1

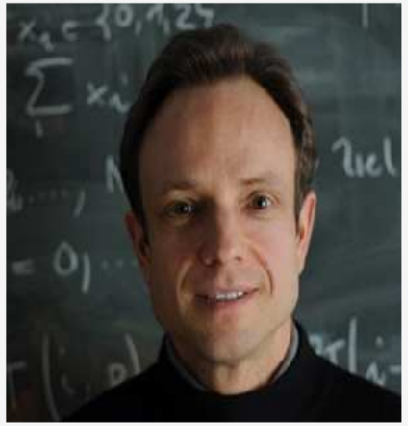
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 1.2.3: Quadratic Minimization Problems

Prof. R. Hiptmair, SAM, ETH Zurich

Date: February 11, 2019



Prerequisites.

- Linear and bilinear forms,
- Norms and inner products,
- Multi-dimensional integrals, [Lecture → Section 0.1.2.5]

Dependency. Depends mildly on unit for [Lecture → Section 1.2.1] and [Lecture → Section 1.2.2].

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

I. Second-Order Scalar Elliptic Boundary Value Problems

1.2. Equilibrium Models

1.2.3. Quadratic Minimization Problems (QMP)

Section 1.2.1.2 [1D, string]

$$u_* = \underset{\substack{u \in C^1_{pw}([a,b]) \\ u(a)=u_a\, u(b)=u_b}}{\operatorname{argmin}} \underbrace{\int_a^b \tfrac{1}{2}\sigma(x)\left|\frac{du}{dx}(x)\right|^2 - f(x)u(x)\,\mathrm{d}x}_{=J_S(u),\ \text{see (1.2.1.18)}} \tag{1.2.3.1a}$$

J_S

Section 1.2.1.2 [2D membrane]

$$u_* = \underset{\substack{u \in C^1_{pw}(\overline{\Omega}) \\ u=g \text{ on } \partial\Omega}}{\operatorname{argmin}} \underbrace{\int_\Omega \tfrac{1}{2}\sigma(\boldsymbol{x})\|\mathbf{grad}\, u(\boldsymbol{x})\|^2 - f(\boldsymbol{x})u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{=:J_M(u),\ \text{see (1.2.1.19)}}\,, \tag{1.2.3.1b}$$

J_M

Section 1.2.2 [2D, 3D electrostatics]

$$u_* = \underset{\substack{u \in C^1_{pw}(\overline{\Omega}) \\ u=U \text{ on } \partial\Omega}}{\operatorname{argmin}} \underbrace{\int_\Omega \tfrac{1}{2}(\epsilon(\boldsymbol{x})\,\mathbf{grad}\, u(\boldsymbol{x}))\cdot \mathbf{grad}\, u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{=:J_E(u),\ \text{see (1.2.2.6)}}\,. \tag{1.2.3.1c}$$

J_E

$\triangleright \quad J_* \ \hat{=}$ quadratic functional

Definition 1.2.3.2. Quadratic functional

A quadratic functional on a *real vector space* V_0 is a mapping $J : V_0 \mapsto \mathbb{R}$ of the form

$$J(u) := \tfrac{1}{2}\mathsf{a}(u,u) - \ell(u) + c\,, \quad u \in V_0\,, \tag{1.2.3.3}$$

where $\mathsf{a} : V_0 \times V_0 \mapsto \mathbb{R}$ is a symmetric bilinear form (→ Def. 0.1.1.7), $\ell : V_0 \mapsto \mathbb{R}$ a linear form, and $c \in \mathbb{R}$.

2

Example: $V_0 = \mathbb{R}^N$

$$J(\vec{\nu}) = \tfrac{1}{2}\vec{\nu}^T A \vec{\nu} - \vec{\beta}^T \vec{\nu} + c \,, \quad A \in \mathbb{R}^{N,N},\ \vec{\beta} \in \mathbb{R}^N$$

On $\mathbb{R}^N$: bilinear form $\Leftrightarrow$ matrix $\in \mathbb{R}^{N,N}$

linear form $\Leftrightarrow$ vector $\in \mathbb{R}^N$

Q: What is $a(\cdot,\cdot)$ and $\ell(\cdot)$ for J_M

Note: $\|z\|^2 = z \cdot z$, $z \in \mathbb{R}^2$

$$a(u,v) = \int_\Omega \sigma(x)\, \mathbf{grad}\, u \cdot \mathbf{grad}\, v \, dx$$

$$\ell(v) = \int_\Omega f(x)\, v(x)\, dx$$

$$u, v \in V_0 := C^1_{pw,0}(\bar{\Omega})$$

Definition 1.2.3.11. Quadratic minimization problem

A minimization problem

$$w_* = \operatorname*{argmin}_{w \in V_0} J(w)$$

is called a quadratic minimization problem, if J is a *quadratic functional* on a real vector space V_0.

Extension to affine space: $\hat{V} = u_0 + V_0 \subset V$

"offset function"

Consider: $\operatorname*{argmin}_{\hat{v} \in \hat{V}} J(\hat{v})$

Write $\hat{v} = u_0 + u$

$$\begin{aligned} J(u+u_0) &= \tfrac{1}{2}\mathrm{a}(u+u_0, u+u_0) - \ell(u+u_0) + c \\ &\overset{(*)}{=} \tfrac{1}{2}\mathrm{a}(u,u) + \tfrac{1}{2}\mathrm{a}(u,u_0) + \tfrac{1}{2}\mathrm{a}(u_0,u) + \tfrac{1}{2}\mathrm{a}(u_0,u_0) - \ell(u) - \ell(u_0) + c \\ &= \tfrac{1}{2}\mathrm{a}(u,u) + \underbrace{\mathrm{a}(u,u_0) - \ell(u)}_{=:\tilde{\ell}(u)} + \underbrace{\tfrac{1}{2}\mathrm{a}(u_0,u_0) - \ell(u_0) + c}_{=:\tilde{c}} =: \tilde{J}(u) . \end{aligned} \quad (1.2.3.14)$$

(*) use (bi-)linearity

Note: for fixed u_0: $u \to a(u,u_0)$ is a linear form!

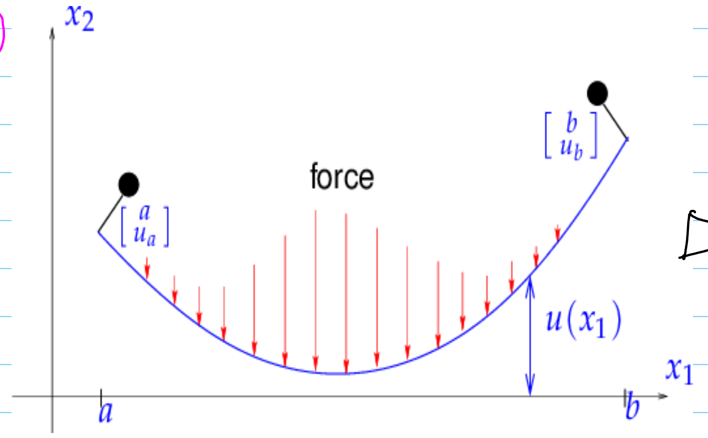
$$\operatorname*{argmin}_{\hat{u} \in \hat{V}} J(\hat{u}) = u_0 + \operatorname*{argmin}_{u \in V_0} \tilde{J}(u)$$

a function satisfying BDC

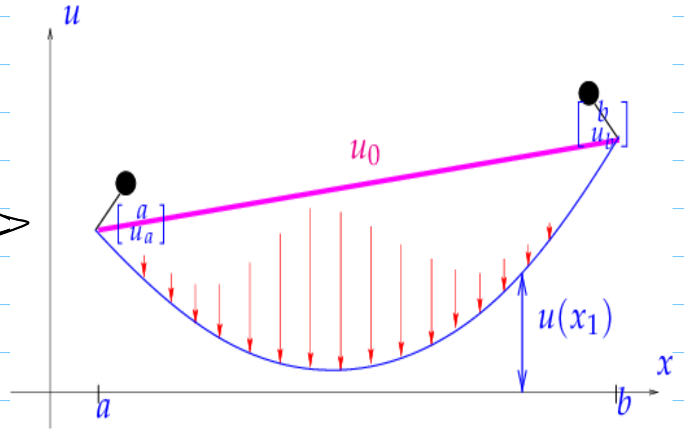
Fits Def 1.2.3.11!

Q: What is u_0 for the elastic string model?

③



▷



1.2.3.2. Existence & Uniqueness of Minimizers

A *necessary* condition for E & U:

Definition 1.2.3.30. Positive definite bilinear form → Def. 0.1.1.21

A (symmetric) bilinear form $\mathsf{a} : V_0 \times V_0 \mapsto \mathbb{R}$ on a real vector space V_0 is positive definite, if

$$u \in V_0 \setminus \{0\} \iff \mathsf{a}(u,u) > 0 .$$

Partial proof (indirect)

Assume: $\exists z \in V_0 \setminus \{0\}: \quad a(z,z) < 0$

$$t \to J(tz) = \tfrac{1}{2} t^2 \underbrace{a(z,z)}_{<0} - t\ell(z) + c \xrightarrow{t\to\infty} -\infty$$

no minimum! $\Box$

A *sufficient* condition for uniqueness:

Theorem 1.2.3.31. Uniqueness of solutions of quadratic minimization problems

If the bilinear form $\mathsf{a} : V_0 \times V_0 \mapsto \mathbb{R}$ is positive definite (→ Def. 1.2.3.30), then any solution of

$$u_* = \operatorname*{argmin}_{u \in V_0} J(u) \quad , \quad J(u) := \tfrac{1}{2}\mathsf{a}(u,u) - \ell(u) + c \,,$$

is unique for any linear form $\ell : V_0 \mapsto \mathbb{R}$.

Proof (indirect) $\exists$ minimizers $u, w \in V_0$, $u \neq w$

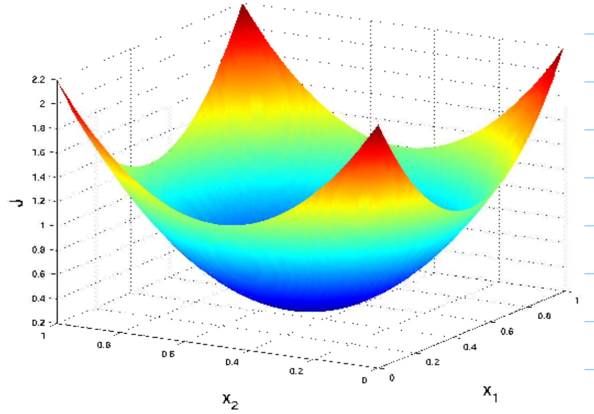
$\varphi(t) := J(tu + (1-t)w)$ has two minima for $t = 0, 1$

$\varphi(t) = \tfrac{1}{2} t^2 a(u-w, u-w) + \dots \overset{\Downarrow}{\equiv}$ const

↳ a parabola! $\quad a(u-w, u-w) = 0$ ←

↯ impossible, since a pos. def. $\Box$

$V_0 = \mathbb{R}^N$ & $a(\cdot,\cdot)$ s.p.d $\Rightarrow$ E & U



◁ $N = 2$

Graph of J for s.p.d. $a(\cdot,\cdot)$

Recall from LA: $A = A^T$ s.p.d $\Rightarrow$

$$(\vec{\mu},\vec{\eta}) \to \vec{\mu}^T A \vec{\eta} \;\hat{=}\; \text{inner product}$$

Definition 1.2.3.35. Energy norm, *cf.*

Let a the symmetric positive definite bilinear form $\mathrm{a} : V_0 \times V_0 \mapsto \mathbb{R}$ ($\rightarrow$ Def. 0.1.1.21) underlying a quadratic functional J. Then the related energy norm is

$$\|u\|_{\mathrm{a}} := (\mathrm{a}(u,u))^{1/2}\,, \quad u \in V_0\,.$$

Ex: Electrostatic energy functional J_E

$$a(u,v) = \int_\Omega \varepsilon(x)\, \operatorname{grad} u \cdot \operatorname{grad} v \, dx$$

$$V_0 := C^1_{pw,0}(\bar{\Omega})$$

Know:

$$\exists\, 0 < \epsilon^- \le \epsilon^+ < \infty:\ \epsilon^- \|\mathbf{z}\|^2 \le (\epsilon(\mathbf{x})\mathbf{z}) \cdot \mathbf{z} \le \epsilon^+ \|\mathbf{z}\|^2 \quad \forall \mathbf{z} \in \mathbb{R}^3, \forall \mathbf{x} \in \Omega, \quad (1.2$$

Q: Is $a(\cdot,\cdot)$ s.p.d.?

$$a(u,u) = \int_\Omega \varepsilon(x)\, \operatorname{grad} u \cdot \operatorname{grad} u \, dx \ge \varepsilon^- \int_\Omega \|\operatorname{grad} u\|^2 dx \ge 0$$

If $a(u,u) = 0 \Rightarrow \operatorname{grad} u = 0$
$\Rightarrow u \equiv \text{const}$

$+$ BDC $\Rightarrow u = 0$

$u = 0$ on $\partial\Omega$

1.2.3.4. A Continuity Condition for $\ell(\cdot)$

A necessary condition for the existence of a minimizer

Lemma 1.2.3.39. Boundedness condition on linear form

The quadratic functional J ($\rightarrow$ Def. 1.2.3.2) based on a symmetric positive definite bilinear form a ($\rightarrow$ Def. 1.2.3.30) is bounded from below on V_0, if and only if

$$\boxed{\exists C > 0:\quad |\ell(u)| \le C\|u\|_{\mathrm{a}} \quad \forall u \in V_0}\,, \qquad (1.2.3.40)$$

where $\|\cdot\|_{\mathrm{a}}$ is the energy norm induced by a, see Def. 1.2.3.38.

$[\,J(u) = \tfrac{1}{2} a(u,u) - \ell(u)\,]$

$\ell(\cdot)$ is continuous / bounded w.r.t $\|\cdot\|_a$

Proof:

① "$\Leftarrow$" $J(u) = \tfrac{1}{2} a(u,u) - \ell(u)$
$\ge \tfrac{1}{2}\|u\|_a^2 - C\|u\|_a \ge -\tfrac{1}{2}C^2$

② "$\Rightarrow$" (indirect)

Assume that $\boxed{}$ does not hold

$\exists\, u_n \in V_0,\ n\in\mathbb{N},\ \|u_n\|_a = 1,\ \ell(u_n) \geq n\|u_n\|_a$

$$J(u_n) = \tfrac{1}{2} - n \rightarrow -\infty \quad \text{for } n\rightarrow\infty$$

$\square$

Example 1.2.3.45: "Needle loading" of a membrane

$$u_* = \operatorname*{argmin}_{\substack{u\in C^1_{\mathrm{pw}}(\overline{\Omega})\\ u=g \text{ on } \partial\Omega}} \underbrace{\int_\Omega \tfrac{1}{2}\sigma(\boldsymbol{x})\|\mathbf{grad}\, u(\boldsymbol{x})\|^2\,\mathrm{d}\boldsymbol{x}}_{=:\frac{1}{2}\mathrm{a}(u,u)} - \underbrace{u(\boldsymbol{y})}_{=:\ell(u)} . \qquad (1.2.3.46)$$

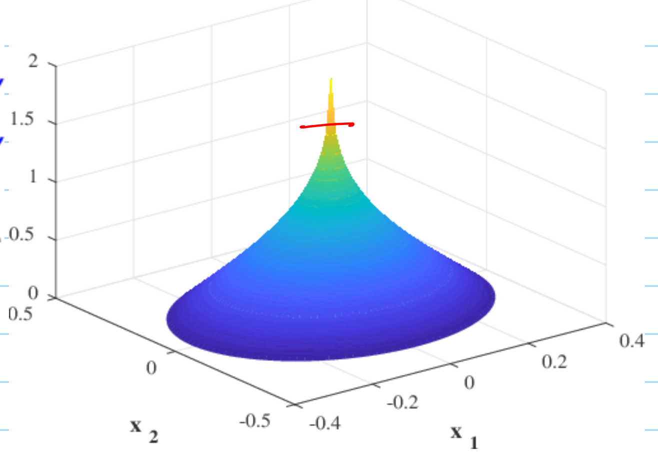
point evaluation functional

$V_0 = C^1_{pw,0}(\overline{\Omega})$

$$v_\epsilon(\boldsymbol{x}) := \begin{cases} \log|\log\|\boldsymbol{x}\|| & \text{, if } \|\boldsymbol{x}\| > \epsilon ,\\ \log|\log\epsilon| & \text{, if } \|\boldsymbol{x}\| \leq \epsilon , \end{cases}$$

$\in V_0$

on the disk domain $\Omega = \{\boldsymbol{x}\in\mathbb{R}^2 : \|\boldsymbol{x}\| < \frac{1}{e}\}$.



v_ε radially symmetric

$\|v\|_a$ by integration in polar coordinates

$$\mathbf{grad}\, v_\epsilon(r,\varphi) = \frac{\partial v_\epsilon}{\partial r}(r,\varphi)\mathbf{e}_r + \cancel{\frac{1}{r}\frac{\partial v_\epsilon}{\partial\varphi}(r,\varphi)\mathbf{e}_\varphi} , \qquad (1.2.3.51)$$

radial unit vector

$$\|v_\epsilon\|_a^2 = \int_\Omega \|\mathbf{grad}\, v(\boldsymbol{x})\|^2\,\mathrm{d}\boldsymbol{x} = \int_\epsilon^{1/e}\int_0^{2\pi}\left\|-\frac{1}{\log r\, r}\mathbf{e}_r\right\|^2 r\,\mathrm{d}\varphi\mathrm{d}r \leq 2\pi\int_0^{1/e}\frac{1}{\log^2 r}\cdot\frac{1}{r}\,\mathrm{d}r$$
$$= 2\pi\left[-1/\log r\right]_0^{1/e} = \frac{2\pi}{\log e} = 2\pi < \infty .$$

Well: $y := 0$

$\ell(v_\varepsilon) = v_\varepsilon(0) = \log|\log\varepsilon| \longrightarrow \infty$ for $\varepsilon\to 0$

arbitrarily big, but $\|v_\varepsilon\|_a \leq \sqrt{2\pi}\ \ \forall\varepsilon$

$\boxed{}$ is violated!

$\triangleright$

⑥ Review questions 1.2.3.50

A:

Let V be a real vector space. Give the formal mathematical definitions of the following concepts:

- of a linear form ℓ on V,
- of a bilinear form a on V,
- of a positive definite bilinar form a on V,
- and of a quadratic functional on V.

B:

The electrostatic field energy

$$J_E(u) := \int_\Omega \tfrac{1}{2}(\epsilon(\boldsymbol{x})\,\mathbf{grad}\,u(\boldsymbol{x}))\cdot\mathbf{grad}\,u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\;.$$

is a quadratic functional. On which space is it defined and what are the involved bilinear form and linear form according to the abstract definition of a quadratic functional?

C:

The general form a quadratic functional on $\mathbb{R}^N$, $N\in\mathbb{N}$, is

$$J(\vec{\eta}) = \tfrac{1}{2}\vec{\eta}^\top\mathbf{A}\vec{\eta} - \vec{\beta}^\top\vec{\eta} + c \quad,\quad \mathbf{A}=\mathbf{A}^\top\in\mathbb{R}^{N,N}\;,\quad \vec{\beta}\in\mathbb{R}^N\;,\quad c\in\mathbb{R}\;. \tag{1.2.3.6}$$

Write down the matrix $\mathbf{A}$, the vector $\vec{\beta}$, and the number c concretely for the quadratic functional

$$\boldsymbol{x}\in\mathbb{R}^2 \mapsto \|\boldsymbol{x}\|^2 - (x_1-1)\;.$$

D:

For what values of $\alpha,\beta,\gamma\in\mathbb{R}$ does the quadratic functional

$$J:\mathbb{R}^2\to\mathbb{R}\quad,\quad J(\boldsymbol{x}) := \boldsymbol{x}^\top\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}\boldsymbol{x} - \gamma x_1 + \alpha\;,$$

possess a unique minimizer.

E:

What is the energy norm induced by a symmetric positive definite bilinear form on a vector space V?

F:

What does it mean that a linear functional ℓ on a vector space V_0 is bounded/continuous with respect to a norm $\|\cdot\|$ on V_0?

G:

Show that every linear functional on $V_0 := \mathbb{R}^2$ is bounded with respect to the Euclidean norm on $\mathbb{R}^2$.

H:

Rephrase the following statement in formal mathematical terms:

> Point evaluation is an unbounded linear functional on $C^1([0,1]^2)$ with respect to the energy norm

$$\|v\|_{\mathsf{a}} := \left(\int_{[0,1]^2}\|\mathbf{grad}\,v(\boldsymbol{x})\|^2\,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{2}}\;.$$