\omega), B) \nabla_z G_{k_m}(x - z) \cdot \nabla u^i(z) + O\Big(\frac{\delta^{d+1}}{\mathrm{dist}(\lambda(\omega), \sigma(\mathcal{K}_D^*))}\Big).$$

- Quasi-static plasmonic resonance: $\mathrm{dist}(\lambda(\omega), \sigma(\mathcal{K}_D^*))$ minimal ($\Re e\, \varepsilon_c(\omega) < 0$).

Resonances for plasmonic nanoparticles

- $M(\lambda(\omega), B) = (\frac{\varepsilon_c(\omega)}{\varepsilon_m} - 1) \int_B \nabla v(y) dy$:

$$\begin{cases} \nabla \cdot \Big(\varepsilon_m \chi(\mathbb{R}^d \setminus \bar{B}) + \varepsilon_c(\omega) \chi(\overline{B}) \Big) \nabla v = 0, \\ v(y) - y \to 0, \qquad |y| \to +\infty. \end{cases}$$

- Corrector v:

$$v(y) = y + \mathcal{S}_B(\lambda(\omega) I - \mathcal{K}_B^*)^{-1}[\nu](y), \quad y \in \mathbb{R}^d.$$

- Inner expansion: $\delta \to 0$, $|x - z| = O(\delta)$,

$$u(x) = u^i(z) + \delta v(\frac{x - z}{\delta}) \cdot \nabla u^i(z) + O(\frac{\delta^2}{\mathrm{dist}(\lambda(\omega), \sigma(\mathcal{K}_D^*))}).$$

- Monitoring of temperature elevation due to nanoparticle heating:

$$\begin{cases} \rho C \frac{\partial T}{\partial t} - \nabla \cdot \tau \nabla T = \frac{\omega}{2\pi} \Im(\varepsilon_c(\omega)) \, |u|^2 \chi(D), \\ T\big|_{t=0} = 0. \end{cases}$$

ρ: mass density; C: thermal capacity; τ: thermal conductivity.

Resonances for plasmonic nanoparticles

- Scattering amplitude:

$$u^s(x) = -ie^{-\frac{\pi i}{4}} \frac{e^{ik_m|x|}}{\sqrt{8\pi k_m |x|}} A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') + o(|x|^{-\frac{1}{2}}),$$

$|x| \to \infty$; θ, θ': incident and scattered directions.

- Scattering cross-section:

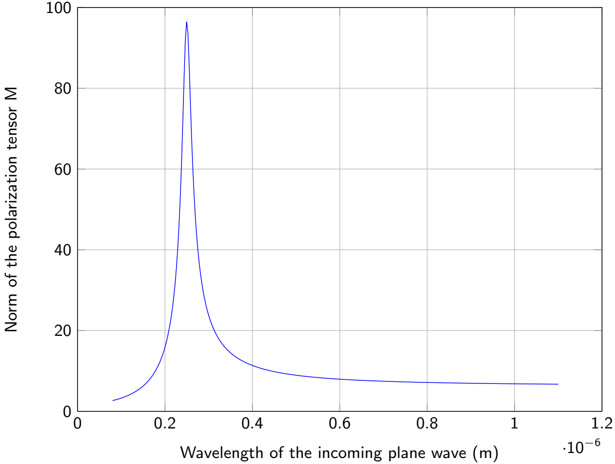
$$Q^s[D, \varepsilon_c, \varepsilon_m, \omega](\theta') := \int_0^{2\pi} \left| A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') \right|^2 d\theta.$$

- Enhancement of the absorption and scattering cross-sections Q^a and Q^s at plasmonic resonances:

$$Q^a + Q^s (= \text{extinction cross-section } Q^e) \propto \Im m \, \mathrm{Trace}(M(\lambda(\omega), D));$$

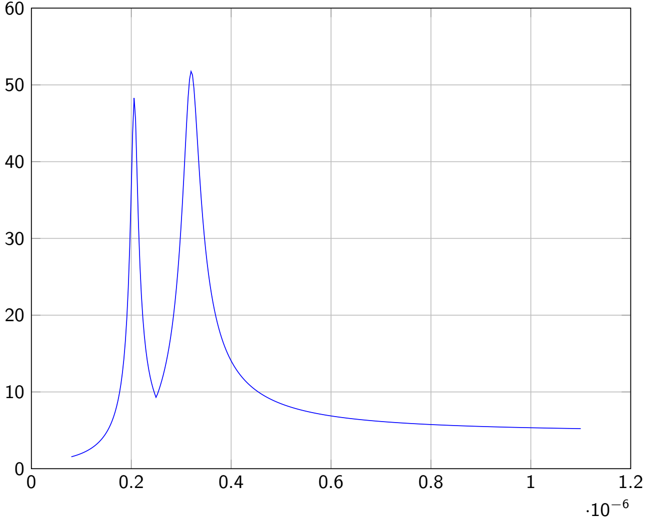
$$Q^s \propto \left| \mathrm{Trace}(M(\lambda(\omega), D)) \right|^2.$$

Resonances for plasmonic nanoparticles



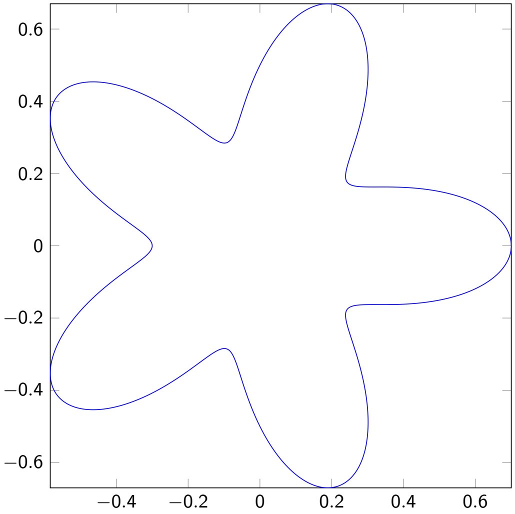
Norm of the polarization tensor for a circular inclusion.

Resonances for plasmonic nanoparticles



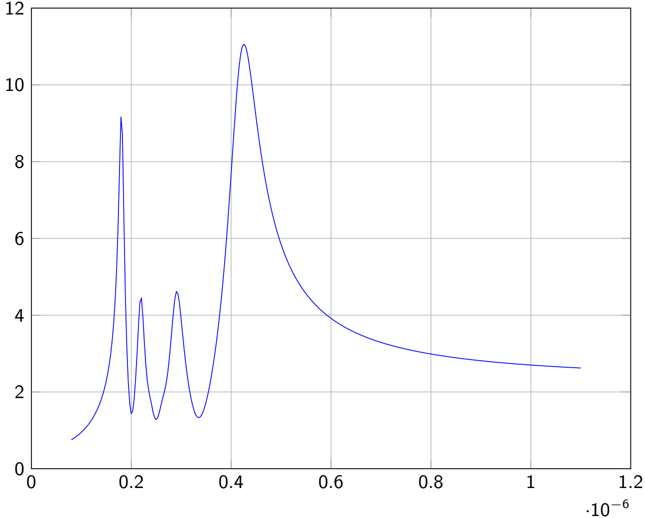
Norm of the polarization tensor for an elliptic inclusion.

Resonances for plasmonic nanoparticles



Shape of the inclusion

Resonances for plasmonic nanoparticles



Norm of the polarization tensor for a flower-shaped inclusion.

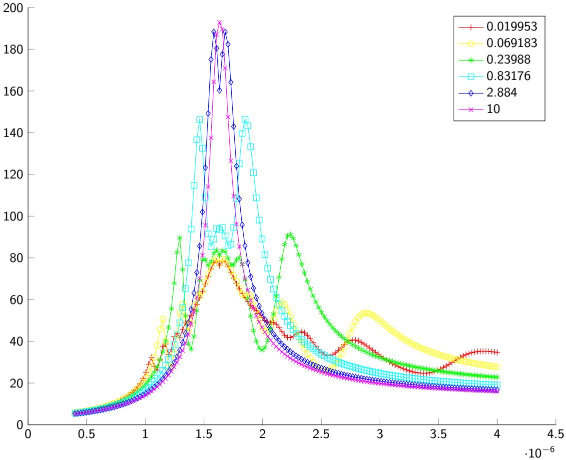
Resonances for plasmonic nanoparticles

- Quasi-plasmonic resonances for multiple particles: D_1 and D_2: $\mathcal{C}^{1,\alpha}$-bounded domains; $\mathrm{dist}(D_1, D_2) > 0$; $\nu^{(1)}$ and $\nu^{(2)}$: outward normal vectors at ∂D_1 and ∂D_2.
- Neumann-Poincaré operator $\mathbb{K}^*_{D_1 \cup D_2}$ associated with $D_1 \cup D_2$:

$$\mathbb{K}^*_{D_1 \cup D_2} := \begin{pmatrix} \mathcal{K}^*_{D_1} & \frac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{D_2} \\ \frac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{D_1} & \mathcal{K}^*_{D_2} \end{pmatrix}.$$

- Symmetrization of $\mathbb{K}^*_{D_1 \cup D_2}$.
- Behavior of the eigenvalues of $\mathbb{K}^*_{D_1 \cup D_2}$ as $\mathrm{dist}(D_1, D_2) \to 0$.

Resonances for plasmonic nanoparticles



Norm of the polarization tensor for two disks for various separating distances.

Resonances for plasmonic nanoparticles

- Algebraic domains: finite number of quasi-static plasmonic resonances:

$$\#\{j : (\nu_l, \varphi_j)_{\mathcal{H}^*} \neq 0\} : \text{finite.}$$

- Algebraic domains: zero level sets of polynomials; dense in Hausdorff metric among all planar domains.
- Blow-up of the polarization tensor for finite number of eigenvalues of the Neumann-Poincaré operator:

$$M_{l,m}(\lambda(\omega), D) = \sum_{j=1}^{\infty} \frac{(\nu_m, \varphi_j)_{\mathcal{H}^*} (\nu_l, \varphi_j)_{\mathcal{H}^*}}{(1/2 - \lambda_j)(\lambda(\omega) - \lambda_j)}.$$

- Two nearly touching disks: infinite number of quasi-static plasmonic resonances.

$$\lambda_j = \pm \frac{1}{2} e^{-2|j|\xi}, \xi = \sinh^{-1}(\sqrt{\frac{\delta}{r}(1 + \frac{\delta}{4r}});$$

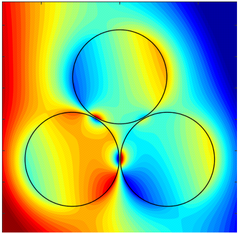
- r: radius of the disks; δ:separating distance.
- Separating distance δ : estimated from the first plasmonic resonance (associated to λ_1).

Resonances for plasmonic nanoparticles

- Singular nature of the interaction between nearly touching plasmonic nanoparticles.
- Applications in nanosensing (beyond the resolution limit).
- Blow-up of ∇u between the disks at plasmonic resonances:

$$\nabla u \propto \frac{r}{\Im(\lambda(\omega))\delta} e^{-2|j|\xi}.$$

- Accurate scheme for computing the field distribution between an arbitrary number of nearly touching plasmonic nanospheres: transformation optics + method of image charges.



Resonances for plasmonic nanoparticles

- (m, l)-entry of the polarization tensor M:

$$M_{l,m}(\lambda(\omega), D) = \sum_{j=1}^{\infty} \frac{\alpha_{l,m}^{(j)}}{\lambda(\omega) - \lambda_j},$$

$$\alpha_{l,m}^{(j)} := \frac{(\nu_m, \varphi_j)_{\mathcal{H}^*}(\nu_l, \varphi_j)_{\mathcal{H}^*}}{(1/2 - \lambda_j)}, \quad \alpha_{l,l}^{(j)} \geq 0, \quad j \geq 1.$$

- Sum rules for the polarization tensor:

$$\sum_{j=1}^{\infty} \alpha_{l,m}^{(j)} = \delta_{l,m}|D|; \qquad \sum_{j=1}^{\infty} \lambda_i \sum_{l=1}^{d} \alpha_{l,l}^{(j)} = \frac{(d-2)}{2}|D|.$$

$$\sum_{j=1}^{\infty} \lambda_j^2 \sum_{l=1}^{d} \alpha_{l,l}^{(j)} = \frac{(d-4)}{4}|D| + \sum_{l=1}^{d} \int_D |\nabla \mathcal{S}_D[\nu_l]|^2 dx.$$

- f holomorphic function in an open set $U \subset \mathbb{C}$ containing $\sigma(\mathcal{K}_D^*)$:

$$f(\mathcal{K}_D^*) = \sum_{j=1}^{\infty} f(\lambda_j)(\cdot, \varphi_j)_{\mathcal{H}^*} \varphi_j.$$

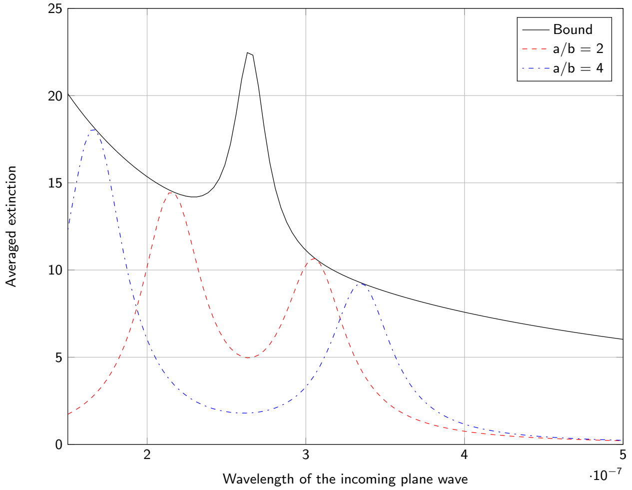
Resonances for plasmonic nanoparticles

- Upper bound for the averaged extinction cross-section Q_m^e of a randomly oriented nanoparticle:

$$
\begin{aligned}
&\left|\Im(\mathrm{Trace}(M(\lambda, D)))\right| \leq \frac{d|\lambda''||D|}{\lambda''^2 + 4\lambda'^2} \\
&+ \frac{1}{|\lambda''|(\lambda''^2 + 4\lambda'^2)}\Big(d\lambda'^2|D| + \frac{(d-4)}{4}|D| \\
&+ \sum_{l=1}^{d} \int_D |\nabla \mathcal{S}_D[\nu_l]|^2 dx + 2\lambda' \frac{(d-2)}{2}|D|\Big) + O\Big(\frac{\lambda''^2}{4\lambda'^2 + \lambda''^2}\Big).
\end{aligned}
$$

$\lambda' = \Re\lambda, \lambda'' = \Im\lambda$.

Resonances for plasmonic nanoparticles



Resonances for plasmonic nanoparticles

Hadamard's formula for $\mathcal{K}_D^*$:

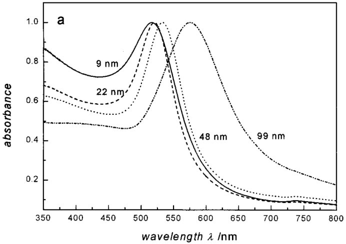
- ∂D: class $\mathcal{C}^2$; $\partial D := \{x = X(t), t \in [a, b]\}$.
- $\Psi_\eta : \partial D \mapsto \partial D_\eta := \{x + \eta h(t)\nu(x)\}$; Ψ_η: diffeomorphism.
- Hadamard's formula for $\mathcal{K}_D^*$:

$$||\mathcal{K}_{D_\eta}^*[\tilde{\phi}] \circ \Psi_\eta - \mathcal{K}_D^*[\phi] - \eta \mathcal{K}_D^{(1)}[\phi]||_{L^2(\partial D)} \leq C\eta^2 ||\phi||_{L^2(\partial D)},$$

  C: depends only on $||X||_{\mathcal{C}^2}$ and $||h||_{\mathcal{C}^1}$; $\phi := \tilde{\phi} \circ \Psi_\eta$.
- $\mathcal{K}_D^{(1)}$: explicit kernel.
- Hadamard's formula for the eigenvalues of $\mathcal{K}_D^*$.
- Shape derivative of plasmonic resonances for nanoparticles.
- Generalization to 3D.

Resonances for plasmonic nanoparticles

- $\mathcal{K}_D^*$: scale invariant $\Rightarrow$ Quasi-static plasmonic resonances: size independent.
- Analytic formula for the first-order correction to quasi-static plasmonic resonances in terms of the particle's characteristic size δ:



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Resonances for plasmonic nanoparticles

- Helmholtz equation:

$$\begin{cases} \nabla \cdot \Big(\varepsilon_m \chi(\mathbb{R}^d \setminus \bar{D}) + \varepsilon_c(\omega) \chi(\overline{D}) \Big) \nabla u + \omega^2 u = 0, \\ u^s := u - u^i \text{ satisfies the outgoing radiation condition.} \end{cases}$$

u^i: incident plane wave; $k_m := \omega\sqrt{\varepsilon_m}$, $k_c := \omega\sqrt{\varepsilon_c(\omega)}$.

- Integral formulation on ∂D:

$$\begin{cases} \mathcal{S}_D^{k_c}[\phi] - \mathcal{S}_D^{k_m}[\psi] = u^i, \\ \varepsilon_c \big(\frac{I}{2} - (\mathcal{K}_D^{k_c})^* \big)[\phi] - \varepsilon_m \big(\frac{I}{2} + (\mathcal{K}_D^{k_m})^* \big)[\psi] = \varepsilon_m \partial u^i / \partial \nu. \end{cases}$$

- Operator-Valued function $\delta \mapsto \mathcal{A}_\delta(\omega) \in \mathcal{L}(\mathcal{H}^*(\partial B), \mathcal{H}^*(\partial B))$:

$$\mathcal{A}_\delta(\omega) = \overbrace{(\lambda(\omega) I - \mathcal{K}_B^*)}^{\mathcal{A}_0(\omega)} + (\omega\delta)^2 \mathcal{A}_1(\omega) + O((\omega\delta)^3).$$

- Quasi-static limit:

$$\mathcal{A}_0(\omega)[\psi] = \sum_{j=0}^{\infty} \tau_j(\omega) (\psi, \varphi_j)_{\mathcal{H}^*} \varphi_j, \quad \tau_j(\omega) := \frac{1}{2}(\varepsilon_m + \varepsilon_c(\omega)) - (\varepsilon_c(\omega) - \varepsilon_m)\lambda_j.$$

Resonances for plasmonic nanoparticles

- Shift in the plasmonic resonance:

$$\arg\min_{\omega} \left| \frac{1}{2}\big(\varepsilon_m + \varepsilon_c(\omega)\big) - \big(\varepsilon_c(\omega) - \varepsilon_m\big)\lambda_j + \big(\omega\delta\big)^2 \tau_{j,1} \right|$$

- $\tau_{j,1} := (\mathcal{A}_1(\omega)[\varphi_j], \varphi_j)_{\mathcal{H}^*}$.
- Gohberg-Sigal theory.

Resonances for plasmonic nanoparticles

- Full Maxwell's equations:

$$\begin{cases} \nabla \times \nabla \times E - \omega^2 \Big(\varepsilon_m \chi(\mathbb{R}^d \setminus \bar{D}) + \varepsilon_c(\omega)\chi(\overline{D}) \Big) E = 0, \\ E^s := E - E^i \text{ satisfies the outgoing radiation condition.} \end{cases}$$

- Small-volume expansion:

$$E^s(x) \quad = \quad -\delta^3 \omega^2 G_{k_m}(x,z) M(\lambda(\omega), B) E^i(z) + O\big(\frac{\delta^4}{\mathrm{dist}(\lambda(\omega), \sigma(\mathcal{K}_D^*))}\big)$$

- G_{k_m}: fundamental (outgoing) solution to Maxwell's equations in free space.
- Shift in the plasmonic resonances due to the finite size of the nanoparticle.

Resonances for plasmonic nanoparticles

- Integral formulation:

$$\begin{pmatrix} I + \mathcal{M}_D^{k_c} - \mathcal{M}_D^{k_m} & \mathcal{L}_D^{k_c} - \mathcal{L}_D^{k_m} \\ \mathcal{L}_D^{k_c} - \mathcal{L}_D^{k_m} & \frac{1}{2}(k_c^2 + k_m^2)I + k_c^2 \mathcal{M}_D^{k_c} - k_m^2 \mathcal{M}_D^{k_m} \end{pmatrix}$$

- Integral operators:

$$\begin{aligned} \mathcal{M}_D^k[\varphi] : H_T^{-\frac{1}{2}}(\mathrm{div}, \partial D) &\longrightarrow H_T^{-\frac{1}{2}}(\mathrm{div}, \partial D) \qquad (\text{compact}) \\ \varphi &\longmapsto \int_{\partial D} \nu(x) \times \nabla_x \times G_k(x,y)\varphi(y) ds(y); \end{aligned}$$

$$\begin{aligned} \mathcal{L}_D^k[\varphi] : H_T^{-\frac{1}{2}}(\mathrm{div}, \partial D) &\longrightarrow H_T^{-\frac{1}{2}}(\mathrm{div}, \partial D) \\ \varphi &\longmapsto \nu(x) \times \Big(k^2 \mathcal{S}_D^k[\varphi](x) + \nabla \mathcal{S}_D^k[\nabla_{\partial D} \cdot \varphi](x) \Big). \end{aligned}$$

- Key identities: $\mathcal{M}_D^{k=0}[\mathrm{curl}_{\partial D}\varphi] = \mathrm{curl}_{\partial D}\mathcal{K}_D[\varphi], \quad \forall \varphi \in H^{\frac{1}{2}}(\partial D)$,

$$\mathcal{M}_D^{k=0}[\nabla_{\partial D}\varphi] = -\nabla_{\partial D}\Delta_{\partial D}^{-1}\mathcal{K}_D^*[\Delta_{\partial D}\varphi] + \mathcal{R}_D[\varphi],$$

$\mathcal{R}_D = -\mathrm{curl}_{\partial D}\Delta_{\partial D}^{-1}\mathrm{curl}_{\partial D}\mathcal{M}_D\nabla_{\partial D}, \ \forall \varphi \in H^{\frac{3}{2}}(\partial D)$.

Resonances for plasmonic nanoparticles

- Quasi-static approximation:

$$\widetilde{\mathcal{M}}_B = \begin{pmatrix} -\Delta_{\partial B}^{-1}\mathcal{K}_B^*\Delta_{\partial B} & 0 \\ \mathcal{R}_B & \mathcal{K}_B \end{pmatrix}.$$

- $H(\partial B) := H_0^{\frac{3}{2}}(\partial B) \times H^{\frac{1}{2}}(\partial B)$, equipped with the inner product

$$(u, v)_{H(\partial B)} = (\Delta_{\partial B} u^{(1)}, \Delta_{\partial B} v^{(1)})_{\mathcal{H}^*} + (u^{(2)}, v^{(2)})_{\mathcal{H}},$$

$$(u, v)_{\mathcal{H}^*} := -(u, \mathcal{S}_D[v])_{-\frac{1}{2},\frac{1}{2}}, \quad (u, v)_{\mathcal{H}} = -(\mathcal{S}_D^{-1}[u], v)_{-\frac{1}{2},\frac{1}{2}}.$$

- The spectrum $\sigma(\widetilde{\mathcal{M}}_B) = \sigma(-\mathcal{K}_B^*) \cup \sigma(\mathcal{K}_B^*)\backslash\{-\frac{1}{2}\}$ in $H(\partial B)$.
- Only $\sigma(\mathcal{K}_B^*)$ can be excited in the quasi-static approximation.

Scattering coefficients

- Scattering coefficients: cloaking structures and dictionary matching approach for inverse scattering.
- Mechanism underlying plasmonic resonances in terms of the scattering coefficients corresponding to the nanoparticle.
- Scattering coefficients of order ± 1: only scattering coefficients iudcing the scattering-cross section enhancement.

Scattering coefficients

- Helmholtz equation:

$$\begin{cases} \nabla\cdot\Big(\varepsilon_m\chi(\mathbb{R}^d\setminus\bar{D})+\varepsilon_c(\omega)\chi(\overline{D})\Big)\nabla u+\omega^2 u=0, \\ u^s := u-u^i \text{ satisfies the outgoing radiation condition.} \end{cases}$$

u^i: incident plane wave; $k_m := \omega\sqrt{\varepsilon_m}, k_c := \omega\sqrt{\varepsilon_c(\omega)}$.

- Scattering coefficients:

$$W_{mn}(D,\varepsilon_c,\varepsilon_m,\omega)=\int_{\partial D}\psi_m(y)J_n(\omega|y|)e^{-in\theta_y}\,ds(y).$$

- ψ_m: electric current density on ∂D induced by the cylindrical wave $J_m(\omega|x|)e^{im\theta_x}$.
- J_n: Bessel function.

Scattering coefficients

Properties of the scattering coefficients:

- W_{mn} decays rapidly:

$$|W_{mn}| \leq \frac{O(\omega^{|m|+|n|})}{\min|\tau_j(\omega)|}\frac{C^{|m|+|n|}}{|m|^{|m|}|n|^{|n|}}, \;\; m,n \in \mathbb{Z},$$

C: independent of ω; $\tau_j = \frac{1}{2}\big(\varepsilon_m + \varepsilon_c(\omega)\big) - \big(\varepsilon_c(\omega) - \varepsilon_m\big)\lambda_j$.

- For any $z \in \mathbb{R}^2, \theta \in [0, 2\pi), s > 0$,

$$W_{mn}(D^z) = \sum_{m',n' \in \mathbb{Z}} J_{n'}(\omega|z|)J_{m'}(\omega|z|)e^{i(m'-n')\theta_z}W_{m-m',n-n'}(D),$$

$$W_{mn}(D^\theta) = e^{i(m-n)\theta}W_{mn}(D),$$

$$W_{mn}(D^s,\omega) = W_{mn}(D, s\omega).$$

Scattering coefficients

- Scattering amplitude:

$$u^s(x) = -ie^{-\frac{\pi i}{4}} \frac{e^{ik_m|x|}}{\sqrt{8\pi k_m|x|}} A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') + o(|x|^{-\frac{1}{2}}),$$

$|x| \to \infty$; θ, θ': incident and scattered directions.

- Graf's formula:

$$A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') = \sum_{n,m\in\mathbb{Z}} (-i)^n i^m e^{in\theta'} W_{nm}(D, \varepsilon_c, \varepsilon_m, \omega) e^{-im\theta}.$$

- Scattering cross-section:

$$Q^s[D, \varepsilon_c, \varepsilon_m, \omega](\theta') := \int_0^{2\pi} \left| A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') \right|^2 d\theta.$$

Cloaking: scattering coefficient cancellation

- Cloaking: make a target invisible when probed by electromagnetic waves.
- Scattering coefficient cancellation technique:
  - Small layered object with vanishing first-order scattering coefficients.
  - Transformation optics:

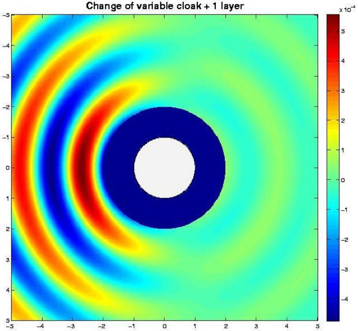
$$(F_\rho)_*[\phi](y) = \frac{DF_\rho(x)\phi(x)DF_\rho(x)^t}{\det(DF_\rho(x))}, \quad x = F_\rho^{-1}(y).$$

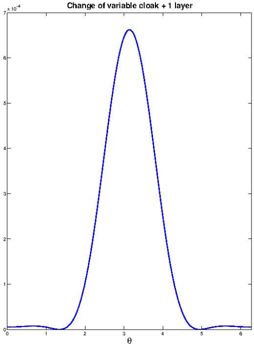
  - Change of variables F_ρ sends the annulus $[\rho, 2\rho]$ onto a fixed annulus.
- Scattering coefficients vanishing structures of order N:

$$Q^s\left[D, (F_\rho)_*(\varepsilon \circ \Psi_{\frac{1}{\rho}}), \varepsilon_m, \omega\right](\theta') = o(\rho^{4N}), \quad \Psi_{1/\rho}(x) = (1/\rho)x.$$

  ρ: size of the small object; N: number of layers.
- Anisotropic permittivity distribution.
- Invisibility at $\omega \Rightarrow$ invisibility at all frequencies $\leq \omega$.

Cloaking: scattering coefficient cancellation





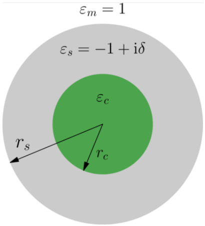
Cancellation of the scattered field and the scattering cross-section: 4 orders of magnitude (with wavelength of order 1, $\rho = 10^{-1}$, and $N = 1$).

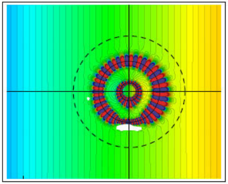
Cloaking: anomalous resonance

- Ω: bounded domain in $\mathbb{R}^2$; $D \Subset \Omega$. Ω and D of class $\mathcal{C}^{1,\mu}$, $0 < \mu < 1$. For a given loss parameter $\delta > 0$, the permittivity distribution in $\mathbb{R}^2$ is given by

$$\varepsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ -1 + i\delta & \text{in } \Omega \setminus \overline{D}, \\ 1 & \text{in } D. \end{cases}$$

- Configuration (plasmonic structure): core with permittivity 1 coated by the shell $\Omega \setminus \overline{D}$ with permittivity $-1 + i\delta$.

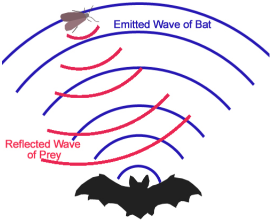




Dictionary matching approach

Dictionary matching approach:

- Form an image from the echo due to targets.
- Identify and classify the target, knowing by advance that it belongs to a learned dictionary of shapes.
    - Extract the features from the data.
    - Construct invariants with respect to rigid transformations and scaling.
    - Compare the invariants with precomputed ones for the dictionary.



Dictionary matching approach

- Feature extraction:
  - Extract $\mathbf{W}$ by solving a least-squares method

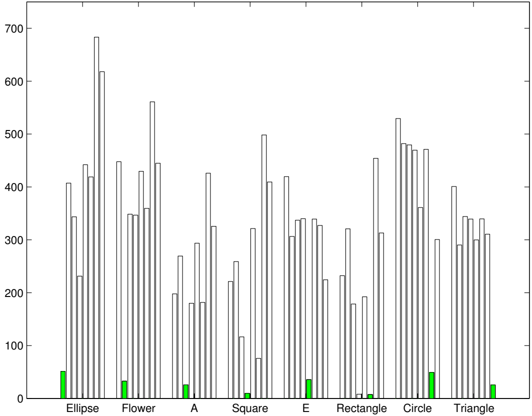
$$\mathbf{W} = \arg\min_{\mathbf{W}} \|\mathbf{L}(\mathbf{W}) - \mathbf{V}\|.$$

  - $\mathbf{L}$ is ill-conditioned ($\mathbf{W}$ decays rapidly).
  - Maximum resolving order K:

$$K^{K+1/2} = C(\omega)\text{SNR}.$$

- Form a multi-frequency shape descriptor.
- Match in a multi-frequency dictionary.

Dictionary matching approach



Shape descriptor matching in a multi-frequency dictionary.

Resonances for plasmonic nanoparticles

- Asymptotic expansion of the scattering amplitude:

$$A_\infty\left(\frac{x}{|x|}, d\right) = \frac{x}{|x|}^t W_1 d + O(\omega^2),$$

d: incident direction; $x/|x|$: observation direction;

$$W_1 = \begin{pmatrix} W_{-11} + W_{1-1} - 2W_{11} & i\big(W_{1-1} - W_{-11}\big) \\ i\big(W_{1-1} - W_{-11}\big) & -W_{-11} - W_{1-1} - 2W_{11} \end{pmatrix}.$$

- Blow up of the scattering coefficients:

$$W_{\pm 1 \pm 1} = \pm \pm \frac{k_m^2}{4} \frac{\big(\varphi_j, |x| e^{\mp i\theta_x}\big)_{-\frac{1}{2},\frac{1}{2}} \big(e^{\pm i\theta_\nu}, \varphi_j\big)_{\mathcal{H}^*}}{\lambda - \lambda_j} + O(1).$$

Super-resolution

- Super-resolution for plasmonic nanoparticles:
  - Sub-wavelength resonators;
  - High contrast: effective medium theory;
  - Single nanoparticle imaging.

Super-resolution

- Resolution: determined by the behavior of the imaginary part of the Green function. Helmholtz-Kirchhoff identity:

$$\Im m\, G_{k_m}(x, x_0) = k_m \int_{|y|=R} \overline{G_{k_m}(y, x_0)} G_{k_m}(x, y) ds(y), \quad R \to +\infty.$$

- The sharper is $\Im m\, G_{k_m}$, the better is the resolution.
- Local resonant media used to make shape peaks of $\Im m\, G_{k_m}$.
- Mechanism of super-resolution in resonant media:
  - Interaction of the point source x_0 with the resonant structure excites high-modes.
  - Resonant modes encode the information about the point source and can propagate into the far-field.
  - Super-resolution: only limited by the resonant structure and the signal-to-noise ratio in the data.

Super-resolution

- System of weakly coupled plasmonic nanoparticles.
- Size of the nanoparticle $\delta \ll$ wavelength $2\pi/k_m$; distance between the nanoparticles of order one.
- $\Im G^\delta = \Im G_{k_m} +$ exhibits sub-wavelength peak with width of order one.
- Break the resolution limit.

S. Nicosia & C. Ciraci, Cover, Science 2012

Super-resolution

- Sub-wavelength resonator:

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- Asymptotic expansion of the Green function (δ: size of the resonator openings; z_j: center of aperture for jth resonator; J: number of resonators; $\omega = O(\sqrt{\delta})$):

$$\Im m\, G^{\delta}(x, x_0, \omega) \approx \frac{\sin \omega |x - x_0|}{2\pi |x - x_0|} + \sqrt{\delta} \sum_{j=1}^{J} \frac{c_j}{|x - z_j|\, |x_0 - z_j|}.$$

Super-resolution

- Effective medium theory:

$$\varepsilon_{\text{eff}}(\omega) = \varepsilon_m \big(I + fM(\lambda(\omega), B)(I - \frac{f}{3} M(\lambda(\omega), B))^{-1}\big) + O\big(\frac{f^{8/3}}{\mathrm{dist}(\lambda(\omega), \sigma(\mathcal{K}_D^*))^2}\big).$$

- f: volume fraction; B: rescaled particle.
- $\varepsilon_{\text{eff}}(\omega)$: anisotropic.
- Validity of the effective medium theory:

$$f \ll \mathrm{dist}(\lambda(\omega), \sigma(\mathcal{K}_D^*))^{3/5}.$$

Super-resolution

- High contrast effective medium at plasmonic resonances:

$$\nabla \times \nabla \times E - \omega^2 \Big(\varepsilon_m \chi(\mathbb{R}^d \setminus \bar{\Omega}) + \varepsilon_{\text{eff}}(\omega) \chi(\overline{\Omega}) \Big) E = 0.$$

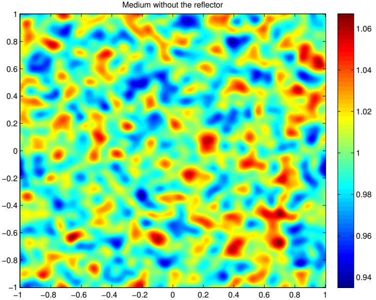
- $E|_\Omega \mapsto \displaystyle\int_\Omega (\varepsilon_{\text{eff}}(\omega) - \varepsilon_m) E(y) G_{k_m}(x, y)\, dy, \quad x \in \Omega.$
- Mixing of resonant modes: intrinsic nature of non-hermitian systems.
- Sub-wavelength resonance modes excited $\Rightarrow$ dominate over the other ones in the expansion of the Green function.
- Imaginary part of the Green function may have sharper peak than the one of G due to the excited sub-wavelength resonant modes.
- Sub-wavelength modes: determine the super-resolution.

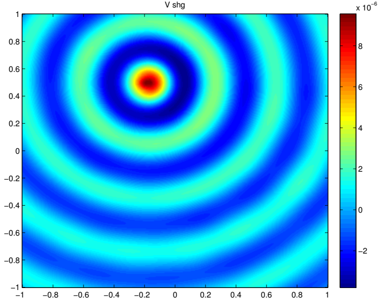
Super-resolution

- Single nanoparticle imaging:

$$\max_{z^S} I(z^S, \omega)$$

- $I(z^S, \omega)$: imaging functional; z^S: search point.
- Resolution: limited only by the signal-to-noise-ratio.
- Cross-correlation techniques: robustness with respect to medium noise.





Plan

- Part I: Mathematical and computational tools
  - Gohberg-Sigal theory
  - Layer potentials, Green's functions (free space, grating, quasi-periodic), integral formulations, Helmholtz-Kirchhoff identities, scattering coefficients, Floquet theory, Muller's method, Ewald's method for grating and quasi-periodic Green's functions.
- Part II: Diffraction gratings and photonic crystals
  - Diffraction gratings: radiation condition, existence and uniqueness of a solution, optimal design problem.
  - Photonic crystals: sensitivity of band gaps, analysis of photonic crystal cavities.

Plan

- Part III: Sub-wavelength resonators and super-resolution
    - Plasmonic nanoparticles.
    - Scattering and absorption enhancement.
    - Resolution enhancement.
    - Super-resolution in high contrast media.
    - Effective medium theory for sub-wavelength resonators.
    - Near-field optics.
- Part IV: Metamaterials, metasurfaces, and sub-wavelength photonic crystals
    - Metamaterials and cloaking.
    - Metasurfaces with superabsorption effect: layers of periodically distributed plasmonic nanoparticles.
    - Sub-wavelength photonic crystals.

Plan

- Part V: Minnaert bubbles
  - Minnaert resonance for bubbles.
  - Acoustic metasurfaces.
  - Effective medium theory and super-resolution.
  - Sub-wavelength phononic crystals.
  - Double-negative acoustic metamaterials.