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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Numerical Methods for Partial Differential Equations

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Introduction

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Relevance:

PDE-based mathematical models are ubiquitous

- Fluid mechanics
- Electromagnetics
- Astrophysics
- Quantum mechanics
- Mathematical finance
- Social systems

$\triangleright$ continuum models

Examples:

- Heat equation: $\frac{\partial u}{\partial t} - \Delta u = f$
- (Stationary, incompressible) Navier-Stokes equations:

$$\begin{aligned} -\nu\Delta\mathbf{u} + D\mathbf{u}\cdot\mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{0.4.3.1}$$

$\nu \;\hat{=}\;$ dynamic viscosity
$\mathbf{f}:\Omega\mapsto\mathbb{R}^3 \;\hat{=}\;$ given external force field
$\mathbf{u}:\Omega\mapsto\mathbb{R}^3 \;\hat{=}\;$ velocity field (unknown)
$p:\Omega\mapsto\mathbb{R} \;\hat{=}\;$ pressure (unknown)

$$\operatorname{grad} u = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_3} \end{bmatrix}$$

$$\operatorname{div} \underline{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$$

Review question(s) 0.4.9.3.

(Q0.4.9.3.A) The following is a PDE for a vector field $\mathbf{u} : \Omega \to \mathbb{R}^3$: [u smooth]

$$\mathbf{grad}\,\mathrm{div}\,\mathbf{u} = [f_1, f_2, f_3]^\top, \quad f_i : \Omega \to \mathbb{R}. \tag{0.4.9.4}$$

Write this PDE in detail for the components of $\mathbf{u}$.

(Q0.4.9.4.B) Compute the Hessian according to (0.3.2.18) for the function $u(x_1, x_2) = \exp(x_1^2 + x_2^2)$.

(Q0.4.9.4.C) Compute the rotation $\mathbf{curl}\,\mathbf{u}$ and the divergence $\mathrm{div}\,\mathbf{u}$ for the vector field $\mathbf{u}(\boldsymbol{x}) = -\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^2}$, $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$.

Hint: Write $\mathbf{u}$ in components and compute partial derivatives.

(Q0.4.9.4.D) Compute the Laplacian of the function $\mathbf{u}(x_1, x_2) = \sin(x_1)\cos(x_2)$. What do you observe?

(Q0.4.9.4.E) If $f \in C^1(\mathbb{R})$, what is the gradient of $\boldsymbol{x} \mapsto f(\|\boldsymbol{x}\|)$.

△

A:
$$\begin{bmatrix} \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2 \partial x_1} + \frac{\partial^2 u_3}{\partial x_3 \partial x_1} \\ \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3 \partial x_2} \\ \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{\partial^2 u_2}{\partial x_2 \partial x_3} + \frac{\partial^2 u_3}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 u}{\partial x_j \partial x_i}$$

C:
$$\mathrm{div}\left\{x \to -\frac{x}{\|x\|^2}\right\} = \mathrm{div}\left[\frac{-x_i}{x_1^2+x_2^2+x_3^2}\right]_{i=1}^{3} =$$

[product rule : $\mathrm{div}(f u) = f \cdot \mathrm{div}\, u + \mathrm{grad} f \cdot u$]

$$= \underbrace{-\frac{1}{\|x\|^2}}_{f}\underbrace{3}_{\mathrm{div}\, u} + \underbrace{\frac{2}{\|x\|^3}\frac{x}{\|x\|}}_{\mathrm{grad} f \text{ by chain rule}} \cdot x = -\frac{1}{\|x\|^2}$$

E:
$$\mathrm{grad}\{x \to f(\|x\|)\} = f'(\|x\|)\,\mathrm{grad}\{x \to \|x\|\}$$
$$= f'(\|x\|)\frac{x}{\|x\|}$$

Philosophy :

PDE $\neq$ just an equation $F(u, Du) = 0$

$=$ (part of) a mathematical model

$\Downarrow$

encodes structural properties

inherited from "physical system"

should be preserved by numerical method

$\Leftrightarrow$ Course outline guided by models

I. Second-order elliptic boundary value problem

$\Leftrightarrow$ heat conduction (stationary)

II. Finite element methods (FEM)

III. FEM : Convergence & Accuracy

VII. 2nd-order linear evolution problems
→ time-dependent models

VIII. Conservation laws
→ transport-dominated phenomena

Style:

Focus: not on theory
not on recipes (using codes)
on algorithms (→ substantial C++-coding)
on properties/behavior/limitations of methods

Format: Flipped-classroom

A flipped-classroom course

This course will follow the **flipped-classroom** paradigm:

Learning by **self-study** guided by

instruction videos tablet notes	lecture notes	interactive Q&A sessions	tutorial classes