ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 1.2.2: Electrostatic Fields

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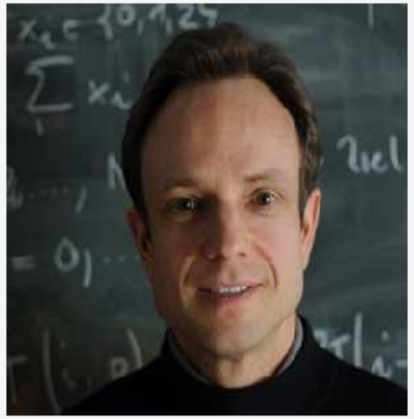


Prerequisites.

- Derivatives in 1D and 2D, in particular the gradient, see [Lecture → § 0.1.2.18]
- Piecewise continuous functions [Lecture → Section 0.1.2.3]
- Integration in 1D and 2D [Lecture → Section 0.1.2.5]

Dependency. Unit on [Lecture → Section 1.2.1] should be covered before.

Note: Possible minor *mismatch of video and tablet notes*!

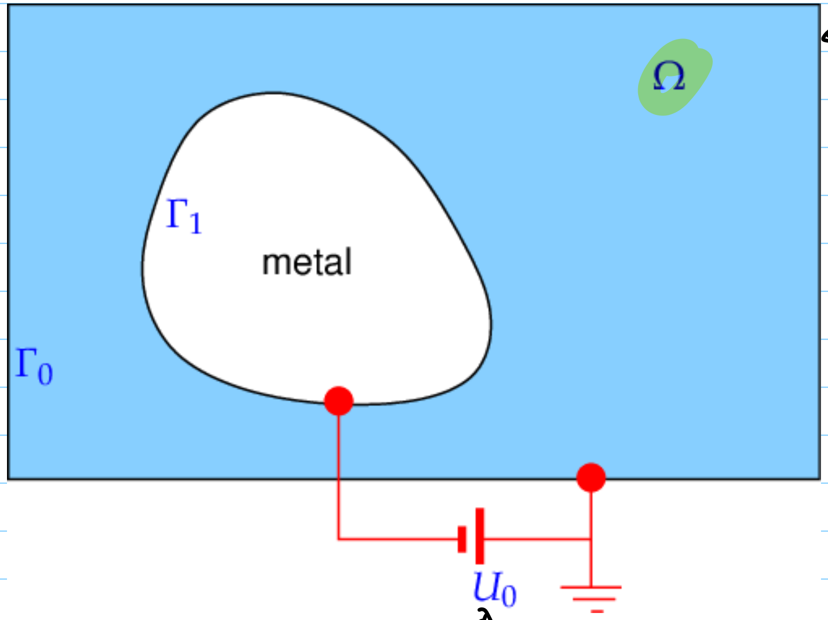
[Corrections and updates can be incorporated into tablet notes only]

I. Second-Order Scalar Elliptic Boundary Value Problems

1.2 Equilibrium Models: Examples

1.2.1. Elastic Membranes

1.2.2. Electrostatic Fields



voltage source

← conducting box

Sought: Electric Field

$E : \Omega \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$d = 3\,(2)$

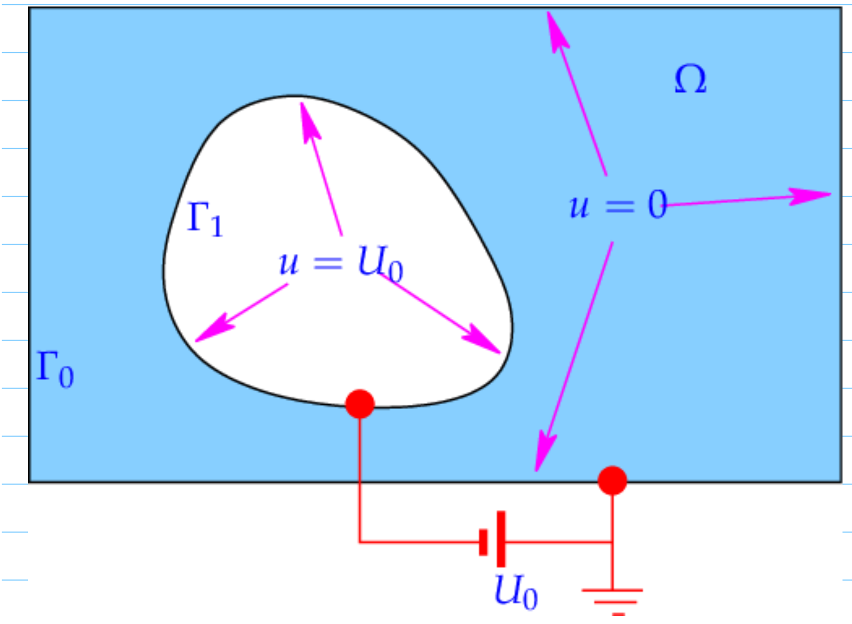
Maxwell equ.:

$E = -\operatorname{grad} u$

scalar potential $\Omega \to \mathbb{R}$

(main unknown)

② BDC: Surfaces of conductors are equipotential surfaces, ($u \equiv$ const)



$u = 0$ on Γ_0
$u = U_0$ on Γ_1

▷ (Tentative) cfg.-spc.

$$\hat{V}_E = \{ u \in C^0(\overline{\Omega}) : \begin{array}{l} u = 0 \text{ on } \Gamma_0 \\ u = U_0 \text{ on } \Gamma_1 \end{array} \}$$

Meaningful gradient ?

Physics :

Electromagnetic field energy: (electrostatic setting with scalar potential u)

$$J_E(u) = \frac{1}{2} \int_\Omega (\boldsymbol{\epsilon}(\boldsymbol{x}) \, \mathbf{grad}\, u(\boldsymbol{x})) \cdot \mathbf{grad}\, u(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \,, \tag{1.2.2.6}$$

where $\boldsymbol{\epsilon} : \Omega \mapsto \mathbb{R}^{3,3}$ is the dielectric tensor, $\boldsymbol{\epsilon}(\boldsymbol{x})$ is symmetric, with units $[\epsilon] = \frac{\mathrm{A\,s}}{\mathrm{V\,m}}$.

$\underline{\underline{\epsilon}} \hat{=}$ macroscopic material coefficient

$$(\boldsymbol{\epsilon}(\boldsymbol{x}) \, \mathbf{grad}\, u(\boldsymbol{x})) \cdot \mathbf{grad}\, u(\boldsymbol{x}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \epsilon_{ij}(\boldsymbol{x}) \frac{\partial u}{\partial x_i}(\boldsymbol{x}) \frac{\partial u}{\partial x_j}(\boldsymbol{x}) \,.$$

Essential : $J_E(u) \geq 0$

$\Rightarrow$ $\underline{\underline{\epsilon}}$ is uniformly positive definite

$$\exists 0 < \epsilon^- \leq \epsilon^+ < \infty: \; \epsilon^- \|\mathbf{z}\|^2 \leq (\boldsymbol{\epsilon}(\boldsymbol{x})\mathbf{z}) \cdot \mathbf{z} \leq \epsilon^+ \|\mathbf{z}\|^2 \quad \forall \mathbf{z} \in \mathbb{R}^3, \forall \boldsymbol{x} \in \Omega \,. \tag{1.2.2.8}$$

[apply with $\mathbf{z} := \mathbf{grad}\, u$]

Meaningful potentials must allow evaluation of $J_E(u)$

Configuration space for electrostatic field problems

As configuration space for scalar potentials we use the affine space

$$\hat{V}_E := \left\{ u \in C^1_{\mathrm{pw}}(\overline{\Omega}) \,,\, u \text{ satisfies (1.2.2.4)} \right\} \,. \tag{1.2.2.13}$$

Note: $\hat{V}_E \subset C^0(\overline{\Omega})$

Equilibrium condition for electrostatics

The physical field will attain minimal electromagnetic field energy

$$u_* = \operatorname*{argmin}_{u \in \hat{V}_E} J_E(u) \,. \tag{1.2.2.16}$$

③ Review questions 1.2.2.17

A:

Explain, why surfaces of conducting bodies are equipotential surfaces in electrostatics.

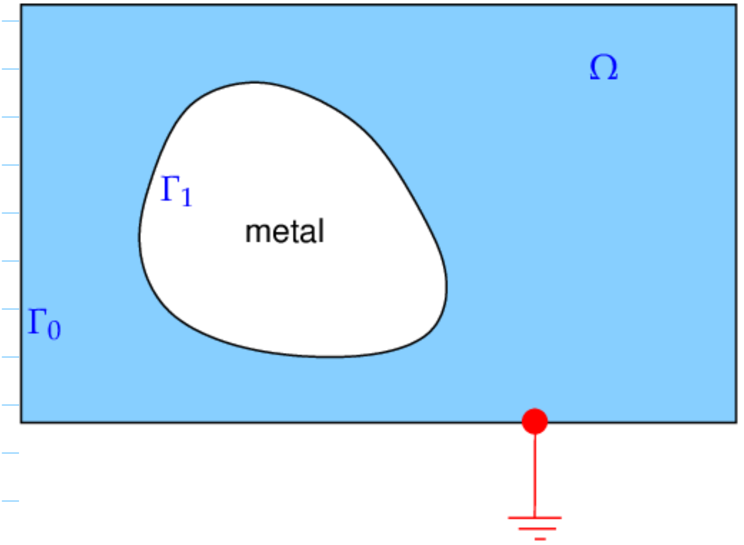
B:

What are the physical units for the electric scalar potential u, the electric field $\mathbf{E}$, the dielectric tensor ϵ, and the electromagnetic field energy

$$J_E(u) = \frac{1}{2}\int_\Omega (\epsilon(x)\,\mathbf{grad}\,u(x)) \cdot \mathbf{grad}\,u(x)\,\mathrm{d}x \text{ ?} \qquad (1.2.2.6)$$

C:

What is a suitable configuration space for describing electrostatics in the following configuration:



◁ A metallic (conducting) body is located inside a (grounded) metal box, but it is not connected to any wire, which means that its scalar potential is not known a priori.
This situation is often referred to as floating potential.