ETH Lecture 401-0663-00L Numerical Methods for PDEs

Endterm Exam

Spring Term 2018

29. May 2018, 15:15, HG F 1

Don't panic!

Family Name		%
First Name		
Department		
Legi Nr.		
Date	29. May 2018	

Points:

	1	2	3	4	Total
max	3	3	3	3	12
achvd					

- This is a **closed-book exam.**
- Keep only writing material and Legi on the table.
- Keep mobile phones, tablets, smartwatches, etc. turned off in your bag.
- Fill in this cover sheet first.
- Turn the cover sheet only when instructed to do so.
- Then write your name and Legi Nr. on each page.
- **Write your answers in the appropriate fields on these problem sheets.**
- **Wrong ticks in multiple-choice boxes will lead to points being subtracted.**
- **Anything written outside the answer boxes will not be taken into account.**
- Do not write with red/green color or with pencil.
- Make sure to hand in every sheet.
- Two blank pages at the end of the exam: space for notes
- **Duration: 30 minutes.**

Problem 1: Convergence of finite element Galerkin solutions (3 pts)

Problem related to [Lecture → Chapter 5] "Convergence and Accuracy" (of finite element solutions).

This is a **multiple-choice task**. Wrong ticks incur point penalty.

On the unit square $\Omega :=]0,1[^2$ we consider a sequence of nested triangular meshes

$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_j \prec \ldots$$

generated by global regular refinement of an initial mesh $\mathcal{M}_0$.

We write $u_N^{(j)} \in \mathcal{S}^0_[(\mathcal{M})\mathcal{M}_j]2$, where j is for mesh $\mathcal{M}_j$, for the finite element Galerkin solution of the Dirichlet problem

$$\Delta u = f \quad \text{in} \quad \Omega\,, \quad u = g \quad \text{on} \quad \partial\Omega\,, \tag{0.1.1}$$

where f and g is chosen so that we get as exact solution

$$u(\boldsymbol{x}) = \exp(x_1^2 + x_2^2)\,, \quad \boldsymbol{x} \in \Omega\,.$$

Which is the asymptotic behavior of

1. $\epsilon_1(j) := \left| u - u_N^{(j)} \right|_{H^1(\Omega)}$ in terms of the meshwidth $h_{\mathcal{M}} \to 0$:

   $\epsilon_1 =$ ○ $O(h)$, ○ $O(h^2)$, ○ $O(h^3)$, **?**

2. $\epsilon_2(j) := \left| u_N^{(j)} \right|^2_{H^1(\Omega)} - \left| u_N^{(j-1)} \right|^2_{H^1(\Omega)}$ in terms of the meshwidth $h_{\mathcal{M}} \to 0$:

   $\epsilon_2 =$ ○ $O(h)$, ○ $O(h^2)$, ○ $O(h^3)$, **?**

3. $\epsilon_3(j) := \left\| u_N^{(j)} - u_N^{(j-1)} \right\|_{L^2(\Omega)}$ in terms of $N := \dim \mathcal{S}^0_{2,0}(\mathcal{M}) \to \infty$:

   $\epsilon_3 =$ ○ $O(N^{-\frac{1}{2}})$, ○ $O(N^{-1})$, ○ $O(N^{-\frac{3}{2}})$, ○ $O(N^{-2})$ **?**

HINT 1 for (1.): Recall that $|v|^2_{H^1(\Omega)} = \int_\Omega \|\mathbf{grad}\, v(\boldsymbol{x})\|^2 \,\mathrm{d}\boldsymbol{x}$.

SOLUTION of (1.):

1. The norm $|\cdot|_{H^1(\Omega)}$ is the energy norm induced by the variational problem underlying (0.1.1). Thus we can count on the optimality of the finite element Galerkin solution according to [Lecture → Thm. 5.1.15].

   To estimate the best approximation error we rely on [Lecture → Thm. 5.3.56] and this theorem gives $O(h^2)$, because the exact solution u is smooth for this problem, such that the rate of convergence is limited by the polynomial degree of the finite element space.

2. Recall the formula "$a^2 - b^2 = (a+b)(a-b)$" for symmetric bilinear forms from [Lecture → § 5.8.9], which yields

$$\left| u_N^{(j)} \right|^2_{H^1(\Omega)} - \left| u_N^{(j-1)} \right|^2_{H^1(\Omega)} = \int_\Omega \mathbf{grad}(u_N^{(j)} + u_N^{(j-1)}) \cdot \mathbf{grad}(u_N^{(j)} - u_N^{(j-1)}) \,\mathrm{d}\boldsymbol{x}\,.$$

Then apply the Cauchy-Schwarz inequality for integrals

$$\left| \left| u_N^{(j)} \right|_{H^1(\Omega)}^2 - \left| u_N^{(j-1)} \right|_{H^1(\Omega)}^2 \right| \leq \left| u_N^{(j)} + u_N^{(j-1)} \right|_{H^1(\Omega)} \left| u_N^{(j)} - u_N^{(j-1)} \right|_{H^1(\Omega)}$$

together with the result of 1, which gives

$$\left| \left| u_N^{(j)} \right|_{H^1(\Omega)}^2 - \left| u_N^{(j-1)} \right|_{H^1(\Omega)}^2 \right| = O(h^2) \quad \text{for} \quad h \to 0 \,.$$

3. On the unit square the Dirichlet problem for $-\Delta$ is 2-regular, *cf.* [Lecture $\to$ Ass. 5.6.23]. Therefore, we can use the $L^2(\Omega)$-estimates of [Lecture $\to$ § 5.6.24], which yields

$$\left\| u - u_N^{(j)} \right\|_{L^2(\Omega)} = O(h^3) \quad \text{for} \quad h \to 0 \,.$$

Combine this with the triangle inequality and the fact that $N \approx h^{-2}$ for uniform refinement in two dimensions and obtain $\left\| u_N^{(j)} - u_N^{(j-1)} \right\|_{L^2(\Omega)} = O(N^{-\frac{3}{2}})$ for $h \to 0$.

End Problem 1

Problem 2: Output functional (3 pts)

Problem related to [Lecture → Section 5.6] "Duality techniques".

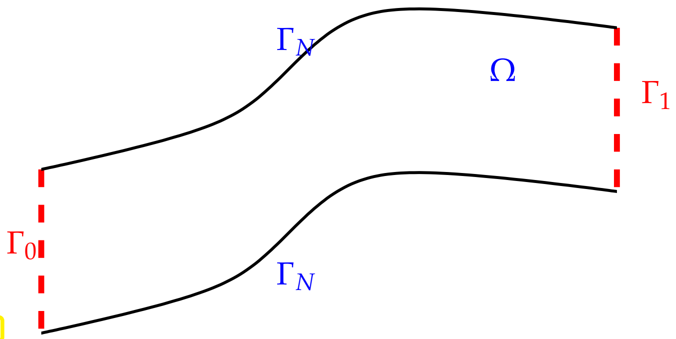


Fig. 1

For a domain $\Omega \subset \mathbb{R}^2$ we assume a partitioning of the boundary

$$\partial\Omega = \overline{\Gamma}_N \cup \overline{\Gamma}_0 \cup \overline{\Gamma}_1 ,$$

where Γ_0 and Γ_1 do not touch, see Fig. 1 beside.

If $u \in H^1(\Omega)$ solves

$$-\Delta u = f \in L^2(\Omega) \quad \text{in} \quad \Omega ,$$
$$u = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_1 , \quad \mathbf{grad}\, u \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_N ,$$

then we have

$$\int_{\Gamma_0} \mathbf{grad}\, u \cdot \boldsymbol{n} \,\mathrm{d}S = \int_{\Omega} \mathbf{grad}\, u \cdot \mathbf{grad}\, \Psi - f\,\Psi \,\mathrm{d}x , \tag{0.2.1}$$

for *suitable* functions $\Psi : \Omega \to \mathbb{R}$.

(2.a) (1 pts) What is the largest possible Sobolev space that Ψ can belong to?

$$\Psi \in \boxed{} .$$

SOLUTION of (2.a):

On one hand, the right-hand side integral in (0.2.1) must be well-defined for any $u \in H_0^1(\Omega)$. This is guaranteed, if $\mathbf{grad}\, \Psi \in \boldsymbol{L}^2(\Omega)$ and $\Psi \in L^2(\Omega)$. On the other hand, if $\mathbf{grad}\, \Psi \notin L^2(\Omega)$ we can find $u \in H_0^1(\Omega)$ such that the integral will blow up.

As a consequence, we need $\Psi \in H^1(\Omega)$.

▲

(2.b) (2 pts) Which boundary conditions does Ψ have to satisfy necessarily to make (0.2.1) hold?

$$\Psi|_{\partial\Omega} = \begin{cases} \boxed{} & \text{on} \quad \Gamma_0 , \\ \boxed{} & \text{on} \quad \Gamma_1 , \\ \boxed{} & \text{on} \quad \Gamma_N . \end{cases}$$

HINT 1 for (2.b): Green's formula reads

$$\int_{\partial\Omega} \Psi\, \mathbf{grad}\, u \cdot \boldsymbol{n} \,\mathrm{d}S = \int_{\Omega} \mathbf{grad}\, u \cdot \mathbf{grad}\, \Psi + \Delta u\, \Psi \,\mathrm{d}x . \tag{0.2.2}$$

SOLUTION of (2.b):

The arguments are the same as in [Lecture → Section 5.6.2], where we derived and equivalent continuous boundary flux functional [Lecture → Eq. (5.6.15)]. From (0.2.2) we see that (0.2.1) will only hold, if $\Psi|_{\Gamma_0} \equiv 1$ and $\Psi|_{\Gamma_1} \equiv 0$. On Γ_N the value of Ψ does not matter, because there $\mathbf{grad}\, u \cdot \boldsymbol{n} \equiv 0$.

Summing up,

$$\Psi|_{\partial\Omega} = \begin{cases} 1 & \text{on } \Gamma_0 \,, \\ 0 & \text{on } \Gamma_1 \,, \\ \text{any value} & \text{on } \Gamma_N \,. \end{cases}$$

▲

End Problem 2

Problem 3: Unconditionally stable timestepping (3 pts)

Problem related to [Lecture → Section 6.1.7] "Timestepping for method-of-lines ODE".

A class of Runge-Kutta single-step methods (RK-SSM) parameterized by $\theta \in [0,1]$ has the stability function (→ [Lecture → Eq. (6.1.127a)])

$$S_\theta(z) = \frac{1+(1-\theta)z}{1-\theta z}\,, \quad 0 \leq \theta \leq 1\,.$$

We use these RK-SSM for timestepping in a finite-element method-of-lines framework for the parabolic evolution equation $\frac{\partial u}{\partial t} - \Delta u = f(\mathbf{x}, t)$.

For which values of θ is there **no** stability-induced timestep constraint?

$$\theta \in \boxed{}\,.$$

HINT 2 for (3.): A Runge-Kutta single-step method (RK-SSM) applied to the scalar ODE $\dot{y} = \lambda y$, $\lambda < 0$, with uniform timestep $\tau > 0$ produces a sequence $(y^{(j)})_{j=0}^{\infty}$ according to

$$y^{(j)} = S(\lambda\tau)y^{(j-1)}\,, \quad j = 1, 2, \ldots\,,$$

where S is the stability function of the RK-SSM. ⌟

SOLUTION of (3.):

As explained in [Lecture → § 6.1.126] we have to aim for $|S_\theta(z)| < 1$ for all $z = \lambda\tau < 0$. This is true, if and only if

$$|1+(1-\theta)z| < 1 - \theta z \quad \forall z < 0\,,$$

where we have used that $1 - \theta z > 0$ for all $z < 0$.

In order to resolve the modulus, we have to discuss two cases

1. $1 + (1-\theta) \geq 0 \quad \Leftrightarrow \quad z \geq -\frac{1}{1-\theta}$:

   Leads to $1 + z < 1$, which is satisfied for all $z < 0$. Hence, this case does not give any information about θ.

2. $1 + (1-\theta) < 0 \quad \Leftrightarrow \quad z < -\frac{1}{1-\theta}$:

   We end up with $(2\theta - 1)z < 3$, which is true for all $z < 0$ if and only if $2\theta - 1 \geq 0 \quad \Leftrightarrow \quad \theta \geq \frac{1}{2}$.

   Hence, we have found and sufficient and necessary condition for unconditional stability $\theta \in [\frac{1}{2}, 1]$.

A not enirely rigorous, but fast, way of finding the suitable range for θ is to examine $S(z)$ for $z \ll -1$:

$$\lim_{z\to-\infty} |S(z)| = \frac{1-\theta}{\theta} \overset{!}{\leq} 1 \quad \Leftrightarrow \quad \theta \geq \frac{1}{2}\,.$$

End Problem 3

Problem 4: Finite difference stencil (3 pts)

Problem related to [Lecture → Section 4.1] “Finite differences”.

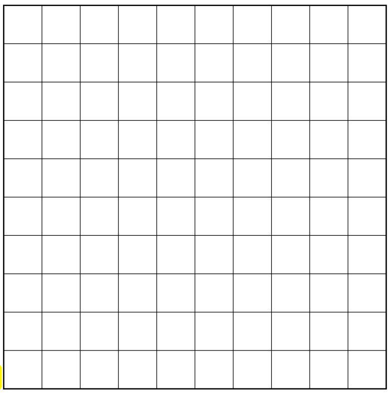


Fig. 2

The unit square domain $\Omega =]0,1[^2$ is equipped with a structured mesh consisting of equal squares with edge length $h := M^{-1}$, $M \in \mathbb{N}$, see Fig. 2 beside.

The Galerkin finite element discretization of a scalar second-order elliptic boundary value problem based on the *bilinear* Lagrangian finite element space $\mathcal{S}_2^0(\mathcal{M})$ with the customary choice of (global) shape functions yields the element matrix

$$\mathbf{A}_K = \begin{bmatrix} 3 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ -1 & 0 & -2 & 3 \end{bmatrix} + h \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where a counterclockwise numbering of the local shape functions, which are associated with vertices, is assumed, see figure ▷

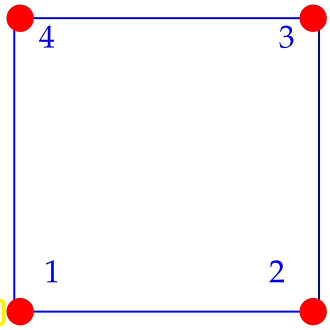
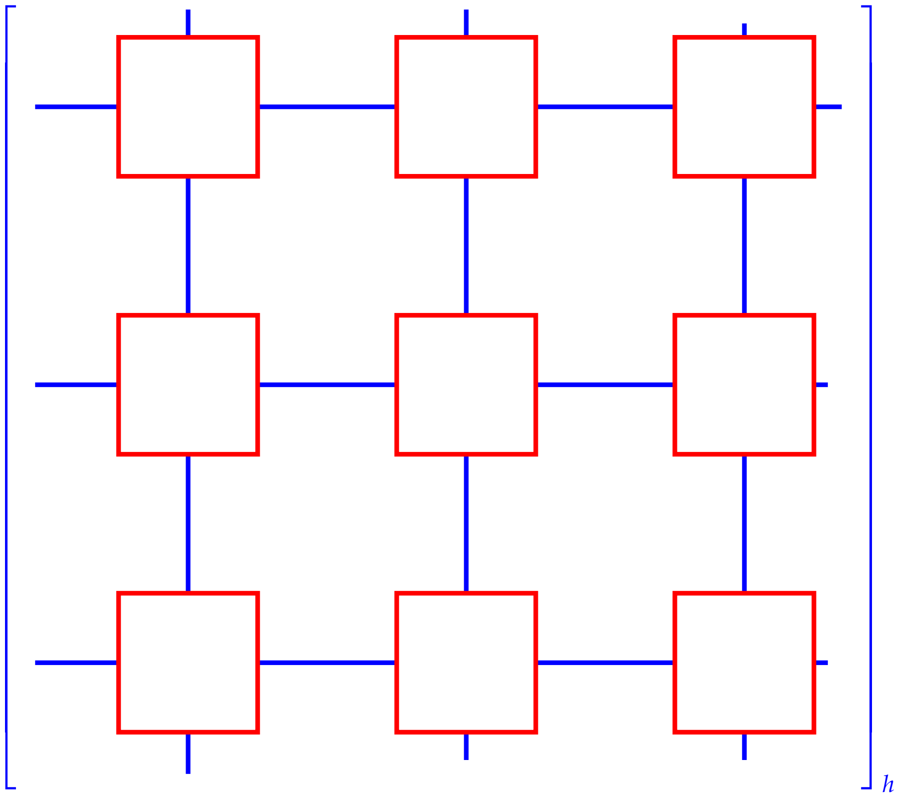


Fig. 3

Write the weights of the resulting finite difference stencil in the boxes below:



HINT 3 for (4.): The entries of a finite difference stencil agree with the entries of the Galerkin matrix connecting the “central” node and “peripheral” nodes. ⌟

SOLUTION of (4.):

We perform “assembly by pencil and paper” as was demonstrated in [Lecture → Section 4.1.2] for a structured triangular mesh. We obtain entries of the Galerkin matrix by adding up suitable entries of element matrices, see [Lecture → Fig. 105] and [Lecture → Fig. 106].

This procedure leads to the element stencil

$$\begin{bmatrix} 0 & -2 & -h \\ -4 & 12+2h & -4 \\ -h & -2 & 0 \end{bmatrix}_h .$$

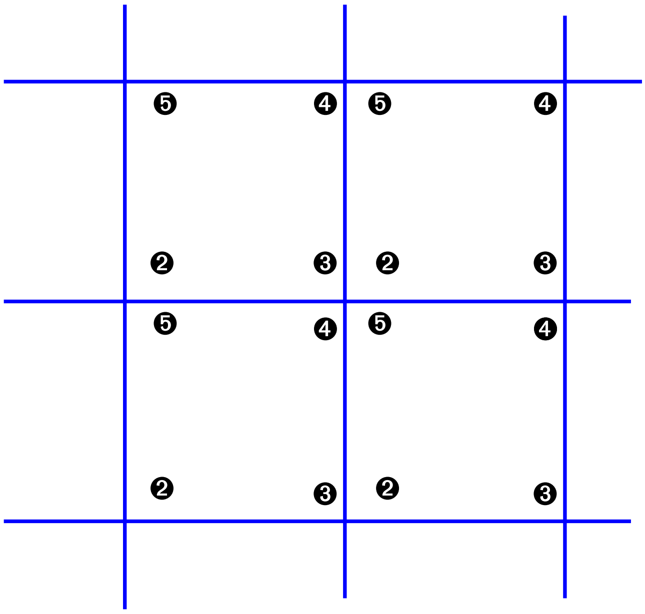


Fig. 4

End Problem 4

Scratch space (will not be evaluated)

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