1

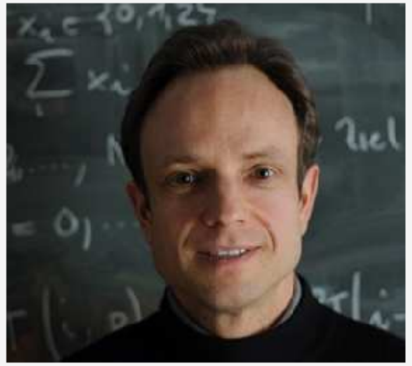
ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video

Section 1.5: Equilibrium Models: Boundary Value Problems

Prof. R. Hiptmair, SAM, ETH Zurich

Date: February 12, 2019



Prerequisites.

- Quadratic minimization problems, [Lecture → Section 1.2.3]
- Linear variational problems, [Lecture → Section 1.4]
- Multi-dimensional integration, [Lecture → Section 0.1.2.5] and

Dependency. Depends on units for [Lecture → Section 1.2.3] and [Lecture → Section 1.4]. Sobolev spaces from [Lecture → Section 1.3] will be used.

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

I. Second-Order Scalar Elliptic BVPs

1.5. Equilibrium Models: Boundary Value Problems

1.5.1. Two-Point BVP (1D)

LVP for elastic string model

$$u_* \in H^1(]a,b[)\,,\quad u(a)=u_a,\; u(b)=u_b : \quad \int_a^b \sigma(x)\frac{du}{dx}(x)\frac{dv}{dx}(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1(]a,b[)\,, \qquad (1.4.2.1)$$

with a uniformly positive stiffness coefficient function $\sigma \in C^0_{pw}([a,b])$ and $f \in L^2(]a,b[)$.

We require/assume more smoothness

Assumption 1.5.1.2. Smoothness required for two-point boundary value problems

The following smoothness requirements have to be satisfied

$$u \in C^2([a,b])\;,\quad \sigma \in C^1([a,b])\;,\quad f \in C^0([a,b])\,. \qquad (1.5.1.3)$$

vs.: for LVP:

$$u \in C^1_{pw}([a,b])\;,\quad \sigma \in C^0_{pw}([a,b])\;,\quad f \in C^0_{pw}([a,b])\,. \qquad (1.5.1.4)$$

Tease out a differential equation from LVP using integration by parts (IbP)

② Recall IbP in 1D: [IbP ≙ integration by parts]

Fund. Thm:
$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)\,, \qquad (1.5.1.6)$$

+ product rule:
$$F(x) = f(x)\cdot g(x) \;\Rightarrow\; F'(x) = f'(x)g(x) + f(x)g'(x)\,: \qquad (1.5.1.7)$$
$$\int_a^b f'(x)g(x) + f(x)g'(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a)\,,$$
$$\Downarrow$$
$$\int_a^b f(\xi)v'(\xi)\,\mathrm{d}\xi = -\int_a^b f'(\xi)v(\xi) + \underbrace{(f(b)v(b) - f(a)v(a))}_{\text{boundary terms}} \quad \forall f,v \in C^1_{\mathrm{pw}}([a,b])\,. \qquad (1.5.1.8)$$

↳ Note $\partial]a,b[= \{a,b\}$

$$(1.4.2.1) \Rightarrow \int_a^b \underbrace{\sigma u'}_{f} \cdot v'\,dx = \int_a^b f v\,dx$$
$$\Downarrow \text{ IbP}$$
$$-\int_a^b (\sigma u')' v\,dx + \underbrace{(\sigma u' v)(b) - (\sigma u' v)(a)}_{=0,\ \text{since } v(a)=v(b)=0} = \int_a^b f v\,dx$$

▷ $\int_a^b \underbrace{(-(\sigma u')' - f)}_{\in C^0([a,b])} v\,dx = 0 \quad \forall v \in H_0^1(\Omega)$

▷ ODE: $-(\sigma u')' = f$ in $]a,b[$

$u(a) = u_a \quad u(b) = u_b$

≙ 2-pt BVP (a Dirichlet problem)
in strong form (vs. "weak form" = LVP)

1.5.2. Integration by parts in higher dimension

I. "Fundamental Theorem" in d-dim., $\Omega \subset \mathbb{R}^d$

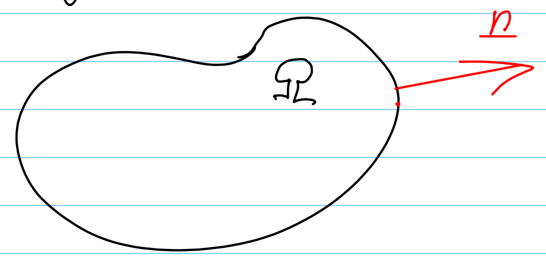
Theorem 1.5.2.4. Gauss' theorem →

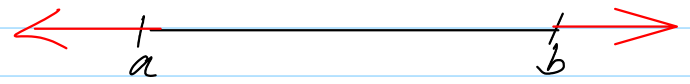
With $\boldsymbol{n} : \partial\Omega \mapsto \mathbb{R}^d$ denoting the exterior unit normal vectorfield on $\partial\Omega$ and $\mathrm{d}S$ indicating integration over a surface, we have
$$\int_\Omega \operatorname{div} \mathbf{j}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \mathbf{j}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) \quad \forall \mathbf{j} \in (C^1_{\mathrm{pw}}(\overline{\Omega}))^d\,. \qquad (1.5.2.5)$$

↳ a vectorfield $j : \Omega \to \mathbb{R}^d$

$$\operatorname{div} j = \frac{\partial j_1}{\partial x_1} + \dots + \frac{\partial j_d}{\partial x_d}$$

exterior unit normal vectorfield $n : \partial\Omega \to \mathbb{R}^d$





II. Product rule in dD

Lemma 1.5.2.1. General product rule

For all $\mathbf{j} \in (C^1(\overline{\Omega}))^d$, $v \in C^1(\overline{\Omega})$ holds
$$\operatorname{div}(\mathbf{j}v) = v \operatorname{div} \mathbf{j} + \mathbf{j} \cdot \mathbf{grad}\, v \quad \textit{in every point of } \Omega\,. \qquad (1.5.2.2)$$

Theorem 1.5.2.7: Green's first formula

For all vector fields $\mathbf{j} \in (C^1_{pw}(\overline{\Omega}))^d$ *and functions* $v \in C^1_{pw}(\overline{\Omega})$ *holds*

$$\int_\Omega \mathbf{j} \cdot \mathbf{grad}\, v \,\mathrm{d}\mathbf{x} = -\int_\Omega \mathrm{div}\,\mathbf{j}\, v \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{j} \cdot \mathbf{n}\, v \,\mathrm{d}S\,. \tag{1.5.2.8}$$

boundary term : a surface integral

1.5.3. Linear Scalar 2nd-Order Elliptic PDEs

LVP $\hat{=}$ 2nd-order Dirichlet problem in weak form

$$u \in H^1(\Omega),\ u = g \text{ on } \partial\Omega: \quad \int_\Omega (\alpha(x)\,\mathbf{grad}\, u(x)) \cdot \mathbf{grad}\, v(x) \,\mathrm{d}x = \int_\Omega f(x) v(x) \,\mathrm{d}x \quad \forall v \in H^1_0(\Omega), \tag{1.4.2.4}$$

[Assumption 1.5.3.1]

Assume extra smoothness : $\alpha\, \mathrm{grad}\, u \in C^1_{pw}(\overline{\Omega})$

Then IbP [apply Green's first formula]

$$\int_\Omega \alpha(x)\,\mathrm{grad}\, u \cdot \mathrm{grad}\, v \,dx = -\int_\Omega \mathrm{div}(\alpha\, \mathrm{grad}\, u) \cdot v \,dx + \int_{\partial\Omega} ((\alpha\, \mathrm{grad}\, u) \cdot n)\, v \,dS$$

boundary term

LVP $\Rightarrow$ $-\int_\Omega \mathrm{div}(\alpha\, \mathrm{grad}\, u) \cdot v \,dx + \int_{\partial\Omega} ((\alpha\, \mathrm{grad}\, u)\cdot n)\, v\, dS = \int_\Omega f v \,dx \quad \forall v \in C^1_{pw,0}(\Omega)$

$v|_{\partial\Omega} = 0 \Rightarrow$ drop boundary terms

$\triangleright$ $\displaystyle\int_\Omega \underbrace{(-\mathrm{div}(\alpha\, \mathrm{grad}\, u) - f)}_{\in C^0_{pw}(\overline{\Omega})} v \,dx = 0 \quad \forall v \in C^1_{pw,0}(\Omega)$

Lemma 1.5.3.4. Fundamental lemma of calculus of variations in higher dimensions

If $f \in L^2(\Omega)$ *satisfies*

$$\int_\Omega f(\mathbf{x}) v(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 0 \quad \forall v \in C^\infty_0(\Omega),$$

then $f \equiv 0$ *can be concluded.*

$\triangleright$ $-\mathrm{div}(\alpha(x)\,\mathrm{grad}\, u) = f(x)$ in Ω

a PDE

+ BDC $u = g$ on $\partial\Omega$ $\Rightarrow$ Dirichlet BVP

Remark :

Minimization problem (1.4.2)		Variational problem (1.4.2.4) weak form		BVP for PDE (1.5.3.8) strong form
$\mathbf{u}_* = \operatorname{argmin}_{\mathbf{v}\in V} J(\mathbf{v})$	❶ $\Leftrightarrow$	$\mathsf{a}(u,v) = \ell(v)\ \forall v$	❷ $\Rightarrow$	$\mathcal{L}u = f$ in Ω, $u\vert_{\partial\Omega} = g$.

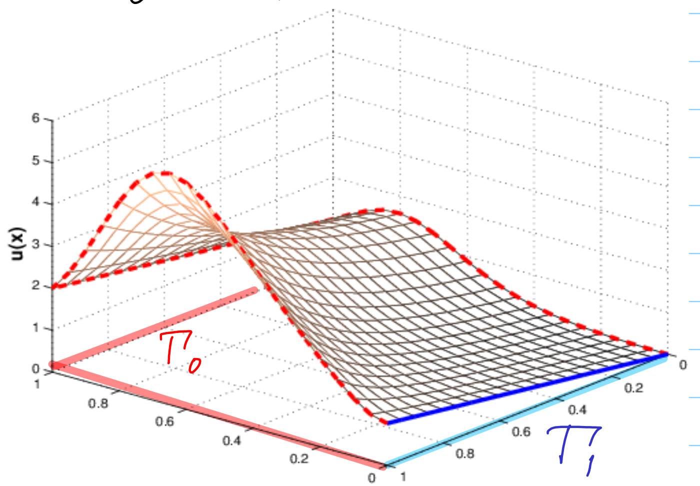
extra smoothness requirements

{ Weak solution } $\supset$ { strong solutions }

(4) Ex 1.5.3.11: Extension: Partly supported membrane

- - - : prescribed boundary values here (Γ_0)

—: "free boundary" (gap of frame)



▶ Configuration space $\hat{V} := \{u \in H^1(\Omega) : u_{|\Gamma_0} = g\}$ → Def. 1.3.4.8

The expression for the total potential energy remains the same as in (1.2.1.19):

$$\boxed{J_M(u) := \int_\Omega \tfrac{1}{2}\sigma(\mathbf{x})\|\mathbf{grad}\,u\|^2 - f(\mathbf{x})u(\mathbf{x})\,\mathrm{d}\mathbf{x}} \,. \qquad (1.2.1.19)$$

Towards LVP: Trial space $\hat{V}$ (affine space)

Test space $V_0 = \{v \in H^1(\Omega) : v_{|\Gamma_0} = 0\}$

▶ Variational formulation, *c.f.* (1.4.2.2)

$$\begin{array}{l} u \in H^1(\Omega)\,, \\ u = g \text{ on } \Gamma_0 \end{array} : \int_\Omega \sigma(\mathbf{x})\,\mathbf{grad}\,u(\mathbf{x}) \cdot \mathbf{grad}\,v(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_\Omega f(\mathbf{x})v(\mathbf{x}) \quad \forall v \in V_0 \,. \qquad (1.5.3.12)$$

Next: IbP

$$\int_\Omega -\mathrm{div}(\sigma\,\mathrm{grad}\,u)\cdot v\,dx + \int_{\partial\Omega\setminus\Gamma_0} ((\sigma\,\mathrm{grad}\,u)\cdot n)\,v\,dS = \int_\Omega f v\,dx \qquad v_{|\Gamma_0} = 0$$

Assumption: $u \in C^2_{pw}(\overline{\Omega})$, $\sigma \in C^1_{pw}(\overline{\Omega})$

Trick ①: First test with $v \in C^1_0(\overline{\Omega}) \subset V_0$

$$\triangleright \quad \int_\Omega (-\mathrm{div}(\sigma\,\mathrm{grad}\,u) - f)\,v\,dx = 0$$

Lemma 1.5.3.4. Fundamental lemma of calculus of variations in higher dimensions

If $f \in L^2(\Omega)$ *satisfies*

$$\int_\Omega f(\mathbf{x})v(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \quad \forall v \in C_0^\infty(\Omega)\,,$$

then $f \equiv 0$ *can be concluded.*

▷ PDE: $-\mathrm{div}(\sigma\,\mathrm{grad}\,u) = f$ in Ω

Trick ②: Test with general $v \in V_0$ & use the PDE

$$\cancel{\int_\Omega -\mathrm{div}(\sigma\,\mathrm{grad}\,u)\cdot v\,dx} + \int_{\partial\Omega\setminus\Gamma_0} ((\sigma\,\mathrm{grad}\,u)\cdot n)\,v\,dS = \cancel{\int_\Omega f v\,dx} = 0$$

5

$$\int_{\partial\Omega\setminus\Gamma_0} ((\sigma \operatorname{grad} u)\cdot n)\, v \, dS = 0 \quad \forall v,\ v_{|\Gamma_0} = 0$$
$$v \neq 0 \text{ on } \partial\Omega\setminus\Gamma_0$$

$$\triangleright \quad \underbrace{\sigma \operatorname{grad} u)\cdot n = 0 \text{ on } \partial\Omega\setminus\Gamma_0}_{\text{a "hidden" boundary condition}}$$

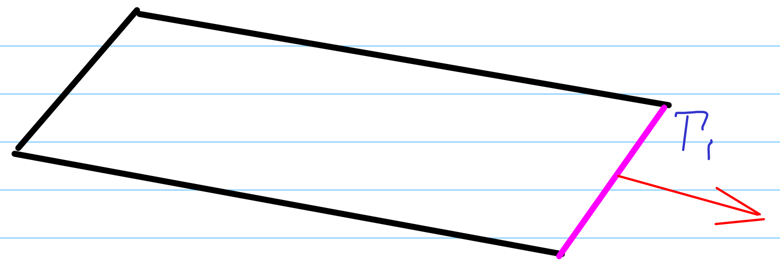
When removing pinning conditions on $\partial\Omega \setminus \Gamma_0$ the equilibrium conditions imply the (homogeneous) **Neumann boundary conditions** $(\sigma(\boldsymbol{x}) \, \mathbf{grad}\, u(\boldsymbol{x})) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0$ on $\partial\Omega \setminus \Gamma_0$.

Boundary value problem for membrane clamped at $\Gamma_0 \subset \partial\Omega$

$$-\operatorname{div}(\sigma(\boldsymbol{x}) \, \mathbf{grad}\, u) = f \quad \text{in } \Omega\,, \qquad \begin{aligned} u &= g && \text{on } \Gamma_0\,, \\ (\sigma(\boldsymbol{x}) \, \mathbf{grad}\, u) \cdot \boldsymbol{n} &= 0 && \text{on } \partial\Omega \setminus \Gamma_0\,. \end{aligned} \tag{1.5.3.18}$$

$\downarrow$

Q: Physical meaning



At $x \in \Gamma_1$: Membrane is flat in the direction perpendicular to its edge.

Supplement

$\Leftarrow$ For this argument we rely on the following version of Lemma 1.5.3.4:

Lemma 1.5.3.16. Fundamental lemma of calculus of variations on boundaries

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary and $\Gamma_0 \subset \partial\Omega$ a part of $\partial\Omega$ with non-zero measure. If $g \in C^0(\partial\Omega)$ satisfies

$$\int_{\Gamma_0} g(\boldsymbol{x}) v(\boldsymbol{x}) \, \mathrm{d}S(\boldsymbol{x}) = 0 \quad \forall v \in C^\infty(\overline{\Omega})\,,$$

then g vanishes on Γ_0: $g|_{\Gamma_0} \equiv 0$.

(6) Review questions 15.3.19.

A:

State **Gauss' theorem** for a vector field $\mathbf{j} \in (C^1(\overline{\Omega}))^d$ on a domain $\Omega \subset \mathbb{R}^d$.

B:

What is meant by the **weak form** and the **strong form** of an elliptic boundary value problem?

C:

What 2-point boundary value problem corresponds to the variational problem

$$\begin{aligned} u_* &\in H^1(]a,b[)\,, \\ u(a) = u_a&,\ u(b) = u_b \end{aligned} : \quad \int_a^b \frac{du}{dx}(x)\frac{dv}{dx}(x) + u(x)v(x)\,\mathrm{d}x = \int_a^b f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1(]a,b[)\,,$$

D:

State the 2-point boundary value problem satisfied by the solution of the variational equation

$$u \in H^1(]0,1[): \quad \int_0^1 (1+x^2)\frac{du}{dx}(x)\Big(\frac{dv}{dx}(x) - v(x)\Big) = v(0) \quad \forall v \in H^1(]0,1[)\,.$$

E:

Give an example for a linear variational problem on $H_0^1(]0,1[)$ with continuous and s.p.d. bilinear form and continuous right-hand side linear form whose solution will not be the solution of a two-point boundary value problem in the sense of classical calculus.

F:

State the boundary value problem satisfied by the solution of the following variational problen

$$\begin{aligned} u &\in H^1(\Omega)\,, \\ u &= g \text{ on } \partial\Omega \end{aligned} : \quad \int_\Omega \frac{\partial u}{\partial x_1}(\boldsymbol{x})\frac{\partial v}{\partial x_1} + 2\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall v \in H_0^1(\Omega)\,.$$

G:

For a bounded domain $\Omega \subset \mathbb{R}^3$ state the second-order elliptic boundary value problem associated with the linear variational problem

$$u \in H^1(\Omega): \quad \int_\Omega \mathbf{grad}\, u(\boldsymbol{x}) \cdot \mathbf{grad}\, v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega} v(\boldsymbol{x})\,\mathrm{d}S(\boldsymbol{x}) \quad \forall v \in H^1(\Omega)\,.$$

H:

We consider the variational problem

$$\mathbf{u} : \Omega \to \mathbb{R}^2: \quad \int_\Omega \operatorname{div} \mathbf{u}(\boldsymbol{x}) \operatorname{div} \mathbf{v}(\boldsymbol{x}) + \mathbf{u}(\boldsymbol{x}) \cdot \mathbf{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \mathbf{v}(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall \mathbf{v} : \Omega \to \mathbb{R}^2\,.$$

What is a suitable Sobolev space and what boundary value problem is satisfied by the vector field $\mathbf{u}$?

7

I :

Which boundary value problem does the minimizer of the functional

$$J(v) = \int_{\Omega} \left| \frac{\partial u}{\partial x_1}(\mathbf{x}) - \frac{\partial u}{\partial x_2}(\mathbf{x}) \right|^2 + |u(\mathbf{x})|^2 - \|\mathbf{x}\| u(\mathbf{x}) \, \mathrm{d}\mathbf{x} \,, \quad v \in H_0^1(\Omega) \,,$$

solve? Here, $\Omega \subset \mathbb{R}^2$ is a bounded domain.