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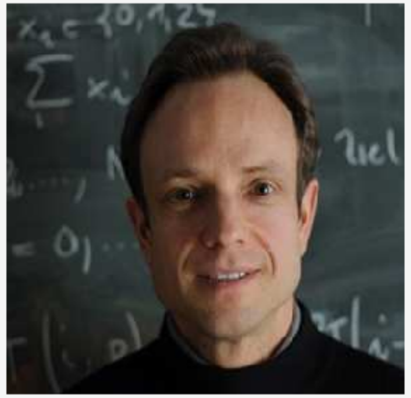
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 1.4: Linear Variational Problems



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Date: February 12, 2019

(C) Seminar für Angewandte Mathematik, ETH Zürich

Prerequisites.

- Quadratic minimization problems, [Lecture → Section 1.2.3]
- Differential operators *grad* and div
- Multi-dimensional integration, [Lecture → Section 0.1.2.5]

Dependency. Depends on units for [Lecture → Section 1.2.3] and [Lecture → Section 1.3].

⚠ Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

I. Second-Order Scalar Elliptic Boundary Value Problems

1.4. Linear Variational Problems

Minimizer of (smooth) function = zero of derivative

Ex. 1.4.0.1: QMP in $\mathbb{R}^N$ for $J : \mathbb{R}^N \to \mathbb{R}$

$$J(\vec{\eta}) := \tfrac{1}{2}\vec{\eta}^\top \mathbf{A}\vec{\eta} - \vec{\beta}^\top \vec{\eta}\,, \quad \mathbf{A} = \mathbf{A}^\top \in \mathbb{R}^{N,N}\,, \quad \vec{\beta},\vec{\eta} \in \mathbb{R}^N\,, \tag{1.4.0.2}$$

↳ s.p.d. ⇒ invertible

$\vec{\mu}$ minimizer of J $\Rightarrow$ $\operatorname{grad} J(\vec{\mu}) = 0$

$$\operatorname{grad} J(\vec{\mu})\cdot\vec{\eta} = \lim_{h\to 0} \frac{J(\vec{\mu}+h\vec{\eta}) - J(\vec{\mu})}{h} = \dots$$

$$= (A\vec{\mu} - \vec{\beta})\cdot\vec{\eta}\,, \quad \vec{\eta} \in \mathbb{R}^N$$

$\vec{\mu}$ minimizer of J $\Rightarrow$ $\operatorname{grad} J(\vec{\mu}) = 0$

$\Leftrightarrow$ $(A\vec{\mu} - \vec{\beta})\cdot\vec{\eta} = 0 \quad \forall \vec{\eta} \in \mathbb{R}^N$

$\Leftrightarrow$ $A\vec{\mu} - \vec{\beta} = 0$

(2) $\hat{=}$ A linear system of equations (LSE)

1.4.1. Quadratic variational calculus

What does "$\operatorname{grad} J = 0$" mean in ∞ dimensions?

$\Rightarrow$ Variational calculus

$$J(v) = \tfrac{1}{2}\,a(v,v) - \ell(v)\,, \quad v \in V$$

s.p.d. $\quad$ cont. w.r.t. $\|\cdot\|_a$ on $V_0 \subset V$

QMP: $\quad u_* = \operatorname*{argmin}_{v \in \hat{V}} J(v)\,, \quad \hat{V} = g + V_0$ $\quad$ (1.4.1.2)

Pick arbitrary $v \in V_0$:

$$\varphi_v(t) := J(u_* + tv)\,, \quad t \in \mathbb{R} \qquad (1.4.1.3)$$

[$\varphi_v(t) = \tfrac{1}{2}t^2 a(v,v) + t\,a(u_*,v) - t\,\ell(v) + \text{const}$
a quadratic polynomial]

$$0 = \frac{d\varphi_v}{dt}(0) = a(u_*,v) - \ell(v) = 0$$

$\Rightarrow$ $\quad a(u_*,v) = \ell(v) \quad \forall v \in V_0$

$\hat{=}$ linear variational equation, a "∞-dimensional LSE"

Definition 1.4.1.6. (Generalized) Linear variational problem (LVP)

With V a vector space, $\hat{V} \subset V$ an affine space, and $V_0 \subset V$ the associated subspace the equation

$$\boxed{u \in \hat{V}: \quad \mathsf{a}(u,v) = \ell(v) \quad \forall v \in V_0}\,, \qquad (1.4.1.7)$$

is called a (generalized) **linear variational problem**, if

- $\mathsf{a} : V \times V_0 \mapsto \mathbb{R}$ is a bilinear form, that is, linear in both arguments ($\to$ Def. 0.1.1.7),
- and $\ell : V_0 \to \mathbb{R}$ is a linear form ($\to$ Def. 0.1.1.5).

trial space $\qquad$ test space

Q: If LVP has a solution then it will be unique if a is s.p.d.

Assume:
$$\begin{array}{rl} & a(u,v) = \ell(v) \\ - & a(w,v) = \ell(v) \\ \hline & a(u,v) - a(w,v) = 0 \end{array} \qquad \forall v \in V_0$$

$$a(\underbrace{u-w}_{\in V_0}, v) = 0$$

Pick $v := u - w \in V_0 \Rightarrow \|u-w\|_a = 0$

$\Rightarrow$ $u = w$

(3)

So far: QMP $\Longrightarrow$ LVP

Assume: LVP has a unique solution u [a s.p.d.]

$$a(u,v) = \ell(v) \quad \forall v \in V_0$$

$$J(v) = \tfrac{1}{2} a(v,v) - \ell(v)$$

$$= \tfrac{1}{2} a(v,v) - a(u,v)$$

$$= \tfrac{1}{2} a(v,v) - a(u,v) + \underbrace{\tfrac{1}{2} a(u,u) - \tfrac{1}{2} a(u,u)}_{=0}$$

$$= \underbrace{\tfrac{1}{2} a(u-v,u-v)}_{>0,\ \text{if } u \neq v} \underbrace{- \tfrac{1}{2} a(u,u)}_{\text{minimal value!}}$$

Theorem 1.4.1.8. Equivalence of quadratic minimization problem and linear variational problem

For a (generalized) quadratic functional $J(v) = \frac{1}{2}\mathsf{a}(v,v) - \ell(v) + c$ on a vector space V and with a symmetric positive definite bilinear form $\mathsf{a} : V \times V \to \mathbb{R}$ are equivalent:

(i) *The quadratic minimization problem for J has unique minimizer $u_* \in \widehat{V}$ over the affine subspace $\widehat{V} = g + V_0$, $g \in V$.*

(ii) *The linear variational problem*

$$u \in \widehat{V}: \quad \mathsf{a}(u,v) = \ell(v) \quad \forall v \in V_0\,,$$

has a unique solution $u_* \in \widehat{V}$.

1.4.2. Linear Second-Order Variational Problems

Now: $J(v) = \tfrac{1}{2} \int_\Omega \sigma(x) \|\operatorname{grad} v\|^2 - f v \, dx$

$$V = H^1(\Omega)\,,\quad V_0 = H_0^1(\Omega)\,,\quad \hat{V} = g + H_0^1(\Omega)$$

$\triangleright$ LVP:

$$u \in \hat{V}: \underbrace{\int_\Omega \sigma(x) \operatorname{grad} u \cdot \operatorname{grad} v}_{= a(u,v)} = \underbrace{\int_\Omega f v \, dx}_{= \ell(v)} \quad \forall v \in H_0^1(\Omega)$$

A generic second-order linear elliptic Dirichlet problem (∗) in variational form seeks

$$\begin{array}{l} u \in H^1(\Omega)\,, \\ u = g \text{ on } \partial\Omega \end{array}: \quad \int_\Omega (\boldsymbol{\alpha}(\mathbf{x}) \,\mathbf{grad}\, u(\mathbf{x})) \cdot \mathbf{grad}\, v(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) \,\mathrm{d}\mathbf{x} \quad \forall v \in H_0^1(\Omega)\,. \tag{1.4.2.4}$$

Symmetric uniformly positive definite material tensor $\boldsymbol{\alpha} : \Omega \mapsto \mathbb{R}^{d,d}$

Loading/source function

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1.4.3. Stability

Problem data:

$u \in H^1(\Omega)$, $u = g$ on $\partial\Omega$: $\displaystyle\int_\Omega (\alpha(x)\,\mathbf{grad}\,u(x)) \cdot \mathbf{grad}\,v(x)\,dx = \int_\Omega f(x)v(x)\,dx \quad \forall v \in H_0^1(\Omega)$.

data (obtained by measurements $\Rightarrow$ perturbed)

Assume: α uniformly positive

$$\alpha : \Omega \to \mathbb{R}\,,\quad \exists 0 < \alpha^- \leq \alpha^+ :\ \alpha^- \leq \alpha(x) \leq \alpha^+ \ \text{ for almost all } x \in \Omega\,, \tag{1.4.3.1}$$

Essential:

Small perturbations of data $\Rightarrow$ small perturbations of u

$\hookrightarrow$ in suitable norms $\qquad \hookrightarrow$ w.r.t. $\|\cdot\|_a$

- For f: L^2-norm $\|\cdot\|_0$
- For α: Maximum / L^∞-norm: $\|\cdot\|_\infty$

$\tilde{u} \in H^1(\Omega)$, $\tilde{u} = g$ on $\partial\Omega$:
$$\int_\Omega ((\alpha + \delta\alpha)(x)\,\mathbf{grad}\,\tilde{u}(x)) \cdot \mathbf{grad}\,v(x)\,dx = \int_\Omega (f + \delta f)(x)v(x)\,dx \quad \forall v \in H_0^1(\Omega)\,, \tag{1.4.3.3}$$

Assume: $\|\delta\alpha\|_\infty \leq \frac{1}{2}\alpha^-$

$\Rightarrow \quad \frac{1}{2}\alpha(x) \leq \alpha(x) + \delta\alpha(x) \leq \frac{3}{2}\alpha(x)$

$\Rightarrow \quad \frac{1}{2}\|v\|_a^2 \leq \int_\Omega (\alpha + \delta\alpha)(x)\,\|\mathbf{grad}\,v\|^2\,dx \leq \frac{3}{2}\|v\|_a^2$

$$\begin{aligned}
\int_\Omega (\alpha + \delta\alpha)(x)\,\mathbf{grad}\,\tilde{u}(x) \cdot \mathbf{grad}\,v(x)\,dx &= \int_\Omega (f + \delta f)(x)v(x)\,dx\,,\\
- \quad \int_\Omega (\alpha(x)\,\mathbf{grad}\,u(x)) \cdot \mathbf{grad}\,v(x)\,dx &= \int_\Omega f(x)v(x)\,dx\,,
\end{aligned}$$

$$\int_\Omega (\alpha + \delta\alpha)\,\mathbf{grad}\,\delta u \cdot \mathbf{grad}\,v\,dx + \int_\Omega \delta\alpha\,\mathbf{grad}\,u \cdot \mathbf{grad}\,v\,dx = \int_\Omega \delta f\,v\,dx\,, \quad \forall v \in H_0^1(\Omega)$$

perturbation of u: $\delta u = \tilde{u} - u \in H_0^1(\Omega)$

▷ Choose δu as test function

▷ $\displaystyle\int_\Omega (\alpha + \delta\alpha)\,\|\mathbf{grad}\,\delta u\|^2\,dx = \int_\Omega \delta f \cdot \delta u - \delta\alpha\,\mathbf{grad}\,u \cdot \mathbf{grad}\,\delta u\,dx$

$\Rightarrow \quad \frac{1}{2}\|\delta u\|_a^2 \leq \left|\int_\Omega \delta f \cdot \delta u\,dx\right| + \left|\int_\Omega \delta\alpha\,\mathbf{grad}\,u \cdot \mathbf{grad}\,\delta u\,dx\right|$

apply Cauchy-Schwarz $\qquad$ pull out $\|\delta\alpha\|_\infty$

$\Rightarrow \quad \frac{1}{2}\|\delta u\|_a^2 \leq \|\delta f\|_0 \|\delta u\|_0 + \|\delta\alpha\|_\infty |u|_1 |\delta u|_1$

Poincaré-Friedrich

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$$\Rightarrow \quad \tfrac{1}{2}\|\delta u\|_a^2 \leq \left(\|\delta f\|_0 \cdot \operatorname{diam}(\Omega) + \|\delta\alpha\|_\infty |u|_1\right) \cdot |\delta u|_1$$

Use $|\delta u|_1^2 \leq 1/\alpha_- \cdot \|\delta u\|_a^2$

$$\triangleright \quad \|\delta u\|_a \leq \frac{2}{\sqrt{\alpha^-}}\left(\|\delta f\|_0 \cdot \operatorname{diam}(\Omega) + \|\delta\alpha\|_\infty |u|_1\right)$$

$$\|\delta u\|_a \lesssim \max\{\|\delta f\|_{L^2(\Omega)}, \|\delta\alpha\|_{L^\infty(\Omega)}\}\ , \tag{1.4.3.7}$$

up to constants indep. of perturbations

Review questions 1.4.3.8

A : What is a linear variational problem posed on a vector space V_0

B :

State the linear variational problem equivalent to the minimization of

$$J : \mathbb{R}^2 \to \mathbb{R} \quad , \quad J(x) := \|x\|^2 + x_1 + x_2 + 1 .$$

C :

Let V be a vector space, $\mathsf{a} : V \times V \to \mathbb{R}$ a symmetric positive definite bilinear form and $\ell : V \to \mathbb{R}$ a linear form bounded w.r.t. to the energy norm $\|\cdot\|_a$ induced by $\mathsf{a}(c;\cdot)$. Assuming that the quadratic functional

$$J(v) := \tfrac{1}{2}\mathsf{a}(v,v) - \ell(v) + c\,, \quad v \in V\,, \tag{1.2.3.3}$$

has a minimizer, what is the minimal value of J expressed

- in terms of $\|u\|_a$, or
- in terms of $\ell(u)$,

where $u \in V$ satisfies

$$\mathsf{a}(u,v) = \ell(v) \quad \forall v \in V .$$

D :

For a bounded domain $\Omega \subset \mathbb{R}^2$ consider the second-order linear elliptic Dirichlet problem

$$u \in H_0^1(\Omega): \quad \int_\Omega (\boldsymbol{\alpha}(\boldsymbol{x})\, \mathbf{grad}\, u(\boldsymbol{x})) \cdot \mathbf{grad}\, v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(\Omega)\,.$$

with uniformly positive definite $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^{2,2}$.

1. For what source functions f will the right-hand side functional still be continuous on $H_0^1(\Omega)$? (Give a reasonably general sufficient condition.)
2. Let $u \in H_0^1(\Omega)$ be the solution of the above variational problem. For which norms introduced in Section 1.3 ($\square \in \{H^1(\Omega), L^2(\Omega)\}$) does the estimate

$$\|u\|_\square \leq C\|f\|_\square$$

hold with a constant $C > 0$ independent of f?

⚠ This list of review questions may not be complete. Additional review questions may be provided in the lecture document.

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