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ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 1.3: Sobolev Spaces

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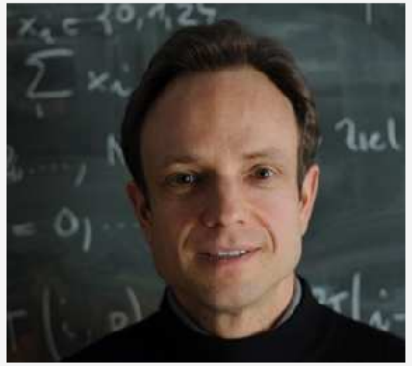

Prerequisites.

- Norms and inner products, [Lecture → Section 0.1.1.3]
- Quadratic minimization problems, [Lecture → Section 1.2.3]
- Cauchy-Schwarz inequality, [Lecture → Thm. 0.1.1.25]

Dependency. Depends on units for [Lecture → Section 1.2.3] and [Lecture → Section 1.2.1].

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

I. Second-Order Scalar Elliptic Boundary Value Problems

1.3. Sobolev Spaces

We will also need the following spaces (see [19]):

$$V_0 = H_{0,\Gamma_\tau}(\mathrm{curl}^0;\Omega) = \left\{ \underline{v} \in V \,\middle|\, \mathrm{curl}\, \underline{v} = 0 \right\} \tag{3}$$

$$H_{0,\Gamma_\nu}(\mathrm{div}^0,\Omega,\epsilon) = \left\{ \underline{v} \in L^2(\Omega)^3 \,\middle|\, \mathrm{div}\,\epsilon \underline{v} = 0, \epsilon \underline{v}\cdot \underline{n}|_{\Gamma_\nu} = 0 \right\} \tag{4}$$

$$H_1 = \epsilon^{-1}\mathrm{curl}\,(H_{0,\Gamma_\nu}(\mathrm{curl},\Omega)) \subset H_{0,\Gamma_\nu}(\mathrm{div}^0,\Omega,\epsilon) \tag{5}$$

$$V_1 = V \cap H_1 \tag{6}$$

$$\mathbb{H} = \mathbb{H}(\Omega,\Gamma_\tau,\epsilon) = H_{0,\Gamma_\tau}(\mathrm{curl}^0,\Omega) \cap H_{0,\Gamma_\nu}(\mathrm{div}^0,\Omega,\epsilon) \tag{7}$$

$$H^1_{0,\Gamma_\tau}(\Omega) = \left\{ \phi \in L^2(\Omega) \,\middle|\, \mathrm{grad}\,\phi \in L^2(\Omega)^3,\ \phi|_{\Gamma_\tau} = 0 \right\}. \tag{8}$$

We will indicate by $\|\ \|_{0,\Omega}$ the norm in H corresponding to $(\ ,\ )_{0,\Omega}$, by $(\ ,\ )_{\mathrm{curl},\Omega}$ and $\|\ \|_{\mathrm{curl},\Omega}$ the standard inner product and norm in $H(\mathrm{curl},\Omega)$, respectively, and by $\|\ \|_{s,\Omega}$, $0 < s \le 1$, the natural norm in $H^s(\Omega)$ or $H^s(\Omega)^3$. For $s = 1$ we will also use the natural seminorm $|\ |_{1,\Omega}$ [16]. Finally, we define the following inner products and norms in H and V:

Example: 1.3.0.1 :

We consider the quadratic functional

$$J(u) := \int_0^1 \tfrac{1}{2}u^2(x) - u(x)\,\mathrm{d}x = \tfrac{1}{2}\int_0^1 \{(u(x)-1)^2 - 1\}\,\mathrm{d}x\,,$$

on $V_0 = C_0^0([0,1])$

▷ Energy norms : $\|u\|_a^2 = \int_0^1 |u(x)|^2 dx$

$\ell(v) = \int_0^1 u(x)dx$

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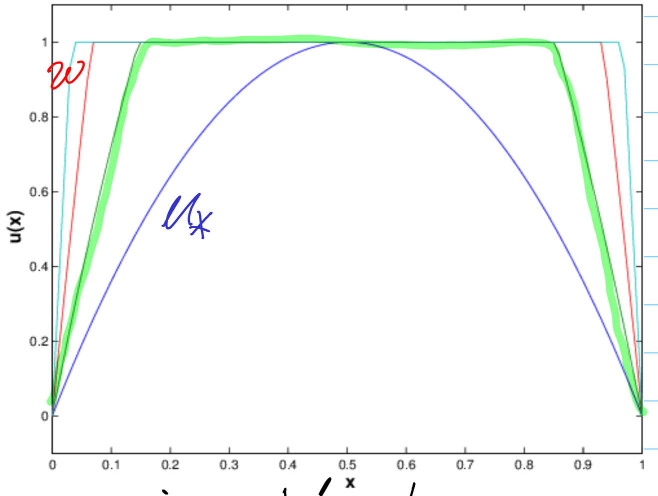
Consider QMP for J:

Assume that $u_* \in V_0$ is a global minimizer of J. Then

$$w(x) := \min\{1, 2\max\{u_*(x), 0\}\}, \quad 0 \le x \le 1,$$

is another function $\in C^0_0([0,1])$, which satisfies

$$u(x) \neq 1 \quad \Rightarrow \quad |w(x) - 1| < |u_*(x) - 1| \quad \Rightarrow \quad J(w) < J(u_*) \quad !$$

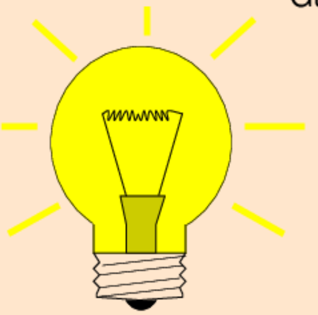


$\triangleright$ Non-existence of minimizers in V_0 !

Related: Find a minimizer $x \to |x^2 - 2|$ in $\mathbb{Q}$

Remedy: $\mathbb{Q} \longrightarrow \mathbb{R}$

1.3.1. Function Spaces for Energy Minimization



Guideline: for a quadratic minimization problem ($\to$ Def. 1.2.3.13) with
- symmetric positive definite (s.p.d.) bilinear form a,
- a linear form ℓ that is continuous w.r.t. $\|\cdot\|_a$, see (1.2.3.43),

posed over a function space follow the advice:

$$\boxed{\text{consider it on the largest space of functions for which a still makes sense !}}$$

Largest possible function space

(and which complies with boundary conditions)

Choose "$V_0 := \{\text{functions } v \text{ on } \Omega : \mathsf{a}(v,v) < \infty\}$"

$\hookrightarrow$ Energy space generated by $a(\cdot,\cdot)$

1.3.2. The Function Space $L^2(\Omega)$

We consider the quadratic functional

$$J(u) := \int_0^1 \tfrac{1}{2}u^2(x) - u(x)\,dx = \tfrac{1}{2}\int_0^1 \{(u(x)-1)^2 - 1\}\,dx\,, \tag{1.3.2.1}$$

$$\triangleright \quad \|u\|_a^2 = \int_0^1 |u(x)|^2\,dx\,, \quad \Omega =]0,1[$$

$$V_0 := \{v : \Omega \to \mathbb{R} : \int_\Omega |v(x)|^2\,dx < \infty\}$$

Definition 1.3.2.3. Space $L^2(\Omega)$ $\to$ **Def. 0.1.2.27**

The function space defined in (1.3.2.2) is the space of square-integrable functions on Ω and denoted by $L^2(\Omega)$.

It is a normed space with norm $\left(\|v\|_0 :=\right) \|v\|_{L^2(\Omega)} := \left(\int_\Omega |v(x)|^2\,dx\right)^{1/2}$.

✎ Notation: $L^2(\Omega)$ $\leftarrow$ superscript "2", because square in the definition of norm $\|\cdot\|_0$

$$\triangleright \quad \operatorname*{argmin}_{u \in L^2(]0,1[)} J(u) = \{x \to 1\}$$

③ # 1.3.4. The Sobolev Space $H^1(\Omega)$

$\Omega \subset \mathbb{R}^d$:

$$J(v) := \tfrac12 \int_\Omega \|\operatorname{grad} v\|^2 dx - f(x)v(x)dx$$

+ zero BDC on $\partial\Omega$

$$\|v\|_a^2 = \int_\Omega \|\operatorname{grad} u\|^2 dx$$

▷ $V_0 := \left\{ v : \Omega \to \mathbb{R} : \begin{array}{l} v = 0 \text{ on } \partial\Omega \\ \|v\|_a < \infty \end{array} \right\}$

Definition 1.3.4.3. Sobolev space $H_0^1(\Omega)$

The space of integrable functions on Ω with square integrable gradient that vanish on the boundary $\partial\Omega$,

$$V_0 := \{v : \Omega \mapsto \mathbb{R} \text{ integrable: } v = 0 \text{ on } \partial\Omega, \int_\Omega |\operatorname{\mathbf{grad}} v(x)|^2 \,\mathrm{d}x < \infty\}, \tag{1.3.4.2}$$

is the Sobolev space $H_0^1(\Omega)$ with norm

$$|v|_{H^1(\Omega)} := \left(\int_\Omega \|\operatorname{\mathbf{grad}} v\|^2 \,\mathrm{d}x\right)^{1/2}.$$

✎ Notation: $H_0^1(\Omega)$

← superscript “1”, because first derivatives occur in norm

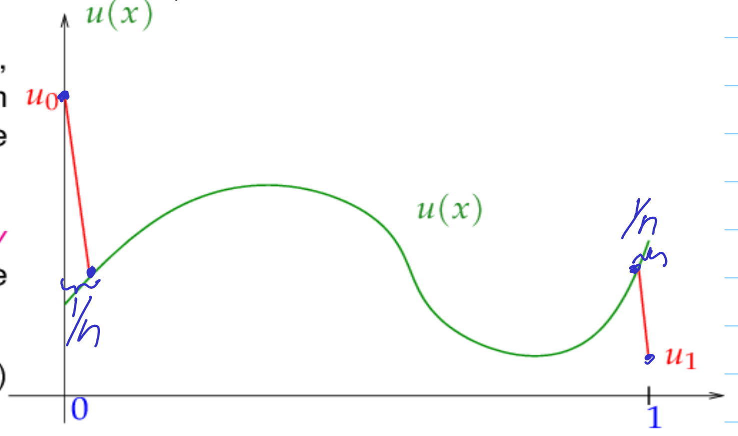
← subscript “0”, because zero on $\partial\Omega$

Ex 1.3.2.5 / Rem 1.3.4.4:

“Thought experiment”, $\Omega =]0,1[$

Now we pursue a more sophisticated reasoning, which will finally tell us that fixing the values in $x = 0, 1$ for functions in $L^2(]0,1[)$ does not make sense.

Consider $u \in C^0([0,1])$ and try to impose *any* boundary values $u_0, u_1 \in \mathbb{R}$ by “altering” u in the following way:



(red parts of the graph belong to $\tilde u_n$.)

$$\tilde u_n(x) := \begin{cases} u(x) + (1-nx)(u_0 - u(0)) & \text{, for } 0 \le x \le \frac1n, \\ u(x) & \text{, for } \frac1n < x < 1 - \frac1n, \\ u(x) - n(1 - \frac1n - x)(u_1 - u(1)) & \text{, for } 1 - \frac1n < x \le 1, \end{cases} \quad n \in \mathbb{N}.$$

$\tilde u_n(0) = u_0$, $\tilde u_n(1) = u_1$

$\|u - \tilde u_n\|_{L^2(\Omega)} \xrightarrow{n\to\infty} 0$ $\qquad$ $\|u - \tilde u_n\|_{H^1(\Omega)} \xrightarrow{n\to\infty} \infty$

↓ Any boundary conditions can be imposed through a negligible change of $\|\cdot\|_{L^2}$

⇓

BDC meaningless in L^2

↓ Changing boundary values inevitably affects $\|\cdot\|_{H^1}$

⇓

BDC can be imposed in H^1

Dropping BDC of $H_0^1(\Omega)$

Definition 1.3.4.8. Sobolev space $H^1(\Omega)$

The Sobolev space

$$H^1(\Omega) := \{v : \Omega \mapsto \mathbb{R} \text{ integrable: } \int_\Omega |\mathbf{grad}\, v(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} < \infty\}$$

is a normed function space with norm

$$\|v\|^2_{H^1(\Omega)} := \|v\|_0^2 + |v|^2_{H^1(\Omega)} .$$

this alone is not a norm on $H^1(\Omega)$

QMP on $H^1(\Omega) / H_0^1(\Omega)$:

Section 1.2.1.2 [1D, string]

$$u_* = \underset{\substack{v \in H^1(]a,b[) \\ v(a)=u_a\; v(b)=u_b}}{\operatorname{argmin}} \underbrace{\int_a^b \tfrac{1}{2}\sigma(x)\left|\frac{dv}{dx}(x)\right|^2 - f(x)v(x)\,\mathrm{d}x}_{=J_S(u),\text{ see (1.2.1.18)}} \tag{1.3.4.9a}$$

Section 1.2.1.2 [2D membrane]

$$u_* = \underset{\substack{v \in H^1(\Omega) \\ v=g \text{ on } \partial\Omega}}{\operatorname{argmin}} \underbrace{\int_\Omega \tfrac{1}{2}\sigma(\boldsymbol{x})\|\mathbf{grad}\, v(\boldsymbol{x})\|^2 - f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{=:J_M(u),\text{ see (1.2.1.19)}} , \tag{1.3.4.9b}$$

Section 1.2.2 [2D, 3D electrostatics]

$$u_* = \underset{\substack{v \in H^1(\Omega) \\ v=U \text{ on } \partial\Omega}}{\operatorname{argmin}} \underbrace{\int_\Omega \tfrac{1}{2}(\epsilon(\boldsymbol{x})\,\mathbf{grad}\, v(\boldsymbol{x})) \cdot \mathbf{grad}\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{=:J_E(u),\text{ see (1.2.2.6)}} . \tag{1.3.4.9c}$$

Here $V_0 = H_0^1(\Omega)$

$$J_M(v) = \underbrace{\tfrac{1}{2}\int_\Omega \|\mathrm{grad}\, v\|^2}_{\to\, a(v,v)} - \underbrace{f(x)v(x)\,dx}_{\to\, \ell(v)}$$

on $V_0 := H_0^1(\Omega) \to a(v,v)$
$\hookrightarrow$ s.p.d.

We need: $\exists C > 0 : \quad |\ell(v)| \le C\|v\|_a$

Key: Cauchy-Schwarz inequality

LA, abstract: $a(\cdot,\cdot)$ s.p.d. $\Rightarrow |a(u,v)| \le \|u\|_a \|v\|_a$

$$\left|\int_\Omega u \cdot v\,dx\right| \le \|u\|_0 \|v\|_0 \quad \forall u,v \in L^2(\Omega)$$

$$|\ell(v)| = \left|\int_\Omega f(x)v(x)dx\right| \le \|f\|_0 \|v\|_0$$

Theorem 1.3.4.17. First Poincaré-Friedrichs inequality

If $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, *is* bounded, *then*

$$\|u\|_0 \le \operatorname{diam}(\Omega)\,\|\mathbf{grad}\, u\|_0 \quad \forall u \in H_0^1(\Omega) .$$

$$\triangleright \quad |\ell(v)| \le \underbrace{\|f\|_0 \operatorname{diam}(\Omega)}_{C} \|v\|_a \quad \forall v \in H_0^1(\Omega)$$

Corollary 1.3.4.19. Admissible loading/source functions linear 2nd-order elliptic problems

If $f \in L^2(\Omega)$, then $\ell(v) := \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ *is a continuous linear functional on* $H_0^1(\Omega)$.

As in Section 1.2.3.4 in this lemma "continuity" has to be read as

$$\boxed{\exists C > 0: \quad |\ell(u)| \le C|u|_{H^1(\Omega)} \quad \forall u \in H_0^1(\Omega)} \,. \tag{1.2.3.43}$$

▷ $f \in L^2(\Omega) \Rightarrow J_\mu$ has a unique minimizer on $H_0^1(\Omega)$

How to "work with" Sobolev spaces

Most concrete results about Sobolev spaces boil down to relationships between their norms. The spaces themselves remain intangible, but the norms are very concrete and can be computed and manipulated as demonstrated above.

Do not be afraid of Sobolev spaces!

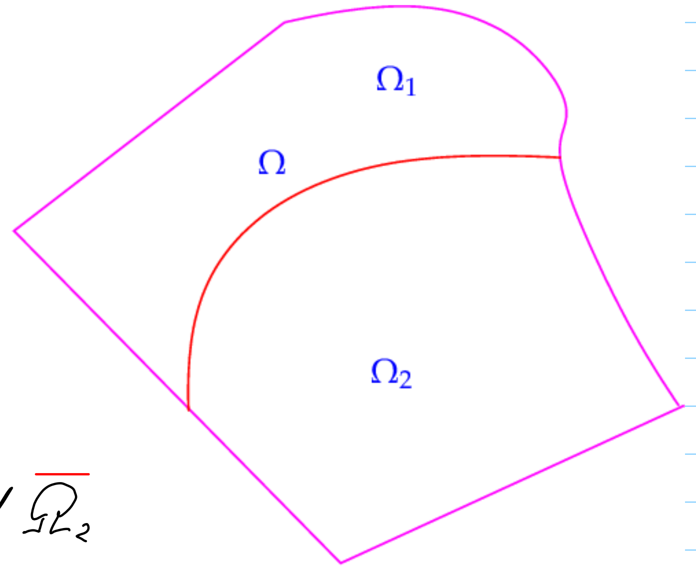
It is only the norms that matter for us, the 'spaces" are irrelevant!

Other information conveyed by $u \in H^1(\Omega)$

$\longrightarrow$ e.g. "continuity"

Theorem 1.3.4.23. Compatibility conditions for piecewise smooth functions in $H^1(\Omega)$

Let Ω be partitioned *into sub-domains Ω_1 and Ω_2. A function u that is continuously differentiable in the closures* of both sub-domains, belongs to $H^1(\Omega)$,* if and only if *u is* continuous *on Ω.*



* derivatives continuous on $\overline{\Omega}_1$ and $\overline{\Omega}_2$ ("up to the boundary")

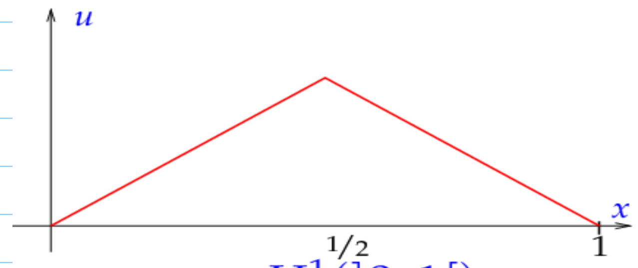
p.w. C^1 & $\in H^1 \Longrightarrow C^0$

▷ $C^1_{pw}(\overline{\Omega}) \subset H^1(\Omega)$

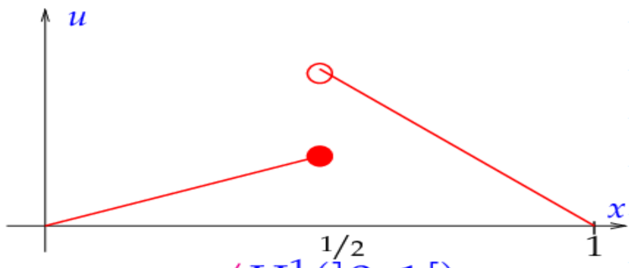
$\uparrow$

"Mental replacement for $H^1(\Omega)$"

Ex 1.3.4.27 :



$u \in H_0^1(]0,1[)$



$u \notin H_0^1(]0,1[)$

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Corollary 1.3.4.26. H^1-norm of piecewise smooth functions

Under the assumptions of Thm. 1.3.4.23 we have for a continuous, piecewise smooth function $u \in C^0(\Omega)$

$$|u|^2_{H^1(\Omega)} = |u|^2_{H^1(\Omega_1)} + |u|^2_{H^1(\Omega_2)} = \int_{\Omega_1} |\mathbf{grad}\, u(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} + \int_{\Omega_2} |\mathbf{grad}\, u(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x}\,.$$

If $u \in C^0(\Omega) \Rightarrow$ compute H^1-norm piecewise

Q :

Which of the following functions belong to the spaces $L^2(]-1,1[)$ and $H^1(]0,1[)$, respectively?

- $f(x) = |x|$
- $f(x) = \log|x|$
- $f(x) = \mathrm{sgn}(x)$
- $f(x) = \sqrt{|x| + x}$.

Hint :
- No "continuity" in $L^2(\Omega)$
- Check finite energy

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Review questions 1.3.4.29

A:

Which of the following functions belong to the spaces $L^2(]-1,1[)$ and $H^1(]0,1[)$, respectively?

1. $f(x) = |x|$,
2. $f(x) = \log |x|$,
3. $f(x) = \mathrm{sgn}(x)$,
4. $f(x) = \sqrt{|x| + x}$.

B:

Explain the statement

> For bounded domains $\Omega \subset \mathbb{R}^d$ the restriction of functions to the boundary $\partial\Omega$ makes sense in $H^1(\Omega)$

in terms of $H^1(\Omega)$-norms of functions.

C:

Show that the point evaluation $v \mapsto v(\frac{1}{2})$ is an unbounded linear functional on $L^2(]0,1[)$.

D:

For a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, which of the spaces $C^0_{\mathrm{pw}}(\overline{\Omega})$, $C^1_{\mathrm{pw}}(\overline{\Omega})$, and $C^2_{\mathrm{pw}}(\overline{\Omega})$ is/is not contained in $L^2(\Omega)$ and $H^1(\Omega)$, respectively?

E:

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Define the Sobolev space fitting the quadratic minimization problem for the functional

$$J(\mathbf{v}) := \int_\Omega |\operatorname{div} \mathbf{v}(\boldsymbol{x})|^2 + \|\mathbf{v}\|^2 \, d\boldsymbol{x} \,, \quad \mathbf{v} = (C^1(\overline{\Omega}))^2 \,.$$

F:

Which Sobolev space, call it W, fits minimization problems for the functional

$$J(v) := \int_\Omega |\mathbf{d} \cdot \mathbf{grad}\, u|^2 + u^2 \, d\boldsymbol{x} \,, \quad v \in C^\infty(\overline{\Omega}) \,,$$

where $\mathbf{d} \in \mathbb{R}^2$ is a fixed unit vector, and $\Omega =]0,1[^2$.

- Give an example of a function belonging to W, but not to $H^1(\Omega)$.
- Show that $H^1(\Omega) \subset W$.

This list of review questions may not be complete. Additional review questions may be provided in the lecture document.