ETH Lecture 401-0674-00L Numerical Methods for Partial Differential Equations

Course Video
Section 1.7: Boundary Conditions

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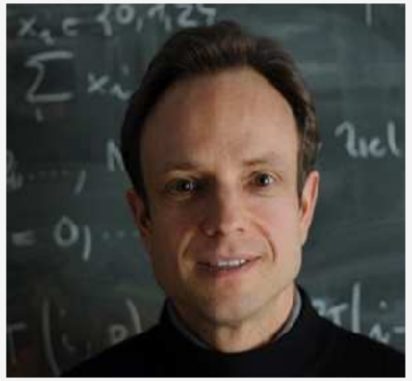


Prerequisites.

- Concept of a flux field
- Differential operators **grad** and div

Dependency. Relies on unit on [Lecture → Section 1.6].

Note: Possible minor *mismatch of video and tablet notes*!

[Corrections and updates can be incorporated into tablet notes only]

I. Second-Order Scalar Elliptic BVPs

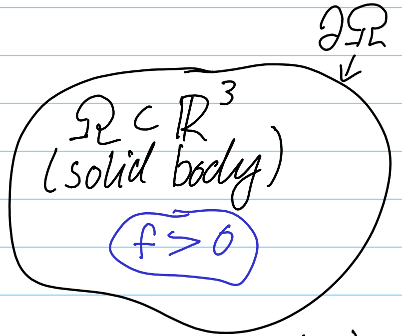
1.7. Boundary Conditions (BDC)

PDE + BDC ⟹ BVP

this can well-posed (↑ E & U, stability)

Context: Heat flow (stationary)

What are good BDC for
- temperature u
- heat flux j



(i) Fix temperature on surface $\partial\Omega$

$u = g$ on $\partial\Omega$, $g : \partial\Omega \to \mathbb{R}$ given

(ii) $j(x) \cdot n(x) = -h(x)$ for $x \in \partial\Omega$ for given $h : \partial\Omega \to \mathbb{R}$

(iii) $j(x) \cdot n(x) = \Psi(u(x))$ on $\partial\Omega$: heat flux ~ temp.

↳ $\Psi : \mathbb{R} \longrightarrow \mathbb{R}$

(2)

Fundamental boundary conditions for 2nd-order elliptic BVPs

Boundary conditions on surface/boundary $\partial\Omega$ of Ω:

(i) Temperature u is fixed: with $g : \partial\Omega \mapsto \mathbb{R}$ prescribed

$$u = g \quad \text{on } \partial\Omega . \tag{1.7.0.2}$$

▶ **Dirichlet boundary conditions**

(ii) Heat flux $\mathbf{j}$ through $\partial\Omega$ is fixed: with $h : \partial\Omega \mapsto \mathbb{R}$ prescribed ($\boldsymbol{n} : \partial\Omega \mapsto \mathbb{R}^3$ exterior unit normal vectorfield) on $\partial\Omega$

$$\mathbf{j}\cdot\boldsymbol{n} = -h \quad \text{on} \quad \partial\Omega . \tag{1.7.0.3}$$

▶ **Neumann boundary conditions**

(iii) Heat flux through $\partial\Omega$ depends on (local) temperature: with increasing function $\Psi : \mathbb{R} \mapsto \mathbb{R}$

$$\mathbf{j}\cdot\boldsymbol{n} = \Psi(u) \quad \text{on } \partial\Omega \tag{1.7.0.4}$$

▶ **radiation boundary conditions**
[generically nonlinear]

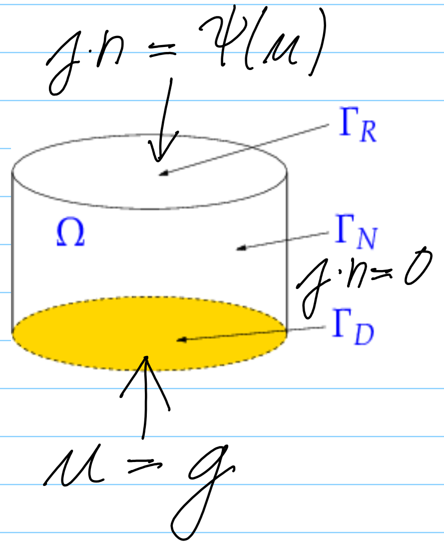
By Fourier law: $j = -\kappa \operatorname{grad} u$
Neumann BDC $\Rightarrow$ $(\kappa \cdot \operatorname{grad} u)\cdot n = h$ on $\partial\Omega$

Linear radion BDC*: $j\cdot n = q(x)\,(u - u_0)$
[convective cooling impedance BDC] ↳ $q : \partial\Omega \to \mathbb{R}$ uniformly positive

* also called "impedance BDC"

Mixed BVPs $\hat{=}$ different types of BDC are imposed on different parts of the boundary

Example 1.7.0.10 ("Wrapped rock on a stove")



- Non-homogeneous Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$
- Homogeneous Neumann boundary conditions on $\Gamma_N \subset \partial\Omega$
- Convective cooling boundary conditions on $\Gamma_R \subset \partial\Omega$

Partition: $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R$, $\Gamma_D, \Gamma_N, \Gamma_R$ mutually disjoint

Γ_N covered with insulating material

Another example: Partly clamped membrane
[Example 1.5.3.11]

For second order elliptic boundary value problems *exactly one* boundary condition is needed on *every* part of $\partial\Omega$.

③

Review questions 1.7.0.12

Review question(s) 1.7.0.12(Boundary conditions for stationary heat conduction)

(Q1.7.0.12.A) Give the formula for the exterior unit normal vectorfield $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^2$ for $\Omega := \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\| < \frac{1}{2}\}$.

(Q1.7.0.12.B) For the second-order elliptic partial differential equation

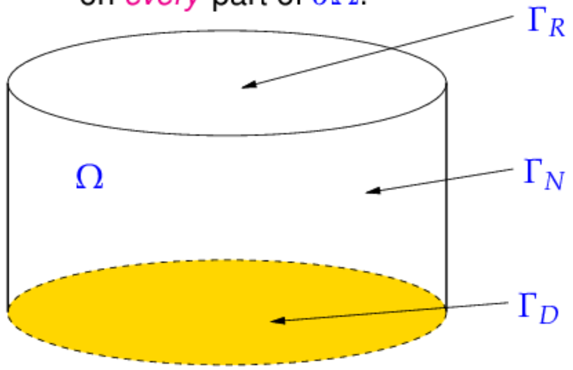
$$-2\frac{\partial^2 u}{\partial x_1{}^2} - \frac{\partial^2 u}{\partial x_2{}^2} + u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^3 \, .$$

give the formulas for

1. **Dirichlet boundary conditions** on $\partial\Omega$,
2. **Neumann boundary conditions** on $\partial\Omega$,
3. and **impedance boundary conditions** on $\partial\Omega$.

(Q1.7.0.12.C) We learned that

For second order elliptic boundary value problems *exactly one* boundary condition is needed on *every* part of $\partial\Omega$.



Describe in your own words what this rule means for the setting discussed in Ex. 1.7.0.10, see figure beside.

△

Review question(s) 1.7.0.13(Boundary conditions for 2nd-order elliptic BVPs)

In this quizz we consider *stationary electric currents* in a conducting body occupying $\Omega \subset \mathbb{R}^3$. In this model a vector field $\mathbf{j} : \Omega \to \mathbb{R}^3$ describes the electric current density (units $[\mathbf{j}] = \frac{\mathrm{A}}{\mathrm{m}^2}$) obeying *Ohm's law* $\mathbf{j} = -\sigma \,\mathbf{grad}\, u$, which corresponds to Fourier's law (1.6.0.5). Here, u is the electric potential, *cf.* (1.2.2.2) (units $[u] = \mathrm{V}$), and $\sigma : \Omega \to \mathbb{R}^+$ stands for the uniformly positive conductivity (units $[\sigma] = \frac{\mathrm{A}}{\mathrm{Vm}}$).

(Q1.7.0.13.A) What is the meaning of $\operatorname{div} \mathbf{j}$?

(Q1.7.0.13.B) Argue, why the *normal component* of $\mathbf{j}$ has to be continuous across any smooth surface.

(Q1.7.0.13.C) What is the physical meaning of Dirichlet and Neumann boundary conditions in the stationary current model?

(Q1.7.0.13.D) What could be described by a linear radiation boundary condition

$$\mathbf{j} \cdot \boldsymbol{n} = -h \quad \text{on} \quad \partial\Omega \, , \tag{1.7.0.3}$$

for the stationary current model?

△

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